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Double Ward quasigroups

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Abstract

In this short note, we prove a one-to-one correspondence between groups and a variety of quasigroups that we call double Ward quasigroups analogous to the correspondence between groups and Ward quasigroups.

1. Introduction

A quasigroup consists of a non-empty set Q equipped with a binary operation $*$ such that for all $a, b \in Q$, there exist unique $x, y \in Q$ such that $a * x = b$ and $y * a = b$. Alternatively, a quasigroup is an algebra $(Q; *, \backslash, /)$ of type $(2, 2, 2)$ such that $x\backslash(x * y) = y$, $(x * y)/y = x$, $x * (x\backslash y) = y$, and $(x/y) * y = x$.

Given a group $(G; \circ, ^{-1}, e)$, we can construct a quasigroup by defining $x * y = x \circ y^{-1}$. The operation $*$ on G is sometimes referred to as *right division* in G. Clearly, this quasigroup satisfies the identity $(x * z) * (y * z) = x * y$. Quasigroups satisfying the above identity are referred to as *Ward quasigroups*. Conversely, given a Ward quasigroup Q, it can be shown that Q is *unipotent* ($x * x = y * y$), so we may write $x * x = e$, and defining $x^{-1} = e * x$ and $x \circ y = x * y^{-1}$ makes $(Q; \circ, ^{-1}, e)$ a group. Writing $W(G)$ for the Ward quasigroup constructed from the group G and $Gr(Q)$ for the group constructed from the Ward quasigroup Q, it can also be shown that $Gr(W(G)) = G$ and $W(Gr(Q)) = Q$. Therefore, there is a one-to-one correspondence between groups and Ward quasigroups. This seems to have first been noticed in [1] and [3].

Similarly, given a group $(G; \circ, ^{-1}, e)$, we can construct a quasigroup by defining $x * y = x^{-1} \circ y^{-1}$. The operation $*$ on G is sometimes referred to as *double division* in G. Clearly, this quasigroup satisfies the Ward-like identity

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$((e*e)*(x*z))*((e*y)*z) = x*y$. For lack of a better term, we will refer to quasigroups with an element e that satisfy the above identity as *double Ward quasigroups* (not to be confused with the Ward double quasigroups of [4]). In this short note, we prove an analogous one-to-one correspondence between groups and double Ward quasigroups.

Remark 1.1. The author gratefully acknowledges the assistance of the automated theorem-prover Prover9 [2]. Theorem 2.2 was found and proved with the aid of Prover9. As such, the proof has been suppressed. However, the proof is available from the author or can quickly be regenerated with Prover9.

2. Results

Theorem 2.2. *Let Q be a double Ward quasigroup. Define $x^{-1} = e*x$ and $x \circ y = x^{-1} * y^{-1}$. Then $(Q; \circ, ^{-1}, e)$ is a group.*

Theorem 2.3. *Let G be a group and let Q be a double Ward quasigroup. Denote by $DW(G)$ the double Ward quasigroup constructed from G and denote by $Gr(Q)$ the group constructed from Q. Then $Gr(DW(G)) = G$ and $DW(Gr(Q)) = Q$.*

Problem 2.4. Characterize double Ward quasigroups by a shorter more appealing identity such as is the case for Ward quasigroups.

Problem 2.5. Prove similar results for other well-known varieties of loops with two-sided inverses, such as (left, right) inverse property loops, anti-automorphic inverse property loops, extra loops, Moufang loops, left (right) Bol loops, C-loops, LC-loops, RC-loops, and flexible loops.

References

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