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Intuitionistic (S, T)-fuzzy Lie ideals of Lie algebras

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Abstract

In this paper we introduce the notion of an intuitionistic (S, T)-fuzzy Lie ideal of a Lie algebra and investigate some related properties. Nilpotency of intuitionistic (S, T)-fuzzy Lie ideals is introduced. Intuitionistic (S, T)-fuzzy of adjoint representation of Lie algebras is introduced and the relation between this representation and nilpotent intuitionistic (S, T)-fuzzy Lie ideals is discussed. Killing form in the intuitionistic (S, T)-fuzzy case is defined and some of its properties are studied.

1. Introduction

Lie algebras were first discovered by Sophus Lie (1842-1899) when he attempted to classify certain "smooth" subgroups of general linear groups. The groups he considered are now called Lie groups. By taking the tangent space at the identity element of such a group, he obtained the Lie algebra and hence the problems on groups can be reduced to problems on Lie algebras so that it becomes more tractable. Lie algebra is applied in different domains of physics and mathematics, such as spectroscopy of molecules, atoms, nuclei, hadrons, hyperbolic and stochastic differential equations.

After the introduction of fuzzy sets by L. Zadeh [14], various notions of higher-order fuzzy sets have been proposed. Among them, intuitionistic fuzzy sets, introduced by K. Atanassov [2, 3], have drawn the attention of many researchers in the last decades. This is mainly due to the fact that intuitionistic fuzzy sets are consistent with human behavior, by reflecting and modeling the hesitancy present in real-life situations. In fact, the fuzzy

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sets give the degree of membership of an element in a given set, while intuitionistic fuzzy sets give both a degree of membership and a degree of non-membership. As for fuzzy sets, the degree of membership is a real number between 0 and 1. This is also the case for the degree of nonmembership, and furthermore the sum of these two degrees is not greater than 1. Fuzzy and anti fuzzy Lie ideals in Lie algebras have been studied in [1, 4, 7, 8, 9, 12].

In this paper, we introduce the notion of an intuitionistic (S,T)-fuzzy Lie ideal of a Lie algebra and investigate some of related properties. Nilpotency of intuitionistic (S,T)-fuzzy Lie ideals is introduced. Intuitionistic (S,T)-fuzzy of adjoint representation of Lie algebras is introduced and the relation between this representation and nilpotent intuitionistic (S,T)-fuzzy Lie ideals is proved. Killing form in the intuitionistic (S,T)-fuzzy case is defined and some of its properties are studied.

2. Preliminaries

A *Lie algebra* is a vector space L over a field F (equal to $\mathbf{R}$ or $\mathbf{C}$) on which $L\times L\to L$ denoted by $(x,y)\to[x,y]$ is defined satisfying the following axioms:

(L_1) $[x,y]$ is bilinear,

(L_2) $[x,x]=0$ for all $x\in L$,

(L_3) $[[x,y],z]+[[y,z],x]+[[z,x],y]=0$ for all $x,y,z\in L$ (Jacobi identity).

In this paper by L will be denoted a Lie algebra. We note that the multiplication in a Lie algebra is not associative, i.e., it is not true in general that $[[x,y],z]=[x,[y,z]]$. But it is *anti commutative*, i.e., $[x,y]=-[y,x]$.

A subspace H of L closed under $[\ ,\ ]$ will be called a *Lie subalgebra*. A subspace I of L with the property $[I,L]\subseteq I$ will be called a *Lie ideal* of L. Obviously, any Lie ideal is a subalgebra. Let γ be a *fuzzy set* on L, i.e., a map $\gamma:L\to[0,1]$. A *fuzzy set* $\gamma:L\to[0,1]$ is called a *fuzzy Lie subalgebra* of L if

(a) $\gamma(x+y)\geqslant\min\{\gamma(x),\gamma(y)\}$,

(b) $\gamma(\alpha x)\geqslant\gamma(x)$,

(c) $\gamma([x,y])\geqslant\min\{\gamma(x),\gamma(y)\}$

hold for all $x, y \in L$ and $\alpha \in F$. A fuzzy subset $\gamma : L \to [0,1]$ satisfying (a), (b) and

(d) $\gamma([x,y]) \geqslant \gamma(x)$

is called a *fuzzy Lie ideal* of L. The addition and the commutator $[\ ,\ ]$ of L are extended by Zadeh's extension principle [15], to two operations on I^L in the following way:

$$(\mu \oplus \lambda)(x) = \sup\{\min\{\mu(y), \lambda(z)\} \mid y, z \in L,\ y + z = x\},$$

$$\ll \mu, \lambda \gg (x) = \sup\{\min\{\mu(y), \lambda(z)\} \mid y, z \in L,\ [y,z] = x\},$$

where μ, λ are fuzzy sets on I^L and $x \in L$. The scalar multiplication αx for $\alpha \in F$ and $x \in L$ is extended to an action of the field F on I^L denoted by $\odot$ as follows for all $\mu \in I^L$, $\alpha \in F$ and $x \in L$:

$$(\alpha \odot \mu)(x) = \begin{cases} \mu(\alpha^{-1}x) & \text{if } \alpha \neq 0, \\ 1 & \text{if } \alpha = 0,\ x = 0, \\ 0 & \text{if } \alpha = 0,\ x \neq 0. \end{cases}$$

The two operations of the field F can be extended to two operations on I^F in the same way. The operations are denoted by $\oplus$ and $\circ$ as well [15]. The zeros of L and F are denoted by the same symbol 0. Obviously $0 \odot \mu = 1_0$ for every $\mu \in I^L$ and every $\mu \in I^F$, where 1_x is the fuzzy subset taking 1 at x and 0 elsewhere.

Let L be a Lie algebra. A fuzzy subset γ of L is called an *anti fuzzy Lie ideal* of L if the following axioms are satisfied:

(AF_1) $\gamma(x+y) \leqslant \max(\gamma(x), \gamma(y))$,

(AF_2) $\gamma(\alpha x) \leqslant \gamma(x)$,

(AF_3) $\gamma([x,y]) \leqslant \gamma(x)$

for all $x, y \in L$ and $\alpha \in F$.

A *t-norm* is a mapping $T : [0,1] \times [0,1] \to [0,1]$ such that

(T_1) $T(x,1) = x$,

(T_2) $T(x,y) = T(y,x)$,

(T_3) $T(x, T(y,z)) = T(T(x,y), z)$,

(T_4) $T(x,y) \leqslant T(x,z)$ whenever $y \leqslant z$, where $x, y, z \in [0, 1]$. Replacing 1 by 0 in condition (T_1), we obtain the concept of *s-norm* S.

A mapping $A = (\mu_A, \lambda_A) : L \to [0, 1] \times [0, 1]$ is called an *intuitionistic fuzzy set* (IFS, in short) in L if $\mu_A(x) + \lambda_A(x) \leqslant 1$, for all $x \in L$, where the mappings $\mu_A : L \to [0, 1]$ and $\lambda_A : L \to [0, 1]$ denote the *degree of membership* (namely $\mu_A(x)$) and the *degree of non-membership* (namely $\lambda_A(x)$) of each element $x \in L$ to A respectively. In particular, $0_\sim$ and $1_\sim$ denote the *intuitionistic fuzzy empty set* and the *intuitionistic fuzzy whole set* in a set L defined by $0_\sim(x) = (0, 1)$ and $1_\sim(x) = (1, 0)$ for each $x \in L$ respectively.

3. Intuitionistic (S, T)-fuzzy Lie ideals

Definition 3.1. An intuitionistic fuzzy set $A = (\mu_A, \lambda_A)$ on L is called an *intuitionistic fuzzy Lie ideal* of L with respect to the t-norm T and the s-norm S (shortly, intuitionistic (S, T)-fuzzy Lie ideals of L) if

(1) $\mu_A(x + y) \geqslant T(\mu_A(x), \mu_A(y))$ and $\lambda_A(x + y) \leqslant S(\lambda_A(x), \lambda_A(y))$,

(2) $\mu_A(\alpha x) \geqslant \mu_A(x)$ and $\lambda_A(\alpha x) \leqslant \lambda_A(x)$,

(3) $\mu_A([x, y]) \geqslant \mu_A(x)$ and $\lambda_A([x, y]) \leqslant \lambda_A(x)$

is satisfied for all $x, y \in L$ and $\alpha \in F$.

From (2) it follows that

(4) $\mu_A(0) \geqslant \mu_A(x)$ and $\lambda_A(0) \leqslant \lambda_A(x)$,

(5) $\mu_A(-x) = \mu_A(x)$ and $\lambda_A(-x) = \lambda_A(x)$

for all $x \in L$.

Example 3.2. Let $\Re^2 = \{(x, y) \,|, x, y \in R\}$ be the set of all 2-dimensional real vectors. Then $\Re^2$ with the bracket $[\ ,\ ]$ defined as usual cross product, i.e., $[x, y] = x \times y$, is a real Lie algebra. We define an intuitionistic fuzzy set $A = (\mu_A, \lambda_A) : L \to [0, 1] \times [0, 1]$ as follows:

$$\mu_A(x, y) = \begin{cases} m_1 & \text{if } x = y = 0, \\ m_2 & \text{otherwise,} \end{cases} \qquad \lambda_A(x, y) = \begin{cases} m_2 & \text{if } x = y = 0, \\ m_1 & \text{otherwise,} \end{cases}$$

where $m_1 > m_2$ and $m_1, m_2 \in [0, 1]$. Let T be a t-norm which is defined by $T(x, y) = \max\{x + y - 1, 0\}$ and S an s-norm which is defined by $S(x, y) = \min\{x + y, 1\}$ for all $x, y \in [0, 1]$. Then by routine computation, we see that $A = (\mu_A, \lambda_A)$ is an intuitionistic (S, T)-fuzzy Lie ideal of L.

The following proposition is obvious.

Proposition 3.3. *If A is an intuitionistic (S,T)-fuzzy Lie ideal of L, then*

(i) $\mu_A([x,y]) \geqslant S(\mu_A(x), \mu_A(y))$,

(ii) $\lambda_A([x,y]) \leqslant T(\lambda_A(x), \lambda_A(y))$

for all $x, y \in L$. □

Theorem 3.4. *Let $G_0 \subset G_1 \subset G_2 \subset \ldots G_n = L$ be a chain of Lie ideals of a Lie algebra L. Then there exists an intuitionistic (S,T)-fuzzy Lie ideal A of L for which level subsets $U(\mu_A, \alpha)$ and $L(\lambda_A, \beta)$ coincide with this chain.*

Proof. Let $\{\alpha_k \,|\, k = 0, 1, \ldots, n\}$ and $\{\beta_k \,|\, k = 0, 1, \ldots, n\}$ be finite decreasing and increasing sequences in $[0,1]$ such that $\alpha_i + \beta_i \leqslant 1$, for $i = 0, 1, \ldots, n$. Let $A = (\mu_A, \lambda_A)$ be an intuitionistic fuzzy set in L defined by $\mu_A(G_0) = \alpha_0$, $\lambda_A(G_0) = \beta_0$, $\mu_A(G_k \setminus G_{k-1}) = \alpha_k$ and $\lambda_A(G_k \setminus G_{k-1}) = \beta_k$ for $0 < k \leqslant n$. Let $x, y \in L$. If $x, y \in G_k \setminus G_{k-1}$, then $x + y, \alpha x, [x,y] \in G_k$ and

$$\mu_A(x+y) \geqslant \alpha_k = T(\mu_A(x), \mu_A(y)),$$
$$\lambda_A(x+y) \leqslant \beta_k = S(\lambda_A(x)), \lambda_A(y)),$$
$$\mu_A(\alpha x) \geqslant \alpha_k = \mu_A(x), \quad \lambda_A(\alpha x) \leqslant \beta_k = \lambda_A(x),$$
$$\mu_A([x,y]) \geqslant \alpha_k = \mu_A(x), \quad \lambda_A([x,y]) \leqslant \beta_k = \lambda_A(x).$$

For $i > j$, if $x \in G_i \setminus G_{i-1}$ and $y \in G_j \setminus G_{j-1}$, then $\mu_A(x) = \alpha_i = \mu_A(y)$, $\lambda_A(x) = \beta_j = \lambda_A(y)$ and $x + y, \alpha x, [x,y] \in G_i$. Thus

$$\mu_A(x+y) \geqslant \alpha_i = T(\mu_A(x), \mu_A(y)),$$
$$\lambda_A(x+y) \leqslant \beta_j = S(\lambda_A(x)), \lambda_A(y)),$$
$$\mu_A(\alpha x) \geqslant \alpha_i = \mu_A(x), \quad \lambda_A(\alpha x) \leqslant \beta_j = \lambda_A(x),$$
$$\mu_A([x,y]) \geqslant \alpha_i = \mu_A(x), \quad \lambda_A([x,y]) \leqslant \beta_j = \lambda_A(x).$$

So, $A = (\mu_A, \lambda_A)$ is an intuitionistic (S,T)-fuzzy Lie ideal of a Lie algebra L and all its nonempty level subsets are Lie ideals. Since $\mathrm{Im}(\mu_A) = \{\alpha_0, \alpha_1, \ldots, \alpha_n\}$, $\mathrm{Im}(\lambda_A) = \{\beta_0, \beta_1, \ldots, \beta_n\}$, level subsets of A form chains:

$$U(\mu_A, \alpha_0) \subset U(\mu_A, \alpha_1) \subset \ldots \subset U(\mu_A, \alpha_n) = L$$

and

$$L(\lambda_A, \beta_0) \subset L(\lambda_A, \beta_1) \subset \ldots \subset L(\lambda_A, \beta_n) = L,$$

respectively. Indeed,

$$U(\mu_A, \alpha_0) = \{x \in L \mid \mu_A(x) \geqslant \alpha_0\} = G_0,$$

$$L(\lambda_A, \beta_0) = \{x \in L \mid \lambda_A(x) \leqslant \beta_0\} = G_0.$$

We now prove that

$$U(\mu_A, \alpha_k) = G_k = L(\lambda_A, \beta_k) \quad \text{for } 0 < k \leqslant n.$$

Clearly, $G_k \subseteq U(\mu_k, \alpha_k)$ and $G_k \subseteq L(\lambda_A, \beta_k)$. If $x \in U(\mu_A, \alpha_k)$, then $\mu_A(x) \geqslant \alpha_k$ and so $x \notin G_i$ for $i > k$. Hence

$$\mu_A(x) \in \{\alpha_0, \alpha_1, \ldots, \alpha_k\},$$

which implies $x \in G_i$ for some $i \leqslant k$. Since $G_i \subseteq G_k$, it follows that $x \in G_k$. Consequently, $U(\mu_A, \alpha_k) = G_k$ for some $0 < k \leqslant n$. Now if $y \in L(\lambda_A, \beta_k)$, then $\lambda_A(x) \leqslant \beta_k$ and so $y \notin G_i$ for $j \leqslant k$. Thus

$$\lambda_A(x) \in \{\beta_0, \beta_1, \ldots, \beta_k\},$$

which implies $x \in G_j$ for some $j \leqslant k$. Since $G_j \subseteq G_k$, it follows that $y \in G_k$. Consequently, $L(\lambda_A, \beta_k) = G_k$ for some $0 < k \leqslant n$. This completes the proof. □

Definition 3.5. Let $f : L_1 \to L_2$ be a homomorphism of Lie algebras. Let $A = (\mu_A, \lambda_A)$ be an IFS of L_2. Then we can define an IFS $f^{-1}(A)$ of L_1 by

$$f^{-1}(A)(x) = A(f(x)) = (\mu_A(f(x)), \lambda_A(f(x))) \quad \forall\, x \in L_1.$$

Proposition 3.6. *Let $f : L_1 \to L_2$ be an epimorphism of Lie algebras. Then A is an intuitionistic (S,T)-fuzzy Lie ideal of L_2 if and only if $f^{-1}(A)$ is an intuitionistic (S,T)-fuzzy Lie ideal of L_1.*

Proof. Straightforward. □

Definition 3.7. Let $f : L_1 \to L_2$ be a homomorphism of Lie algebras. Let $A = (\mu_A, \lambda_A)$ be an intuitionistic fuzzy set of L_1. Then IFS $f(A) = (f(\mu_A), f(\lambda_A))$ in L_2 is defined by

$$f(\mu_A)(y) = \begin{cases} \sup\{\mu_A(t) \mid t \in L_1, f(t) = y\}, & \text{if } f^{-1}(y) \neq \emptyset\ , \\ 0, & \text{otherwise,} \end{cases}$$

$$f(\lambda_A)(y) = \begin{cases} \inf\{\lambda_A(t) \mid t \in L_1, f(t) = y\}, & \text{if } f^{-1}(y) \neq \emptyset\ , \\ 1, & \text{otherwise.} \end{cases}$$

Definition 3.8. Let L_1 and L_2 be any sets and $f : L_1 \to L_2$ any function. Then, we call an intuitionistic fuzzy set $A = (\mu_A, \lambda_A)$ of L_1 *f-invariant* if $f(x) = f(y)$ implies $A(x) = A(y)$, i.e., $\mu_A(x) = \mu_A(y)$, $\lambda_A(x) = \lambda_A(y)$ for $x, y \in L_1$.

Theorem 3.9. *Let $f : L_1 \to L_2$ be an epimorphism of Lie algebras. Then $A = (\mu_A, \lambda_A)$ is an f-invariant intuitionistic (S, T)-fuzzy Lie ideal of L_1 if and only if $f(A)$ is an intuitionistic (S, T)-fuzzy ideal of L_2.*

Proof. Let $x, y \in L_2$. Then there exist $a, b \in L_1$ such that $f(a) = x$, $f(b) = y$ and $x + y = f(a + b)$, $\alpha x = \alpha f(a)$. Since A is f-invariant, by straightforward verification, we have

$$f(\mu_A)(x + y) = \mu_A(a + b) \geqslant T(\mu_A(a), \mu_A(b)) = T(f(\mu_A)(x), f(\mu_A)(y)),$$
$$f(\lambda_A)(x + y) = \lambda_A(a + b) \leqslant S(\lambda_A(a), \lambda_A(b)) = S(f(\lambda_A)(x), f(\lambda_A)(y)),$$
$$f(\mu_A)(\alpha x) = \mu_A(\alpha a) \geqslant \mu_A(a) = f(\mu_A)(x),$$
$$f(\lambda_A)(\alpha x) = \lambda_A(\alpha a) \leqslant \lambda_A(a) = f(\lambda_A)(x),$$
$$f(\mu_A)([x, y]) = \mu_A([a, b]) = [\mu_A(a), \mu_A(b)] \geqslant \mu_A(a) = f(\mu_A)(x),$$
$$f(\lambda_A)([x, y]) = \lambda_A([a, b]) = [\lambda_A(a), \lambda_A(b)] \leqslant \lambda_A(a) = f(\lambda_A)(x).$$

Hence $f(A)$ is an intuitionistic (S, T)-fuzzy Lie ideal of L_2.

Conversely, if $f(A)$ is an intuitionistic (S, T)-fuzzy Lie ideal of L_2, then for any $x \in L_1$

$$\begin{aligned} f^{-1}(f(\mu_A))(x) &= f(\mu_A)(f(x)) = \sup\{\mu_A(t) \mid t \in L_1, f(t) = f(x)\} \\ &= \sup\{\mu_A(t) \mid t \in L_1, \mu_(t) = \mu_A(x)\} = \mu_A(x), \end{aligned}$$

$$\begin{aligned} f^{-1}(f(\lambda_A))(x) &= f(\lambda_A)(f(x)) = \inf\{\lambda_A(t) \mid t \in L_1, f(t) = f(x)\} \\ &= \inf\{\lambda_A(t) \mid t \in L_1, \lambda_(t) = \lambda_A(x)\} = \lambda_A(x). \end{aligned}$$

Hence $f^{-1}(f(A)) = A$ is an intuitionistic (S, T)-fuzzy Lie ideal. □

Lemma 3.10. *Let $A = (\mu_A, \lambda_A)$ be an intuitionistic (S, T)-fuzzy Lie ideal of a Lie algebra L and let $x \in L$. Then $\mu_A(x) = t$, $\lambda_A(x) = s$ if and only if $x \in U(\mu_A, t)$, $x \notin U(\mu_A, s)$ and $x \in L(\lambda_A, s)$, $x \notin L(\lambda_A, t)$, for all $s > t$.*

Proof. Straightforward. □

Definition 3.11. A Lie ideal A of Lie algebra L is said to be *characteristic* if $f(A) = A$, for all $f \in Aut(L)$, where $\mathrm{Aut}(L)$ is the set of all automorphisms of a Lie algebra L. An intuitionistic (S,T)-fuzzy Lie ideal $A = (\mu_A, \lambda_A)$ of a Lie algebra L is called *characteristic* if $\mu_A(f(x)) = \mu_A(x)$ and $\lambda_A(f(x)) = \lambda_A(x)$ for all $x \in L$ and $f \in Aut(L)$.

Theorem 3.12. *An intuitionistic (S,T)-fuzzy Lie ideal is characteristic if and only if each its level set is a characteristic Lie ideal.*

Proof. Let an intuitionistic (S,T)-fuzzy Lie ideal $A = (\mu_A, \lambda_A)$ be characteristic, $t \in Im(\mu_A)$, $f \in Aut(L)$, $x \in U(\mu_A, t)$. Then $\mu_A(f(x)) = \mu_A(x) \geqslant t$, which means that $f(x) \in U(\mu_A, t)$. Thus $f(U(\mu_A, t)) \subseteq U(\mu_A, t)$. Since for each $x \in U(\mu_A, t)$ there exists $y \in L$ such that $f(y) = x$ we have $\mu_A(y) = \mu_A(f(y)) = \mu_A(x) \geqslant t$, whence we conclude $y \in U(\mu_A, t)$. Consequently $x = f(y) \in f(U(\mu_A, t))$. Hence $f(U(\mu_A, t)) = U(\mu_A, t)$. Similarly, $f(L(\lambda_A, s)) = L(\lambda_A, s)$. This proves that $U(\mu_A, t)$-and $L(\lambda_A, s)$ are characteristic.

Conversely, if all levels of $A = (\mu_A, \lambda_A)$ are characteristic Lie ideals of L, then for $x \in L$, $f \in Aut(L)$ and $\mu_A(x) = t < s = \lambda_A(x)$, by Lemma 3.10, we have $x \in U(\mu_A, t)$, $x \notin U(\mu_A, s)$ and $x \in L(\lambda_A, s)$, $x \notin L(\lambda_A, t)$. Thus $f(x) \in f(U(\mu_A, t)) = U(\mu_A, t)$ and $f(x) \in f(L(\lambda_A, s)) = L(\lambda_A, s)$, i.e., $\mu_A(f(x)) \geqslant t$ and $\lambda_A(f(x)) \leqslant s$. For $\mu_A(f(x)) = t_1 > t$, $\lambda_A(f(x)) = s_1 < s$ we have $f(x) \in U(\mu_A, t_1) = f(U(\mu_A, t_1))$, $f(x) \in L(\lambda_A, s_1) = f(L(\lambda_A, s_1))$, whence $x \in U(\mu_A, t_1)$, $x \in L(\mu_A, s_1)$. This is a contradiction. Thus $\mu_A(f(x)) = \mu_A(x)$ and $\lambda_A(f(x)) = \lambda_A(x)$. So, $A = (\mu_A, \lambda_A)$ is characteristic. □

Using the same method as in the proof of Theorems 4.6 in [5] we can prove the following theorem.

Theorem 3.13. *Let $\{C_\alpha \mid \alpha \in \Lambda \subseteq [0, \frac{1}{2}]\}$ be a collection of Lie ideals of a Lie algebra L such that $L = \bigcup_{\alpha \in \Lambda} C_\alpha$, and for every $\alpha, \beta \in \Lambda$, $\alpha < \beta$ if and only if $C_\beta \subset C_\alpha$. Then an intuitionistic fuzzy set $A = (\mu_A, \lambda_A)$ defined by*

$$\mu_A(x) = \sup\{\alpha \in \Lambda \mid x \in C_\alpha\} \quad \text{and} \quad \lambda_A(x) = \inf\{\alpha \in \Lambda \mid x \in C_\alpha\}$$

is an intuitionistic (S,T)-fuzzy Lie ideal of L. □

Theorem 3.14. *Let $A = (\mu_A, \lambda_A)$ be an intuitionistic (S,T)-fuzzy Lie ideal of Lie algebra L. Define a binary relation $\sim$ on L by*

$$x \sim y \longleftrightarrow \mu_A(x - y) = \mu_A(0) \text{ and } \lambda_A(x - y) = \lambda_A(0).$$

Then $\sim$ is a congruence on L.

Proof. The reflexivity and symmetry is obvious. To prove transitivity let $x \sim y$ and $y \sim z$. Then $\mu_A(x-y) = \mu_A(0)$, $\mu_A(y-z) = \mu_A(0)$ and $\lambda_A(x-y) = \lambda_A(0)$, $\lambda_A(y-z) = \lambda_A(0)$, by (5). Thus

$$\mu_A(x-z) = \mu_A(x-y+y-z) \geqslant T(\mu_A(x-y), \mu_A(y-z)) = \mu_A(0),$$
$$\lambda_A(x-z) = \lambda_A(x-y+y-z) \leqslant S(\lambda_A(x-y), \lambda_A(y-z)) = \lambda(0),$$

whence, by (4), we conclude $x \sim z$.

If $x_1 \sim y_1$ and $x_2 \sim y_2$, then

$$\begin{aligned}\mu_A((x_1+x_2)-(y_1+y_2)) &= \mu_A((x_1-y_1)+(x_2-y_2))\\ &\geqslant T(\mu_A(x_1-y_1), \mu_A(x_2-y_2)) = \mu_A(0),\\ \lambda_A((x_1+x_2)-(y_1+y_2)) &= \lambda_A((x_1-y_1)+(x_2-y_2))\\ &\leqslant S(\lambda_A(x_1-y_1), \lambda_A(x_2-y_2)) = \lambda_A(0),\end{aligned}$$
$$\mu_A((\alpha x_1 - \alpha y_1) = \mu_A(\alpha(x_1-y_1)) \geqslant \mu_A(x_1-y_1) = \mu(0),$$
$$\lambda_A((\alpha x_1 - \alpha y_1) = \lambda_A(\alpha(x_1-y_1)) \leqslant \lambda_A(x_1-y_1) = \lambda_A(0),$$
$$\mu_A([x_1,x_2]-[y_1,y_2]) = \mu_A([x_1-y_1],[x_2-y_2]) \geqslant \mu_A(x_1-y_1) = \mu_A(0),$$
$$\lambda_A([x_1,x_2]-[y_1,y_2]) = \lambda_A([x_1-y_1],[x_2-y_2]) \leqslant \lambda_A(x_1-y_1) = \lambda_A(0).$$

Now, applying (4), it is easily to see that $x_1+x_2 \sim y_1+y_2$, $\alpha x_1 \sim \alpha y_1$ and $[x_1,x_2] \sim [y_1,y_2]$. So, $\sim$ is a congruence. □

4. Nilpotency of intuitionistic (S,T)-fuzzy Lie ideals

Definition 4.1. Let $A = (\mu_A, \lambda_A) \in I^L$, an intuitionistic fuzzy subspace of L generated by A will be denoted by $[A]$. It is the intersection of all intuitionistic fuzzy subspaces of L containing A. For all $x \in L$, we define:

$$[\mu_A](x) = \sup\{\min \mu_A(x_i) : \,|\, x = \sum \alpha_i x_i, \alpha_i \in F, x_i \in L\},$$
$$[\lambda_A](x) = \inf\{\max \lambda_A(x_i) \,|\, x = \sum \alpha_i x_i, \alpha_i \in F, x_i \in L\}.$$

Definition 4.2. Let $f : L_1 \to L_2$ be a homomorphism of Lie algebras which has an extension $f : I^{L_1} \to I^{L_2}$ defined by:

$$f(\mu_A)(y) = \sup\{\mu_A(x), x \in f^{-1}(y)\},$$
$$f(\lambda_A)(y) = \inf\{\lambda_A(x), x \in f^{-1}(y)\},$$

for all $A = (\mu_A, \lambda_A) \in I^{L_1}$, $y \in L_2$. Then $f(A)$ is called the *homomorphic image* of A.

The following two propositions are obvious.

Proposition 4.3. *Let $f : L_1 \to L_2$ be a homomorphism of Lie algebras and let $A = (\mu_A, \lambda_A)$ be an intuitionistic (S, T)-fuzzy Lie ideal of L_1. Then*
(i) $f(A)$ is an intuitionistic (S, T)-fuzzy Lie ideal of L_2,
(ii) $f([A]) \supseteq [f(A)]$.

Proposition 4.4. *If A and B are intuitionistic (S, T)-fuzzy Lie ideals in L, then $[A, B]$ is an intuitionistic (S, T)-fuzzy Lie ideal of L.*

Theorem 4.5. *Let A_1, A_2, B_1, B_2 be intuitionistic (S, T)-fuzzy Lie ideals in L such that $A_1 \subseteq A_2$ and $B_1 \subseteq B_2$, then $[A_1, B_1] \subseteq [A_2, B_2]$.*

Proof. Indeed,

$$\begin{aligned}
\ll \mu_{A_1}, \mu_{B_1} \gg (x) &= \sup\{T(\mu_{A_1}(a), \mu_{B_1}(b)) \mid a, b \in L_1, [a, b] = x\} \\
&\geqslant \sup\{T(\mu_{A_2}(a), \mu_{B_2}(b)) \mid a, b \in L_1, [a, b] = x\} \\
&= \ll \mu_{A_2}, \mu_{B_2} \gg (x), \\
\ll \lambda_{A_1}, \lambda_{B_1} \gg (x) &= \inf\{S(\lambda_{A_1}(a), \lambda_{B_1}(b)) \mid a, b \in L_1, [a, b] = x\} \\
&\leqslant \inf\{S(\lambda_{A_2}(a), \lambda_{B_2}(b)) \mid a, b \in L_1, [a, b] = x\} \\
&= \ll \lambda_{A_2}, \lambda_{B_2} \gg (x).
\end{aligned}$$

Hence $[A_1, B_1] \subseteq [A_2, B_2]$. □

Let $A = (\mu_A, \lambda_A)$ be an intuitionistic (S, T)-fuzzy Lie ideal in L. Putting

$$A^0 = A,\ A^1 = [A, A_0],\ A^2 = [A, A_1],\ \ldots,\ A^n = [A, A^{n-1}]$$

we obtain a descending series of an intuitionistic (S, T)-fuzzy Lie ideals

$$A^0 \supseteq A^1 \supseteq A^2 \supseteq \ldots \supseteq A^n \supseteq \ldots$$

and a series of intuitionistic fuzzy sets $B^n = (\mu_B^n, \lambda_B^n)$ such that

$$\mu_B^n = \sup\{\mu_A^n(x) \,|\, 0 \neq x \in L\}, \quad \lambda_B^n = \inf\{\lambda_A^n(x) \,|\, 0 \neq x \in L\}.$$

Definition 4.6. An intuitionistic (S, T)-fuzzy Lie ideal $A = (\mu_A, \lambda_A)$ is called *nilpotent* if there exists a positive integer n such that $B^n = 0_\sim$.

Theorem 4.7. *A homomorphic image of a nilpotent intuitionistic (S, T)-fuzzy Lie ideal is a nilpotent intuitionistic (S, T)-fuzzy Lie ideal.*

Proof. Let $f : L_1 \to L_2$ be a homomorphism of Lie algebras and let $A = (\mu_A, \lambda_A)$ be a nilpotent intuitionistic (S,T)-fuzzy Lie ideal in L_1. Assume that $f(A) = B$. We prove by induction that $f(A^n) \supseteq B^n$ for every natural n. First we claim that $f([A,A]) \supseteq [f(A), f(A)] = [B,B]$. Let $y \in L_2$, then

$$\begin{aligned}
&f(\ll \mu_A, \mu_A \gg)(y) = \sup\{\ll \mu_A, \mu_A \gg)(x) \mid f(x) = y\}\\
&= \sup\{\sup\{T(\mu_A(a), \mu_A(b)) \mid a,b \in L_1, [a,b] = x, f(x) = y\}\}\\
&= \sup\{T(\mu_A(a), \mu_A(b)) \mid a,b \in L_1, [a,b] = x, f(x) = y\}\\
&= \sup\{T(\mu_A(a), \mu_A(b)) \mid a,b \in L_1, [f(a), f(b)] = x\}\\
&= \sup\{T(\mu_A(a), \mu_A(b)) \mid a,b \in L_1, f(a) = u, f(b)] = v, [u,v] = y\}\\
&\geqslant \sup\{T(\sup_{a\in f^{-1}(u)} \mu_A(a), \sup_{b\in f^{-1}(v)} \mu_A(b)) \mid [u,v] = y\}\\
&= \sup\{T(f(\mu_A)(u), f(\mu_A)(v)) \mid [u,v] = y\} = \ll f(\mu_A), f(\mu_A) \gg (y),
\end{aligned}$$

$$\begin{aligned}
&f(\ll \lambda_A, \lambda_A \gg)(y) = \inf\{\ll \lambda_A, \lambda_A \gg)(x) \mid f(x) = y\}\\
&= \inf\{\inf\{S(\lambda_A(a), \lambda_A(b)) \mid a,b \in L_1, [a,b] = x, f(x) = y\}\}\\
&= \inf\{S(\lambda_A(a), \lambda_A(b)) \mid a,b \in L_1, [a,b] = x, f(x) = y\}\\
&= \inf\{S(\lambda_A(a), \lambda_A(b)) \mid a,b \in L_1, [f(a), f(b)] = x\}\\
&= \inf\{S(\lambda_A(a), \lambda_A(b)) \mid a,b \in L_1, f(a) = u, f(b)] = v, [u,v] = y\}\\
&\leqslant \inf\{S(\inf_{a\in f^{-1}(u)} \lambda_A(a), \inf_{b\in f^{-1}(v)} \lambda_A(b)) \mid [u,v] = y\}\\
&= \inf\{S(f(\lambda_A)(u), f(\lambda_A)(v)) \mid [u,v] = y\} = \ll f(\lambda_A), f(\lambda_A) \gg (y).
\end{aligned}$$

Thus

$$f([A,A]) \supseteq f(\ll A, A \gg) \supseteq \ll f(A), f(A) \gg = [f(A), f(A)].$$

For $n > 1$, we get

$$f(A^n) = f([A, A^{n-1}]) \supseteq [f(A), f(A^{n-1})] \supseteq [B, B^{n-1}] = B^n.$$

Let m be a positive integer such that $A^m = 0_\sim$. Then for $0 \neq y \in L_2$ we have

$$\mu_B^m(y) \leqslant f(\mu_A^m)(y) = f(0)(y) = \sup\{0(a) \mid f(x) = y\} = 0,$$

$$\lambda_B^m(y) \geqslant f(\lambda_A^m)(y) = f(1)(y) = \inf\{1(a) \mid f(x) = y\} = 1.$$

Thus $B^m = 0_\sim$. This completes the proof. □

Let $A = (\mu_A, \lambda_A)$ be an intuitionistic (S,T)-fuzzy Lie ideal in L. Putting

$$A^{(0)} = A,\ A^{(1)} = [A^{(0)}, A^{(0)}],\ A^{(2)} = [A^{(1)}, A^{(1)}], \ldots, A^{(n)} = [A^{(n-1)}, A^{(n-1)}]$$

we obtain series

$$A^{(0)} \subseteq A^{(1)} \subseteq A^{(2)} \subseteq \ldots \subseteq A^{(n)} \subseteq \ldots$$

of intuitionistic (S,T)-fuzzy Lie ideals and a series of intuitionistic fuzzy sets $B^{(n)} = (\mu_B^{(n)}, \lambda_B^{(n)})$ such that

$$\mu_B^{(n)} = \sup\{\mu_A^{(n)}(x) \mid 0 \neq x \in L\}, \quad \lambda_B^{(n)} = \inf\{\lambda_A^{(n)}(x) \mid 0 \neq x \in L\}.$$

Definition 4.8. An intuitionistic (S,T)-fuzzy Lie ideal $A = (\mu_A, \lambda_A)$ is called *solvable* if there exists a positive integer n such that $B^{(n)} = 0_\sim$.

Theorem 4.9. *A nilpotent intuitionistic (S,T)-fuzzy Lie ideal is solvable.*

Proof. It is enough to prove that $A^{(n)} \subseteq A^n$ for all positive integers n. We prove it by induction on n and by the use of Theorem 4.5:

$$A^{(1)} = [A, A] = A^1, \qquad A^{(2)} = [A^{(1)}, A^{(1)}] \subseteq [A, A^{(1)}] = A^2.$$

$$A^{(n)} = [A^{(n-1)}, A^{(n-1)}] \subseteq [A, A^{(n-1)}] \subseteq [A, A^{(n-1)}] = A^n.$$

This completes the proof. □

Definition 4.10. Let $A = (\mu_A, \lambda_A)$ and $B = (\mu_B, \lambda_B)$ be two intuitionistic (S,T)-fuzzy Lie ideals of a Lie algebra L. The sum $A \oplus B$ is called a *direct sum* if $A \cap B = 0_\sim$.

Theorem 4.11. *The direct sum of two nilpotent intuitionistic (S,T)-fuzzy Lie ideals is also a nilpotent intuitionistic (S,T)-fuzzy Lie ideal.*

Proof. Suppose that $A = (\mu_A, \lambda_A)$ and $B = (\mu_B, \lambda_B)$ are two intuitionistic (S,T)-fuzzy Lie ideals such that $A \cap B = 0_\sim$. We claim that $[A, B] = 0_\sim$. Let $x (\neq 0) \in L$, then

$$\ll \mu_A, \mu_B \gg (x) = \sup\{T(\mu_A(a), \mu_B(b)) \mid [a,b] = x\} \leqslant T(\mu_A(x), \mu_B(x)) = 0$$

and

$$\ll \lambda_A, \lambda_B \gg (x) = \inf\{S(\lambda_A(a), \lambda_B(b)) \mid [a,b] = x\} \geqslant S(\lambda_A(x), \lambda_B(x)) = 1.$$

This proves our claim. Thus we obtain $[A^m, B^n] = 0_\sim$ for all positive integers m, n. Now we again claim that $(A \oplus B)^n \subseteq A^n \oplus B^n$ for positive integer n. We prove this claim by induction on n. For $n = 1$,

$$(A \oplus B)^1 = [A \oplus B, A \oplus B] \subseteq [A, A] \oplus [A, B] \oplus [B, A] \oplus [B, B] = A^1 \oplus B^1.$$

Now for $n > 1$,

$$\begin{aligned}(A \oplus B)^n &= [A \oplus B, (A \oplus B)^{n-1}] \subseteq [A \oplus B, A^{n-1} \oplus B^{n-1}] \\ &\subseteq [A, A^{n-1}] \oplus [A, B^{n-1}] \oplus [B, A^{n-1}] \oplus [B, B^{n-1}] = A^n \oplus B^n.\end{aligned}$$

Since there are two positive integers p and q such that $A^p = B^q = 0_\sim$, we have $(A \oplus B)^{p+q} \subseteq A^{p+q} \oplus B^{p+q} = 0_\sim$. □

In a similar way we can prove the following theorem.

Theorem 4.12. *The direct sum of two solvable intuitionistic (S,T)-fuzzy Lie ideals is a solvable intuitionistic (S,T)-fuzzy Lie ideal.*

Definition 4.13. For any $x \in L$ we define the function $adx : L \to L$ putting $adx(y) = [x, y]$. It is clear that this function is a linear homomorphism with respect to y. The set $H(L)$ of all linear homomorphisms from L into itself is made into a Lie algebra by defining a commutator on it by $[f, g] = f \circ g - g \circ f$. The function $ad : L \to H(L)$ defined by $ad(x) = adx$ is a Lie homomorphism (see [6]) which is called the *adjoint representation* of L.

The adjoint representation $adx : L \to L$ is extended to $a\bar{d}x : I^L \to I^L$ by putting

$$a\bar{d}x(\gamma)(y) = \sup\{\gamma(a) : [x, a] = y\}$$

for all $\gamma \in I^L$ and $y \in L$.

Theorem 4.14. *Let $A = (\mu_A, \lambda_A)$ be an intuitionistic (S,T)-fuzzy Lie ideal in a Lie algebra L. Then $A^n \subseteq [A_n]$ for any $n > 0$, where an intuitionistic fuzzy subset $[A_n] = ([\mu_{A_n}], [\lambda_{A_n}])$ is defined by*

$$[\mu_{A_n}](x) = \sup\{\mu_A(a) \,|\, [x_1, [x_2, [\ldots, [x_n, a] \ldots]]] = x, \quad x_1, \ldots, x_n \in L\},$$
$$[\lambda_{A_n}](x) = \inf\{\lambda_A(a) \,|\, [x_1, [x_2, [\ldots, [x_n, a] \ldots]]] = x, \quad x_1, \ldots, x_n \in L\}.$$

Proof. It is enough to prove that $\ll A, A^{n-1} \gg \subseteq [A_n]$. We prove it by induction on n. For n=1 and $x \in L$, we have

$$\begin{aligned}\ll \mu_A, \mu_A \gg (x) &= \sup\{T(\mu_A(a), \mu_A(b)) \,|\, [a, b] = x\} \\ &\geqslant \sup\{\mu_A(b) \,|\, [a, b] = x, a \in L\} = [\mu_{A_1}](x),\end{aligned}$$

$$\begin{aligned} \ll \lambda_A, \lambda_A \gg (x) &= \inf\{S(\mu_A(a), \mu_A(b)) \mid [a,b]=x\} \\ &\leqslant \inf\{\lambda_A(b) : [a,b]=x, a \in L\} = [\lambda_{A_1}](x). \end{aligned}$$

For $n > 1$,

$$\begin{aligned} &\ll \mu_A, \mu_A^{n-1} \gg (x) = \sup\{T(\mu_A(a), \mu_A^{n-1}(b)) \mid [a,b]=x\} \\ &= \sup\{T(\mu_A(a), [\mu_A(b), \mu_A^{n-2}(b)]) \mid [a,b]=x\} \\ &\geqslant \sup\{T(\mu_A(a), \sup\{\ll \mu_A, \mu_A^{n-2} \gg (b_i) \mid b = \textstyle\sum \alpha_i b_i\}) \mid [a,b]=x\} \\ &\geqslant \sup\{T(\mu_A(a), \sup\{[\mu_{A_{n-1}}](b_i) \mid b = \textstyle\sum \alpha_i b_i\}) \mid [a,b]=x\} \\ &\geqslant \sup\{T(\mu_A(a), [\mu_{A_{n-1}}](b_i)) \mid \textstyle\sum \alpha_i [a,b_i]=x\} \\ &\geqslant \sup\{T(\mu_A(a), \sup\{\mu_{A_{n-1}}(c_i) \mid b_i = \textstyle\sum \beta_i c_i\}) \mid \textstyle\sum \alpha_i [a,b_i]=x\} \\ &\geqslant \sup\{T(\mu_A(a), \mu_{A_{n-1}}(c_i)) \mid \textstyle\sum \gamma_i [a,c_i]=x\} \\ &\geqslant \sup\{T(\mu_A(a), \sup\{\mu_A(d_i)) \mid [x_1,[x_2,[\ldots,[x_{n-1},d_i]\ldots]]]=c_i\} \mid \textstyle\sum \gamma_i [a,c_i]=x\} \\ &\geqslant \sup\{T(\mu_A(a), \mu_A(d_i)) \mid \textstyle\sum \gamma_i [a,[x_1,[x_2,[\ldots,[x_{n-1},d_i]\ldots]]]]=x\} \\ &\geqslant \sup\{\mu_{A_n}(d_i) \mid \textstyle\sum \gamma_i [a,[x_1,[x_2,[\ldots,[x_{n-1},d_i]\ldots]]]]=x\} \geqslant [\mu_{A_n}](x), \end{aligned}$$

$$\begin{aligned} &\ll \lambda_A, \lambda_A^{n-1} \gg (x) = \inf\{S(\lambda_A(a), \lambda_A^{n-1}(b)) \mid [a,b]=x\} \\ &= \inf\{S(\lambda_A(a), [\lambda_A(b), \lambda_A^{n-2}(b)]) \mid [a,b]=x\} \\ &\leqslant \inf\{S(\lambda_A(a), \inf\{\ll \lambda_A, \lambda_A^{n-2} \gg (b_i) \mid b = \textstyle\sum \alpha_i b_i\}) \mid [a,b]=x\} \\ &\leqslant \inf\{S(\lambda_A(a), \inf\{[\lambda_{A_{n-1}}](b_i) \mid b = \textstyle\sum \alpha_i b_i\}) \mid [a,b]=x\} \\ &\leqslant \inf\{S(\lambda_A(a), [\lambda_{A_{n-1}}](b_i)) \mid \textstyle\sum \alpha_i [a,b_i]=x\} \\ &\leqslant \inf\{S(\lambda_A(a), \inf\{\lambda_{A_{n-1}}(c_i) \mid b_i = \textstyle\sum \beta_i c_i\}) \mid \textstyle\sum \alpha_i [a,b_i]=x\} \\ &\leqslant \inf\{S(\lambda_A(a), \lambda_{A_{n-1}}(c_i)) \mid \textstyle\sum \gamma_i [a,c_i]=x\} \\ &\leqslant \inf\{S(\lambda_A(a), \inf\{\lambda_A(d_i)) \mid [x_1,[x_2,[\ldots,[x_{n-1},d_i]\ldots]]]=c_i\} \mid \textstyle\sum \gamma_i [a,c_i]=x\} \\ &\leqslant \inf\{S(\lambda_A(a), \lambda_A(d_i)) \mid \textstyle\sum \gamma_i [a,[x_1,[x_2,[\ldots,[x_{n-1},d_i]\ldots]]]]=x\} \\ &\leqslant \inf\{\lambda_{A_n}(d_i) \mid \textstyle\sum \gamma_i [a,[x_1,[x_2,[\ldots,[x_{n-1},d_i]\ldots]]]]=x\} \leqslant [\lambda_{A_n}](x). \end{aligned}$$

This complete the proof. □

Theorem 4.15. *If for an intuitionistic (S,T)-fuzzy Lie ideal $A = (\mu_A, \lambda_A)$ there exists a positive integer n such that*

$$(\overline{ad}x_1 \circ \overline{ad}x_2 \circ \cdots \circ \overline{ad}x_n)(\mu_A) = 0,$$

$$(\overline{ad}x_1 \circ \overline{ad}x_2 \circ \cdots \circ \overline{ad}x_n)(\lambda_A) = 1,$$

for all $x_1, \ldots, x_n \in L$, then A is nilpotent.

Proof. For $x_1,\dots,x_n \in L$ and $x(\neq 0)\in L$, we have

$$(a\overline{d}x_1\circ\cdots\circ a\overline{d}x_n)(\mu_A)(x)=\sup\{\mu_A(a)\,|\,[x_1,[x_2,[\dots,[x_n,a]\dots]]]=x\}=0,$$

$$(a\overline{d}x_1\circ\cdots\circ a\overline{d}x_n)(\lambda_A)(x)=\inf\{\lambda_A(a)\,|\,[x_1,[x_2,[\dots,[x_n,a]\dots]]]=x\}=1.$$

Thus $[A_n]=0_\sim$. From Theorem 4.14, it follows that $A^n=0_\sim$. Hence $A=(\mu_A,\lambda_A)$ is a nilpotent intuitionistic (S,T)-fuzzy Lie ideal. □

5. The intuitionistic (S,T)-fuzzy Killing form

The mapping $K : L\times L\to F$ defined by $K(x,y)=Tr(adx\circ ady)$, where Tr is the *trace* of a linear homomorphism, is a symmetric bilinear form which is called the *Killing form*. It is not difficult to see that this form satisfies the identity $K([x,y],z)=K(x,[y,z])$. The form K can be naturally extended to $\overline{K} : I^{L\times L}\to I^F$ defined by putting

$$\overline{K}(\mu_A)(\beta)=\sup\{\mu_A(x,y)\,|\,Tr((adx\circ ady))=\beta\},$$
$$\overline{K}(\lambda_A)(\beta)=\inf\{\lambda_A(x,y)\,|\,Tr((adx\circ ady))=\beta\}$$

The Cartesian product of two intuitionistic (S,T)-fuzzy sets $A=(\mu_A,\lambda_A)$ and $B=(\mu_B,\lambda_B)$ is defined as

$$(\mu_A\times\mu_B)(x,y)=T(\mu_A(x),\mu_B(y)),$$
$$(\lambda_A\times\lambda_B)(x,y)=S(\lambda_A(x),\lambda_B(y)).$$

Similarly we define

$$\overline{K}(\mu_A\times\mu_B)(\beta)=\sup\{T(\mu_A(x),\mu_B(y))\,|\,Tr((adx\circ ady))=\beta\},$$
$$\overline{K}(\lambda_A\times\lambda_B)(\beta)=\inf\{S(\lambda_A(x),\lambda_B(y))\,|\,Tr((adx\circ ady))=\beta\}.$$

Proposition 5.1. *Let $A=(\mu_A,\lambda_A)$ be an intuitionistic (S,T)-fuzzy Lie ideal of Lie algebra L. Then*

(i) $1_{\sim(x+y)}=1_{\sim x}\oplus 1_{\sim y}$,

(ii) $1_{\sim(\alpha x)}=\alpha\odot 1_{\sim x}$

for all $x,y\in L$, $\alpha\in F$.

Proof. Straightforward. □

Theorem 5.2. *Let $A = (\mu_A, \lambda_A)$ be an intuitionistic (S,T)-fuzzy Lie ideal of Lie algebra L. Then $\overline{K}(\mu_A \times 1_{(\alpha x)}) = \alpha \odot \overline{K}(\mu_A \times 1_x)$ and $\overline{K}(\lambda_A \times 0_{(\alpha x)}) = \alpha \odot \overline{K}(\lambda_A \times 0_x)$ for all $x \in L$, $\alpha \in F$.*

Proof. If $\alpha = 0$, then for $\beta = 0$ we have

$$\begin{aligned}\overline{K}(\mu_A \times 1_0)(0) &= \sup\{T(\mu_A(x), 1_0(y)) \,|\, Tr(adx \circ ady) = 0\} \\ &\geqslant T(\mu_A(0), 1_0(0)) = 0, \\ \overline{K}(\lambda_A \times 0_0)(0) &= \inf\{S(\lambda_A(x), 0_0(y)) : Tr(adx \circ ady) = 0\} \\ &\leqslant S(\lambda_A(0), 0_0(0)) = 1.\end{aligned}$$

For $\beta \neq 0$ $Tr((adx \circ ady) = \beta$ means that $x \neq 0$ and $y \neq 0$. So,

$$\begin{aligned}\overline{K}(\mu_A \times 1_0)(\beta) &= \sup\{T(\mu_A(x), 1_0(y)) \,|\, Tr((adx \circ ady) = \beta\} = 0, \\ \overline{K}(\lambda_A \times 0_0)(\beta) &= \inf\{S(\lambda_A(x), 0_0(y)) \,|\, Tr((adx \circ ady) = \beta\} = 1.\end{aligned}$$

If $\alpha \neq 0$, then for arbitrary β we obtain

$$\begin{aligned}\overline{K}(\mu_A \times 1_{\alpha x})(\beta) &= \sup\{T(\mu_A(y), 1_{\alpha x}(z)) \,|\, Tr((ady \circ adz) = \beta\} \\ &= \sup\{T(\mu_A(y), \alpha \odot 1_x(z)) \,|\, Tr((ady \circ adz) = \beta\} \\ &= \sup\{T(\mu_A(y), 1_x(\alpha^{-1}z)) \,|\, \alpha Tr((ady \circ ad(\alpha^{-1}z)) = \beta\} \\ &= \sup\{T(\mu_A(y), 1_x(\alpha^{-1}z)) \,|\, Tr((ady \circ ad(\alpha^{-1}z)) = \alpha^{-1}\beta\} \\ &= \overline{K}(\mu_A \times 1_x)(\alpha^{-1}\beta) = \alpha \odot \overline{K}(\mu_A \times 1_x)(\beta),\end{aligned}$$

$$\begin{aligned}\overline{K}(\lambda_A \times 0_{\alpha x})(\beta) &= \inf\{S(\lambda_A(y), 0_{\alpha x}(z)) \,|\, Tr((ady \circ adz) = \beta\} \\ &= \inf\{S(\lambda_A(y), \alpha \odot 0_x(z)) \,|\, Tr((ady \circ adz) = \beta\} \\ &= \inf\{S(\lambda_A(y), 0_x(\alpha^{-1}z)) \,|\, \alpha Tr((ady \circ ad(\alpha^{-1}z)) = \beta\} \\ &= \inf\{S(\lambda_A(y), 0_x(\alpha^{-1}z)) \,|\, Tr((ady \circ ad(\alpha^{-1}z)) = \alpha^{-1}\beta\} \\ &= \overline{K}(\lambda_A \times 1_x)(\alpha^{-1}\beta) = \alpha \odot \overline{K}(\lambda_A \times 0_x)(\beta).\end{aligned}$$

This completes the proof. □

Theorem 5.3. *Let $A = (\mu_A, \lambda_A)$ be an intuitionistic (S,T)-fuzzy Lie ideal of a Lie algebra L. Then $\overline{K}(\mu_A \times 1_{(x+y)}) = \overline{K}(\mu_A \times 1_x) \oplus \overline{K}(\mu_A \times 1_y)$ and $\overline{K}(\mu_A \times 0_{(x+y)}) = \overline{K}(\mu_A \times 0_x) \oplus \overline{K}(\mu_A \times 0_y)$ for all $x, y \in L$.*

Proof. Indeed,

$$\begin{aligned}\overline{K}(\mu_A \times 1_{(x+y)})(\beta) &= \sup\{T(\mu_A(z), 1_{x+y}(u)) \,|\, Tr((adz \circ adu) = \beta\} \\ &= \sup\{\mu_A(z) \,|\, Tr(adz \circ ad(x+y)) = \beta\} \\ &= \sup\{\mu_A(z) \,|\, Tr(adz \circ adx) + Tr(adz \circ ady) = \beta\}\end{aligned}$$

$$= \sup\{T(\mu_A(z), T(1_x(v), 1_y(w))) \mid Tr(adz \circ adv) + Tr(adz \circ adw) = \beta\}$$
$$= \sup\{T(\sup\{T(\mu_A(z), 1_x(v)) \mid Tr(adz \circ adv) = \beta_1\},$$
$$\sup\{T(\mu_A(z), 1_y(w)) \mid Tr(adz \circ adw) = \beta_2\} \mid \beta_1 + \beta_2 = \beta)\}$$
$$= \sup\{T(\overline{K}(\mu_A \times 1_x)(\beta_1),\ \overline{K}(\mu_A \times 1_y)(\beta_2)) \mid \beta_1 + \beta_2 = \beta\}$$
$$= \overline{K}(\mu_A \times 1_x) \oplus \overline{K}(\mu_A \times 1_y)(\beta),$$

$$\overline{K}(\lambda_A \times 0_{(x+y)})(\beta) = \inf\{S(\lambda_A(z), 0_{x+y}(u)) \mid Tr((adz \circ adu) = \beta\}$$
$$= \inf\{\lambda_A(z) \mid Tr(adz \circ ad(x+y)) = \beta\}$$
$$= \inf\{\lambda_A(z) \mid Tr(adz \circ adx) + Tr(adz \circ ady) + \beta\}$$
$$= \inf\{S(\lambda_A(z), S(0_x(v), 0_y(w))) \mid Tr(adz \circ adv) + Tr(adz \circ adw) = \beta\}$$
$$= \inf\{S(\inf\{S(\lambda_A(z), 0_x(v)) \mid Tr(adz \circ adv) = \beta_1\},$$
$$\inf\{S(\lambda_A(z), 0_y(w) \mid Tr(adz \circ adw) = \beta_2\} \mid \beta_1 + \beta_2 = \beta)\}$$
$$= \inf\{S(\overline{K}(\lambda_A \times 0_x)(\beta_1),\ \overline{K}(\lambda_A \times 0_y)(\beta_2)) \mid \beta_1 + \beta_2 = \beta\}$$
$$= \overline{K}(\lambda_A \times 0_x) \oplus \overline{K}(\lambda_A \times 0_y)(\beta).$$

This completes the proof. □

As a consequence of the above two theorems we obtain

Corollary 5.4. *For each intuitionistic* (S,T)*-fuzzy Lie ideal* $A = (\mu_A, \lambda_A)$ *and all* $x, y \in L$, $\alpha, \beta \in F$ *we have*

$$\overline{K}(\mu_A \times 1_{(\alpha x + \beta y)}) = \alpha \odot \overline{K}(\mu_A \times 1_x) \oplus \beta \odot \overline{K}(\mu_A \times 1_y),$$
$$\overline{K}(\lambda_A \times 0_{(\alpha x + \beta y)}) = \alpha \odot \overline{K}(\lambda_A \times 0_x) \oplus \beta \odot \overline{K}(\lambda_A \times 0_y).$$

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