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# S-systems of n-ary quasigroups

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**Abstract**

In the theory of binary quasigroups the notions of a right (left) $S$-system and an $S$-system [1] are known. An $S$-system is simultaneously a left and right $S$-system. We introduce $(k)$-$S$-systems and $S$-systems (otherwise than in [10]) of $n$-ary quasigroups for $n \geqslant 2$ and $1 \leqslant k \leqslant n$, give examples of such systems and prove that any $(k)$-$S$-system, given on a finite set, is a pairwise orthogonal set ([3]) of $n$-ary operations.

## 1. Introduction

In the theory of binary quasigroups the notion of a right (left) Stein system (shortly, a right $S$-system or a left $S$-system) is known. Such system is defined in the following way [1].

A system $Q(\Sigma)$, $\Sigma = \{E, A_1^s\}$ $(\Sigma = \{F, A_1^s\}$, where $A_1^s$ denotes the sequence $A_1, A_2, ..., A_s)$, which consists of binary quasigroups and the right (left) identity operation $E$ $(F)$: $E(x,y) = y$ $(F(x,y) = x)$ given on a set $Q$ is called a right (left) $S$-system if $\Sigma$ is a group with respect to the Stein's right (left) multiplication $\cdot$ $(\circ)$ of binary operations:

$$(A \cdot B)(x,y) = A(x, B(x,y)) \ ((A \circ B)(x,y) = A(B(x,y), y)).$$

A system $Q(\Sigma)$, $\Sigma = \{E, F, A_1^s\}$, is called an $S$-*system* if $\Sigma' = \{E, A_1^s\}$ $(\Sigma'' = \{F, A_1^s\})$ is a right (left) $S$-system.

Finite binary $S$-systems are completely described in the works [1], [5], [6] by V.Belousov, G. Belyavskaya and A. Cheban.

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Any two operations $A$ and $B$ on a set $Q$ from a right (left) $S$-system $Q(\Sigma)$ of binary quasigroups are orthogonal, that is the pair of equations $A(x,y) = a, B(x,y) = b$ has a unique solution for any $a, b \in Q$ and any $A, B \in \Sigma$, $A \neq B$.

In this article we introduce $(k)$-$S$-systems of $n$-ary quasigroups for $n \geqslant 2$, $1 \leqslant k \leqslant n$, give some examples of such systems and prove that any finite $(k)$-$S$-system is a pairwise orthogonal set. We also consider $S$-systems of $n$-ary quasigroups in the more natural sense, than the $S$-systems of T. Yakubov [10], and prove that such a finite $S$-system contains only one $n$-quasigroup, whereas $S$-systems of [10] do not at all exist.

## 2. Necessary notions and results

We recall some notations, concepts and results which are used in the article. At first remember the following notations from [2]. By $x_i^j$ we will denote the sequence $x_i, x_{i+1}, \ldots, x_j$, $i \leqslant j$. If $j < i$, then $x_i^j$ is the empty sequence, $\overline{1,n} = \{1, 2, \ldots, n\}$. Let $Q$ be a finite or an infinite set, $n \geqslant 2$ be a positive integer and let $Q^n$ denote the Cartesian power of the set $Q$.

An *$n$-ary operation* $A$ (briefly, an *$n$-operation*) on a set $Q$ is a mapping $A : Q^n \to Q$ defined by $A(x_1^n) \to x_{n+1}$, and in this case we write $A(x_1^n) = x_{n+1}$.

A *finite $n$-groupoid* $(Q, A)$ *of order* $m$ is a set $Q$ with one $n$-ary operation $A$ defined on $Q$, where $|Q| = m \geqslant 2$.

An *$n$-ary quasigroup ($n$-quasigroup)* is an $n$-groupoid such that in the equality

$$A(x_1^n) = x_{n+1}$$

every $n$ elements from $x_1^{n+1}$ uniquely define the $(n+1)$-th element. Usually a quasigroup $n$-operation $A$ is itself considered as an $n$-quasigroup.

The $n$-operation $E_k$, $1 \leqslant k \leqslant n$, on a set $Q$ with $E_k(x_1^n) = x_k$ is called the *$k$-th identity operation* (or the *$k$-th selector) of arity $n$*.

An $n$-operation $A$ on $Q$ is called *$k$-invertible* for some $k \in \overline{1,n}$ if the equation

$$A(a_1^{k-1}, x_k, a_{k+1}^n) = a_{n+1}$$

has a unique solution for each fixed $n$-tuple $(a_1^{k-1}, a_{k+1}^n, a_{n+1}) \in Q^n$.

For a $k$-invertible $n$-operation there exists the *$k$-inverse $n$-operation* ${}^{(k)}A$ defined in the following way:

$${}^{(k)}A(x_1^{k-1}, x_{n+1}, x_{k+1}^n) = x_k \Leftrightarrow A(x_1^n) = x_{n+1}$$

for all $x_1^{n+1} \in Q^{n+1}$.

It is evident that

$$A(x_1^{k-1}, {}^{(k)}A(x_1^n), x_{k+1}^n) = {}^{(k)}A(x_1^{k-1}, A(x_1^n), x_{k+1}^n) = x_k$$

and ${}^{(k)}[{}^{(k)}A] = A$ for $k \in \overline{1,n}$.

Let $\Omega_n$ be the set of all $n$-ary operations on a finite or an infinite set $Q$. On $\Omega_n$ define a binary operation $\underset{k}{\oplus}$ (the $k$-multiplication) in the following way:

$$(A \underset{k}{\oplus} B)(x_1^n) = A(x_1^{k-1}, B(x_1^n), x_{k+1}^n),$$

$A, B \in \Omega_n$, $x_1^n \in Q^n$. Shortly this equality can be written as

$$A \underset{k}{\oplus} B = A(E_1^{k-1}, B, E_{k+1}^n)$$

where $E_k$ is the $k$-th selector.

In [11] it was proved that $(\Omega_n, \underset{k}{\oplus})$ is a semigroup with the identity $E_k$. If $\Lambda_k$ is the set of all $k$-invertible $n$-operations from $\Omega_n$ for some $k \in \overline{1,n}$, then $(\Lambda_k, \underset{k}{\oplus})$ is a group. In this group $E_k$ is the identity, the inverse element of $A$ is the operation ${}^{(k)}A \in \Lambda_k$, since $A \underset{k}{\oplus} E_k = E_k \underset{k}{\oplus} A$, $A \underset{k}{\oplus} {}^{(k)}A = {}^{(k)}A \underset{k}{\oplus} A = E_k$.

An $n$-ary quasigroup $(Q, A)$ (or simply $A$), is an $n$-groupoid with an $k$-invertible $n$-operation for each $k \in \overline{1,n}$ [2].

Let $(x_1^n)_k$ denote the $(n-1)$-tuple $(x_1^{k-1}, x_{k+1}^n) \in Q^{n-1}$ and let $A$ be an $n$-operation, then the $(n-1)$-operation $A_a$:

$$A_a(x_1^n)_k = A(x_1^{k-1}, a, x_{k+1}^n)$$

is called the $(n-1)$-*retract* of $A$, defined by position $k$, $k \in \overline{1,n}$, with the element $a$ in this position (with $x_k = a$).

If in an $n$-operation $A$ we fix all positions except two positions $k$ and $l$ we obtain a binary operation $\overline{A}(x_k, x_l) = A(a_1^{k-1}, x_k, a_{k+1}^{l-1}, x_l, a_{l+1}^n)$ which is called a *binary retract* of $A$ [2].

An $n$-ary operation $A$ on $Q$ is called *complete* if there exists a permutation $\overline{\varphi}$ of $Q^n$ such that $A = E_1\overline{\varphi}$ (that is $A(x_1^n) = E_1\overline{\varphi}(x_1^n)$). If a complete $n$-operation $A$ is finite and has order $m$, then the equation $A(x_1^n) = a$ has exactly $m^{n-1}$ solutions for any $a \in Q$ [11].

Any $k$-invertible $n$-operation $A$, $k \in \overline{1,n}$, is complete, but there exist complete $n$-operations, which are not $k$-invertible for each $k \in \overline{1,n}$ [11].

**Definition 1.** (cf. [3]) Two $n$-ary operations ($n \geqslant 2$) $A$ and $B$ given on a set $Q$ of order $m$ are called *orthogonal* (shortly, $A \perp B$) if the system $\{A(x_1^n) = a, B(x_1^n) = b\}$ has exactly $m^{n-2}$ solutions for any $a, b \in Q$.

A set $\Sigma = \{A_1^s\}$, $s \geqslant 2$, of $n$-operations is called *pairwise orthogonal* if each pair of distinct $n$-operations from $\Sigma$ is orthogonal.

It is an algebraic analog of orthogonality of $n$-dimensional hypercubes which (just as $n$-operations and $n$-quasigroups) are used in various areas including affine and projective geometries, designs of experiments, error-correcting and error-detecting coding theory and cryptology.

In the article [7] a connection between $n$-dimensional hypercubes and $n$-ary operations and different types of their orthogonality were considered. The pairwise orthogonality is the weakest from these types.

In [3] the algebraic approach was first used for study of orthogonality of two $n$-dimensional hypercubes and the following criterion of orthogonality of two finite $k$-invertible $n$-operations was established.

**Theorem 1.** (cf. [3]) *Let $k$ be a fixed number from $\overline{1,n}$. Two finite $k$-invertible $n$-operations $A$ and $B$ on a set $Q$ are orthogonal if and only if the $(n-1)$-retract $C_a$ of the $n$-operation $C = B \underset{k}{\oplus}{}^{(k)} A$, defined by $x_k = a$, is complete for every $a \in Q$.*

## 3. $(k)$-$S$-systems of $n$-quasigroups

For the $n$-ary case, $n \geqslant 2$, we introduce $(k)$-$S$-systems of $n$-quasigroups in the following way.

**Definition 2.** A system $Q(\Sigma_k)$, $\Sigma_k = \{E_k, A_1^s\}$, $s \geqslant 1$, where all $A_i$ are $n$-quasigroups, given on a set $Q$, is called a *$(k)$-$S$-system of $n$-quasigroups* if $(\Sigma_k, \underset{k}{\oplus})$ is a group.

If $n = 2$ and $\Sigma_2 = \{E, A_1^s\}$ $\quad(\Sigma_1 = \{F, A_1^s\})$ we obtain a right (left) $S$-system of binary quasigroups, since $\underset{2}{\oplus} = \cdot$ $(\underset{1}{\oplus} = \circ)$ (the right and the left multiplications of binary operations respectively).

**Examples of $(k)$-$S$-systems**. Let $(Q, +)$ be an elementary abelian group (that is a group which is a direct power of a group of a prime order $p$ [9]) of order $m = p^t$, $p \geqslant 3$, and an $n$-quasigroup $(Q, A)$ has the form:

$$A(x_1^n) = \alpha_1 x_1 + \ldots + \alpha_{k-1}x_{k-1} + x_k + \alpha_{k+1}x_{k+1} + \ldots + \alpha_n x_n \qquad (1)$$

where all $\alpha_i$ are permutations of $Q$. Consider the $(k)$-powers $A, A^2, \ldots, A^{p-1}$, that is the powers of $A$ with respect to the $k$-multiplication of $n$-operations: $A^l = A \underset{k}{\oplus} A \underset{k}{\oplus} \ldots \underset{k}{\oplus} A$ ($l$ times) [4]. By Corollary 6 of [4] all these powers are $n$-quasigroups, $A^p = E_k$ and $(\Sigma'_k, \underset{k}{\oplus})$ where $\Sigma'_k = \{E_k, A, A^2, \ldots, A^{p-1}\}$ is a (cyclic) group. Hence, $Q(\Sigma'_k)$ is a $(k)$-$S$-systems of $n$-quasigroups.

Moreover, if $m = p \geqslant 3$ and in (1) $\alpha_i$ is the identity permutation for each $i \in \overline{1,n}$, that is

$$A(x_1^n) = x_1 + x_2 + \ldots + x_n, \tag{2}$$

then $Q(\Sigma'_k)$ is a $(k)$-$S$-system for any $k \in \overline{1,n}$.

Note, that $n$-quasigroups of $\Sigma'_k$ can be different from $n$-quasigroups of $\Sigma'_l$, if $k \neq l$. So, it is easy to check that if an $n$-quasigroup $A$ of order $p$ has the form (2), then the sets $\Sigma'_k$ and $\Sigma'_l$ are intersected only by the $n$-quasigroup $A$.

Indeed, let $1 \leqslant k \leqslant l \leqslant n$ and the $(k)$-power $A^r$ coincide with the $(l)$-power $A^t$ for $1 \leqslant r < t \leqslant p-1$. Then

$$r(x_1+...+x_{k-1})+x_k+r(x_{k+1}+...+x_n) = t(x_1+...+x_{l-1})+x_l+t(x_{l+1}+...+x_n),$$

whence it follows that

$$(t-r)(x_1+\ldots+x_{k-1})+t(x_k+\ldots+x_{l-1})+x_l-x_k-r(x_{k+1}+\ldots+x_l)+$$
$$+(t-r)(x_{l+1}+\ldots+x_n)=0.$$

Setting $x_1 = \ldots = x_{k-1} = x_{k+1} = \ldots = x_{l-1} = x_{l+1} = \ldots = x_n = 0$, we obtain $tx_k - x_k = rx_l - x_l$ for all $x_k, x_l$ of $Q$ and so $t = r = 1$. □

**Proposition 1.** *Let $Q(\Sigma_k)$, $\Sigma_k = \{E_k, A_1^s\}$, be a $(k)$-$S$-system of $n$-quasigroups, $n \geqslant 3$, $1 \leqslant l < k \leqslant n$ and $u = a_1^{l-1}$, $v = a_{l+1}^{k-1}$, $w = a_{k+1}^n$ be fixed (ordered) tuples of elements from $Q$. Then the system $Q(\Sigma_{u,v,w})$ of binary retracts where $\Sigma_{u,v,w} = \{E, \overline{A}_1^s\}$ with $\overline{A}_i(x_l, x_k) = A_i(u, x_l, v, x_k, w)$, is a right $S$-system of binary quasigroups for any $u \in Q^{l-1}$, $v \in Q^{k-l-1}$, $w \in Q^{n-k}$.*

*Proof.* We must prove that $\Sigma_{u,v,w}$ is a group with respect to the right multiplication of binary operations. At first we note that $E_k(u, x_l, v, x_k, w) = \overline{E}_k(x_l, x_k) = x_k$, that is $\overline{E}_k = E$.

Let $A_i \in \Sigma_k$, then ${}^{(k)}A_i \in \Sigma_k$, $\overline{A}_i \in \Sigma_{u,v,w}$ and ${}^{(k)}\overline{A}_i \in \Sigma_{u,v,w}$. Prove that ${}^{(2)}\overline{A}_i \in \Sigma_{u,v,w}$. Indeed, from $(A_i \underset{k}{\oplus} {}^{(k)} A_i)(x_1^n) = x_k$ it follows

$$(A_i \underset{k}{\oplus} {}^{(k)} A_i)(u, x_l, v, x_k, w) = A_i(u, x_l, v, {}^{(k)} A_i(u, x_l, v, x_k, w), w) = \overline{A}_i(x_l, {}^{(k)} \overline{A}_i(x_l, x_k)) = x_k.$$

But $\overline{A}_i(x_l, {}^{(2)} \overline{A}_i(x_l, x_k)) = x_k$. Hence, ${}^{(k)}\overline{A}_i = {}^{(2)} \overline{A}_i$ and ${}^{(2)}\overline{A}_i \in \Sigma_{u,v,w}$.

Further, if $A_i \underset{k}{\oplus} A_j = A_r \in \Sigma_k$, then

$$\begin{aligned}(A_i \underset{k}{\oplus} A_j)(u, x_l, v, x_k, w) &= A_i(u, x_l, v, A_j(u, x_l, v, x_k, w), w) \\ &= \overline{A}_i(x_l, \overline{A}_j(x_l, x_k)) = (\overline{A}_i \cdot \overline{A}_j)(x_l, x_k) \\ &= A_r(u, x_l, v, x_k, w) = \overline{A}_r(x_l, x_k),\end{aligned}$$

that is $\overline{A}_i \cdot \overline{A}_j = \overline{A}_r \in \Sigma_{u,v,w}$.

It still remains to prove that $\overline{A}_i \neq \overline{A}_j$ if $i \neq j$. Let $\overline{A}_i = \overline{A}_j$, then $A_i(u, x_l, v, x_k, w) = A_j(u, x_l, v, x_k, w)$, ${}^{(k)}A_i(u, x_l, v, A_j(u, x_l, v, x_k, w), w) = x_k$. But $B = {}^{(k)} A_i \underset{k}{\oplus} A_j \in \Sigma_k$, so $B(u, x_l, v, x_k, w) = x_k$ for any $x_l \in Q$ implies that $B$ is not an $n$-quasigroup, so $B = E_k$ and $i = j$.

Therefore, we proved that the set $\Sigma_{u,v,w}$ is a group with respect to the right multiplication of binary operations. □

**Remark.** If in Proposition 1 $k < l$, $u = a_1^{k-1}$, $v = a_{k+1}^{l-1}$, $w = a_{l+1}^n$, $\overline{A}_i(x_k, x_l) = A_i(u, x_k, v, x_l, w)$, then analogously one can prove that $\Sigma_{u,v,w} = = \{F, \overline{A}_1^s\}$ is a left $S$-system of binary quasigroups.

**Theorem 2.** *Let $n \geqslant 3$, $k$ $(1 \leqslant k \leqslant n)$ be a fixed number, $Q$ be a set of order $m$, $Q(\Sigma_k)$, $\Sigma_k = \{E_k, A_1^s\}$, be a $(k)$-$S$-system of $n$-quasigroups. Then $\Sigma_k$ is a pairwise orthogonal set of $n$-operations and $s \leqslant m - 1$.*

*Proof.* Let $A_i, A_j \in \Sigma_k$, $i \neq j$, then ${}^{(k)}A_j \in \Sigma_k$ and $A_i \underset{k}{\oplus} ({}^{(k)}A_j)$ is an $n$-quasigroup of $\Sigma_k$, so any $(n-1)$-retract of this $n$-quasigroup is an $(n-1)$-quasigroup which is always complete. By Theorem 1 $A_i \perp A_j$. Now it is evident that $A_i \perp E_k$, since $A_i \underset{k}{\oplus} E_k = A_i$ and ${}^{(k)}E_k = E_k$. Thus, $\Sigma_k$ is a pairwise orthogonal set of $n$-operations.

But by Proposition 1 $Q(\Sigma_{u,v,w})$, where $\Sigma_{u,v,w} = \{E, \overline{A}_1^s\}$, is a right $S$-system of binary quasigroups which is an orthogonal set and can not contain more than $m - 1$ binary quasigroups (latin squares) of order $m$ [8], so $s \leqslant m - 1$. □

**Definition 3.** A $(k)$-$S$-system $Q(\Sigma_k)$ with $\mid Q \mid = m$ is called *complete* if it contains $m-1$ $n$-quasigroups, that is if $\mid \Sigma_k \mid = m$.

**Proposition 2.** *For any $n \geqslant 3$ and any $k \in \overline{1,n}$ there exist complete $(k)$-$S$-systems of $n$-quasigroups of each prime order $p \geqslant 3$.*

*Proof.* Examples of such $(k)$-$S$-systems are the (cyclic) systems obtained with the help of $n$-quasigroups of the form (2) where $(Q,+)$ is a group of a prime order $p \geqslant 3$. □

Note that (cyclic) $(k)$-$S$-systems which are obtained from an $n$-quasigroup $A$ of the form (1) are not complete if $m = p^t$, $t > 1$.

## 4. $S$-systems of $n$-quasigroups

In [10] $n$-ary $S$-systems were considered in the following sense.

**Definition 4.** (cf. [11]) A system $Q(\Sigma)$, $\Sigma = \{E_1^n, A_1^s\}$, $s \geqslant 1$, where $A_i$ is an $n$-quasigroup for each $i \in \overline{1,s}$, $n \geqslant 2$, is called an *$S$-system of $n$-quasigroups* if $\Sigma$ is closed with respect to the (Menger's) superposition: $C(B_1^n) = C(B_1, B_2, \dots, B_n) \in \Sigma$ $(C(B_1^n)(x_1^n) = C(B_1(x_1^n), \dots, B_n(x_1^n))$ for any $C, B_1, \dots, B_n \in \Sigma$.

T. Yakubov in [10] proved that if $Q(\Sigma)$ is a finite (that is the set $Q$ is finite) $n$-ary $S$-system in this sense, then $\Sigma_k = \{E_k, A_1^s\}$ is a group with respect to the $k$-multiplication of $n$-operations for each $k \in \overline{1,n}$. Using this fact and the definition of $(k)$-$S$-systems it is natural to define an $S$-system of $n$-ary quasigroups in the following way.

**Definition 5.** A system $Q(\Sigma)$, $\Sigma = \{E_1^n, A_1^s\}$, $s \geqslant 1$, $n \geqslant 2$, where all $A_i$ are $n$-quasigroups is called an *$S$-system of $n$-quasigroups* if $\Sigma_k = \{E_k, A_1^s\}$ is a $(k)$-$S$-system for any $k \in \overline{1,n}$.

**Proposition 3.** *Let $Q(\Sigma)$, $\Sigma = \{E_1^n, A_1^s\}$, be an $S$-system of $n$-quasigroups, $n \geqslant 3$, $1 \leqslant p < q \leqslant n$ and $u = a_1^{p-1}$, $v = a_{p+1}^{q-1}$, $w = a_{q+1}^n$ be fixed (ordered) tuples of elements from $Q$. Then the system $Q(\Sigma_{u,v,w})$ of binary retracts where $\Sigma_{u,v,w} = \{E, F, \overline{A}_1^s\}$ with $\overline{A}_i(x_p, x_q) = A_i(u, x_p, v, x_q, w)$, is an $S$-system of binary quasigroups for any $u \in Q^{p-1}$, $v \in Q^{q-p-1}$, $w \in Q^{n-q}$.*

*Proof.* In this case $E_p(u, x_p, v, x_q, w) = x_p = F(x_p, x_q)$, $E_q(u, x_p, v, x_q, w) = x_q = E(x_p, x_q)$. From Definition 5 it follows that $\Sigma_k = \{E_k, A_1^s\}$ is a $(k)$-$S$-system for any $k \in \overline{1,n}$. If $k = q$, then by Proposition 1 $\Sigma_{u,v,w} = \{E, \overline{A}_1^s\}$

of binary retracts is a right $S$-system of binary quasigroups. On the other hand, if $k = p$, then $\Sigma'_{u,v,w} = \{F, \overline{A}_1^s\}$ for the same $u, v, w$ is a left $S$-system of binary quasigroups (see Remark). Thus, $Q(\Sigma_{u,v,w})$ is an $S$-system of binary quasigroups. □

For the binary case Definition 4 and Definition 5 are equivalent (see Theorem 4.1 of [1]). We shall prove that when $n > 2$ it is not true. At first remind that an $n$-quasigroup $(Q, A)$ is called an *$n$-$TS$- quasigroup* if its $k$-inverse $n$-quasigroups coincide with $A$ for each $k \in \overline{1, n}$ (see [2]).

**Theorem 3.** *A finite system $Q(\Sigma)$, $\Sigma = \{E_1^n, A_1^s\}$, $n \geqslant 3$, is an $S$-system of $n$-quasigroups if and only if $s = 1$ and the $n$-quasigroup $A_1$ is an $n$-$TS$-quasigroup.*

*Proof.* By Proposition 3 the system $Q(\Sigma_{u,v,w})$ of binary retracts, where $\Sigma_{u,v,w} = \{F, E, \overline{A}_1^s\}$, $\overline{A}_i(x_p, x_q) = A_i(u, x_p, v, x_q, w)$, is an $S$-system of binary quasigroups. By Theorem 4.2 of [1] all operations of a finite $S$-system of binary quasigroups are idempotent if $s \geqslant 2$ (note that in [1] $s \geqslant 4$ since $s$ designates the number of all operations in an $S$-system), that is $A_i(u, x, v, x, w) = \overline{A}_i(x, x) = x$ for every $x \in Q$. Now we use the idea of the proof from [10].

If $n = 3$, then $\overline{A}_i(a, a) = a$ and $A_i(a, a, w) = a$ (if, for example, $p = 1$, $q = 2$) for any $w$ of $Q$. But it is impossible as $A_i$ is a 3-quasigroup.

Let $n \geqslant 4$, $a \neq b$, the element $a$ be in $A_i$ in positions $p$, $q$ ($p < q$) and the element $b$ is in positions $r, t$ ($q < r < t$), i.e., $A_i(\ldots, a, \ldots, a, \ldots, b, \ldots, b, \ldots)$. Fix tuples $u \in Q^{p-1}$, $v \in Q^{q-p-1}$, $w \in Q^{n-q}$ where in the tuple $w$ the element $b$ is in the positions $r, t$. Then for a binary quasigroup $\overline{A}_i$ of the system $\Sigma_{u,v,w}$ we have

$$\overline{A}_i(x_p, x_q) = A_i(u, x_p, v, x_q, w) = A_i(u, x_p, v, x_q, w_1, b, w_2, b, w_3),$$

if $w = (w_1, b, w_2, b, w_3)$, and

$$\overline{A}_i(a, a) = A_i(u, a, v, a, w) = A_i(u, a, v, a, w_1, b, w_2, b, w_3) = a. \qquad (3)$$

Now consider the system $\Sigma_{u_1,w_2,w_3}$ with $u_1 = (u, a, v, a, w_1)$, then

$$\overline{\overline{A}}_i(x_r, x_t) = A_i(u_1, x_r, w_2, x_t, w_3),$$
$$\overline{\overline{A}}_i(b, b) = A_i(u, a, v, a, w_1, b, w_2, b, w_3) = b.$$

Taking into account the equality (3), we conclude that $a = b$. Thus, the case $s \geqslant 2$ for $n > 2$ is impossible.

It remains only the case $s = 1$. In this case the $n$-quasigroup $A_1$ coincides with all its inverse $n$-quasigroups, that is it is an $n$-$TS$-quasigroup. On the other hand, if an $n$-quasigroup $A$ is an $n$-$TS$-quasigroup, then $A =^{(k)} A$ for any $k \in \overline{1,n}$, $A \underset{k}{\oplus} A = E_k$ and $\Sigma = \{E_1^n, A\}$ is an $S$-system. □

Unfortunately, such $S$-systems of $n$-quasigroups are trivial.

As an example of an $n$-$TS$-quasigroup can be the $n$-quasigroup of the form (2) where $(Q, +)$ is an (abelian) group of exponent two ($2x = 0$ for all $x \in Q$). Such group has order $2^t$ for some natural $t \geqslant 1$.

In [10] it was proved that finite $S$-systems of $n$-quasigroups in the sense of Definition 4 do not exist even for $s = 1$. Taking into account Theorem 3 we conclude that Definition 4 and Definition 5 are not equivalent for $n > 2$.

## References

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