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A note on an Abel-Grassmann's 3-band

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Abstract

An Abel-Grassmann's groupoid is discussed in several papers. In this paper we have investigated AG-3-band and ideal theory on it. An AG-3-band S has associative powers and is fully idempotent. A subset of an AG-3-band is a left ideal if and only it is right and every ideal of S is prime if and only if the set of all ideals of S is totally ordered under inclusion. An ideal of S is prime if and only if it is strongly irreducible. The set of ideals of S is a semilattice.

1. Introduction

An *left almost semigroup* [3], abbreviated as an *LA-semigroup*, is a groupoid S whose elements satisfy for all $a, b, c \in S$ the *invertive law*:

$$(ab)c = (cb)a. \tag{1}$$

In [[1], the same structure is called a *left invertive groupoid* and in [7] it is called an *AG-groupoid*. It is a useful non-associative algebraic structure, midway between a groupoid and a commutative semigroup, with wide applications in the theory of flocks and has a character similar to commutative semigroup.

An AG-groupoid S is *medial* [3], that is,

$$(ab)(cd) = (ac)(bd) \tag{2}$$

holds for all $a, b, c, d, \in S$.

If an AG-groupoid S satisfies for all $a, b, c, d, \in S$ one of the following properties

$$(ab)c = b(ca), \tag{3}$$

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$$(ab)c = b(ac), \tag{4}$$

then it is called an AG*-*groupoid* [9]. It is easy to see that the conditions (3) and (4) are equivalent.

In AG*-groupoid S holds all permutation identities of a next type [9],

$$(x_1x_2)(x_3x_4) = (x_{p(1)}x_{p(2)})(x_{p(3)}x_{p(4)}) \tag{5}$$

where $\{p(1), p(2), p(3), p(4)\}$ means any permutation of the set $\{1, 2, 3, 4\}$.

An AG-groupoid satisfying the identity

$$a(bc) = b(ac) \tag{6}$$

is called an AG**-*groupoid* [6]. An AG-groupoid in which $(aa)a = a(aa) = a$ holds for all a is called an AG-3-*band* [9]. In an AG-3-band S we have $S^2 = S,\ (Sa)S = S(aS)$ and $(SS)S = S(SS)$.

It has been shown in [9], that $(aa)a = a(aa) = a$ and $(bb)b = b(bb) = b$ imply

$$ab = (ab)((ab)(ab)) = ((ab)(ab))(ab).$$

2. AG-3-bands

By an AG**-3-*band* we mean an AG-3-band satisfying identity (6). An AG**-3-band S is a commutative semigroup because using (2), (6) and (1), we get

$$\begin{aligned} xy &= (xy)((xy)(xy)) = (xy)((xx)(yy)) = (xx)((xy)(yy)) \\ &= (xx)((yy)y)x) = ((yy)y)((xx)x) = yx \end{aligned}$$

for all $x, y \in S$. The commutativity and (1) leads us to the associativity.

By an AG*-3-*band* we mean an AG-3-band satisfying (3). If S is an AG-3-band, then $S = S^2$ and by virtue of identity (5), a non-associative AG*-3-band does not exist.

An AG-groupoid S is *paramedial* [2], that is,

$$(ab)(cd) = (db)(ca)$$

holds for all $a, b, c, d, \in S$.

A paramedial AG-3-band becomes a commutative semigroup because

$$ab = (ab)((ab)(ab)) = (ab)((ba)(ba)) = ((ba)(ba))(ba) = ba.$$

Lemma 1. *Every left identity in an AG-3-band is a right identity.*

Proof. Let e be a left identity and a be any element in an AG-3-band S. Then using (1), we get

$$ae = (a(aa))e = (e(aa))a = (aa)a = a.$$

Hence e is right identity. □

As a consequence of Lemma 1, one can see that an AG-3-band with a left identity becomes a commutative monoid, because it has been shown in [5] that every right identity is the unique identity in an AG-groupoid and the identity implies commutativity which further implies associativity.

Lemma 2. *An AG-3-band S is a commutative semigroup if and only if $(xy)^2 = (yx)^2$ holds for all $x, y \in S$.*

Proof. Indeed, using (1), (2), we get

$$\begin{aligned} sa &= ((ss)s)a = (as)(ss) = ((a(aa))s)(ss) = (as)((aa)s)s) \\ &= (as)((ss)(aa)) = (as)((aa)(ss)) = (a(aa))(s(ss)) = as. \end{aligned}$$

The converse is easy. □

Lemma 3. *If S is an AG-3-band, then $aS \subseteq Sa$ for all a in S.*

Proof. Using (1) and (2), we get

$$\begin{aligned} as &= (a(aa))(xy) = (ax)((aa)y) = (ax)(ya)a) \\ &= (a(ya))(xa) = ((xa)(ya))a, \end{aligned}$$

which completes the proof. □

It is easy fact that $(aS)S = Sa$, $S(aS) = (Sa)S$, $(Sa)S \subseteq S(Sa)$ and $Sa \subseteq (Sa)S$.

Lemma 4. *If S is an AG-3-band, then $a^n = a$ and $a^{n+1} = a^2$, where n is a positive odd integer.*

Proof. Obviously $a^3 = (aa)a = a(aa)$. Let the result be true for an odd integer k, that is $a^k = a$. Then using (1), we obtain $a^{k+2} = a^{k+1+1} = a^{k+1}a^1 = (a^k a)a = a^2 a^k = a^2 a = a^3 = a$. Hence $a^n = a$ for all odd integers n. This proves the first identity. To prove the second, observe that $a^4 = a^3 a = aa = a^2$ and assume that $a^s = a^2$ is true for an even integer s. Then using (1), we get $a^{s+2} = a^2 a^s = a^2 a^2 = a^4 = a^2$, which proves that $a^{n+1} = a^2$ is true for a positive odd integer n. □

Lemma 5. *An AG-3-band has associative powers.*

Proof. The proof is easy. □

As a consequence of Lemmas 4 and 5, one can easily see that the sequence of the powers of a has an element a at odd position and a^2 at even position that is, $a, a^2, a, a^2 \ldots$.

The following proposition can be proved easily.

Proposition 1. *In an AG-3-band S for all $a, b \in S$ and all positive integers m, n we have*

$$a^m a^n = a^{m+n}, \quad (ab)^n = a^n b^n, \quad (a^m)^n = a^{mn}.$$

Let $\{S_\alpha : \alpha \in I\}$ be a family of AG-3-bands containing a zero element. We may denote all the zeros elements by common symbol 0. The disjoint union of $\{0\}$ and all $S_\alpha \backslash \{0\}$ becomes an AG-3-band if we define the product of x and y as their product in S_α, if they are in the same S_α, and zero otherwise.

An AG-groupoid S is called *locally associative* if $a(aa) = (aa)a$ holds for all $a \in S$ [4].

Lemma 6. *Every AG-3-band is locally associative AG-groupoid, but the converse is not true.*

Example 1. Let the binary operation on $S = \{a, b, c, d\}$ be defined as follows [4]:

$\cdot$	a	b	c	d
a	d	d	b	d
b	d	d	a	d
c	a	b	c	d
d	d	d	d	d

Then $(S, \cdot)$ is locally associative but it is not AG-3-band because $a(aa) = (aa)a = d \neq a$.

A subset I of an AG-groupoid S is said to be *right* (*left*) *ideal* if $IS \subseteq I$ ($SI \subseteq I$). As usual I is said to be an *ideal* if it is both right and left ideal. An ideal I of an AG-groupoid is called 3-*potent* if $I(II) = (II)I = I$.

An AG-groupoid S without zero is called *simple* (*left simple, right simple*) if it does not properly contain any two sided (left, right) ideal.

An AG-groupoid S with zero is called *zero-simple* if it has no proper ideals and $S^2 \neq \{0\}$.

The existence of non-associative simple and zero-simple AG-3-bands is guaranteed by the following example.

Example 2. The set $S = \{1, 2, 3, 4, 5, 6, 7, 8\}$ with the binary operation defined as follows [9]:

$\cdot$	1	2	3	4	5	6	7	8
1	1	2	7	8	3	4	5	6
2	2	1	8	7	4	3	6	5
3	5	6	3	4	7	8	1	2
4	6	5	4	3	8	7	2	1
5	7	8	1	2	5	6	3	4
6	8	7	2	1	6	5	4	3
7	3	4	5	6	1	2	7	8
8	4	3	6	5	2	1	8	7

is an AG-3-band which has no proper ideals, so it is simple. If we add the element 0 to the set S and extend the binary operation putting $0 \cdot 0 = 0 \cdot s = s \cdot 0 = 0$ for all s in S, then $(S \cup \{0\}, \cdot)$ will be a zero-simple AG-3-band.

Proposition 2. *A subset of an AG-3-band is a right ideal if and only if it is left.*

Proof. Let A be a right ideal of S. Then using (1) we get $sa = ((ss)s)a = (as)(ss)$, which implies that A is a left ideal of S.

The converse follows from Lemma 3. □

A subset M of an AG-groupoid S is called an *m-system* if for $a, b \in M$ there exists $x \in S$ such that $(ax)b \in M$.

A subset B of an AG-groupoid S is called a *p-system* if for every $b \in B$ there exists $x \in S$ such that $(bx)b \in B$.

Proposition 3. *In an AG-groupoid each m-system is a p-system.* □

Lemma 7. *In an AG-3-band every (left, right) ideal is p-system, but the converse is not true.*

Proof. If a, b belongs to an ideal I of an AG-3-band S, then $(as)a \in (IS)I$.

The converse statement follows from Example 2. In this example $B = \{1, 2\}$ is a p-system but not an ideal. □

Two subsets A, B of an AG-groupoid S are called *right* (*left*) *connected* if $AS \subseteq B$ and $BS \subseteq A$ (resp. $SA \subseteq B$ and $SB \subseteq A$) [8]. A and B are *connected* if they are both left and right connected.

Lemma 8. *If A and B are ideal of an AG-3-band S, then AB band BA are right and left connect.*

Proof. Using (1), we get $(AB)S = (SB)A \subseteq BA$. Similarly $(BA)S \subseteq AB$. So, AB and BA are right connected. Also $S(BA) = (SS)(BA) = ((BA)S)S = ((SA)B)S \subseteq AB$, and $S(AB) \subseteq BA$. □

Proposition 4. *If A and B are ideals of an AG-3-band, then AB is an ideal.*

Proof. Using (2), one can easily show that AB is an ideal. □

It is interesting to note that if S is an AG-3-band and I_1, I_2, I_3 are proper ideals of S, then $(I_1I_2)I_3$ is an ideal of S. It can be generalized, that is, if $I_1, I_2, \ldots, I_n$ are ideals, then $(...((I_1I_2)I_3)...)I_n$ is also an ideal and $(...((I_1I_2)I_3)...)I_n \subseteq I_1 \cap I_2 \cap \ldots \cap I_n$.

An AG-groupoid S is said to be *fully idempotent* if every ideal of S is idempotent, i.e., for ecery ideal I of S we have $I^2 = I$.

An AG-groupoid S is said to be *fully semiprime* if every ideal of S is *semiprime*, i.e., for every ideal P of S from $A^2 \subseteq P$, where A is an ideal of S, it follows $A \subseteq P$.

Every AG-3-band is fully idempotent and fully semiprime. Consequently, $A^n = A$ for an ideal A and any positive integer n.

Lemma 9. *$IJ = JI = I \cap J$ for all ideals of an AG-3-band.*

Proof. If $x \in I \cap J$, then $x = x(xx) \in IJ$, whence $IJ = I \cap J$. So, $IJ = JI$. □

An ideal I of an AG-groupoid S is said to be *strongly irreducible* if and only if for ideals H and K of S, $H \cap K \subseteq I$ implies either $H \subseteq I$ or $K \subseteq I$.

An AG-groupoid S is called *totally ordered* if for all ideals A, B of S either $A \subseteq B$ or $B \subseteq A$.

An ideal P of an AG-groupoid S is called *prime* if and only if $AB \subseteq P$ implies that either $A \subseteq P$ or $B \subseteq P$ for all ideals A and B in S.

Using Lemma 9, one can prove the following Theorems.

Theorem 1. *In an AG-3-band an ideal is strongly irreducible if and only if it is prime.*

Theorem 2. *An ideal of an AG-3-band S is prime if and only if the set of all ideals of S is totally ordered under inclusion.*

Theorem 3. *The set of ideals of an AG-3-band S form a semilattice, $(L_S, \wedge)$, where $A \wedge B = AB$, A and B are ideals of S.*

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