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A note on Belousov quasigroups

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Abstract

A Belousov identity is a balanced identity which is a consequence of commutativity. It is proved that a quasigroup is Belousov iff it has a permutation π satisfying $\pi(xy) = yx$ and a weak (anti)automorphism–like property depending on Belousov identities the quasigroup satisfies.

A *balanced* (also called *linear*) *identity* is one in which each variable appears precisely twice, once on each side of the equality symbol. Instead of identity the word *equation* is sometimes used. We note that, although quasigroups might be defined equationally, using multiplication $(\cdot)$ and both division operations ($\backslash$ and $/$), the identities which we consider contain the multiplication symbol only. The dual operation $*$ of $\cdot$ is defined by $x * y = y{\cdot}x$. The symbol $*$ is considered not to belong to the language of quasigroups. When unambiguous, the term $x \cdot y$ is usually shortened to xy.

The product symbol ($\prod$) is used *but only for products of* 2^n *factors*. Formally: $\prod_{i=m}^{m} x_i = x_m$ and $\prod_{i=m}^{m+2^n-1} x_i = (\prod_{i=m}^{m+2^{n-1}-1} x_i)(\prod_{i=m+2^{n-1}}^{m+2^n-1} x_i)$.

V. D. Belousov defined in [1] an important class of balanced identities which were named *Belousov equations* by A. Krapež and M. A. Taylor in [3]. A balanced identity $s = t$ is *Belousov* if for every subterm p of s (t) there is a subterm q of t (s) such that p and q have exactly the same variables.

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Examples of Belousov identities are:

$$x = x \tag{B_0}$$
$$xy = xy$$
$$xy = yx \tag{B_1}$$
$$x \cdot yz = zy \cdot x$$
$$xy \cdot uv = vu \cdot yx \tag{B_{11}}$$
$$xy \cdot (zu \cdot vw) = (uz \cdot wv) \cdot yx$$

The identity (B_0) and all identities $t = t$ are *trivial*. Belousov identities not equivalent to (B_0) are *nontrivial*. A quasigroup satisfying a set of Belousov identities, not all of them trivial, is a *Belousov quasigroup*.

The characteristic property of Belousov identities is:

Theorem 1. (partially in Krapež [2]) *A balanced quasigroup identity $s = t$ is Belousov:*

- *iff $s = t$ is a consequence of the theory of commutative quasigroups,*
- *iff there is an identity $Eq(\cdot, *)$ which is true in all quasigroups and $s = t$ is $Eq(\cdot, \cdot)$,*
- *iff the trees of terms s, t are isomorphic.*

Their importance stems from:

Theorem 2. (Krapež [2], Belousov [1]) *A quasigroup satisfying a balanced but not Belousov identity is isotopic to a group.*

Belousov identities are described in [4] using polynomials from $\mathbb{Z}_2[x]$.

Theorem 3. (Krapež, Taylor [4]) *Every set of Belousov identities is equivalent to a single normal Belousov identity.*

For the reduction algorithm and the proof consult [4]. Below we just give the definition of a normal Belousov identity.

Definition 1. A quasigroup for which there is a permutation π such that $\pi(xy) = yx$ is called *almost commutative*. The permutation π is called a *swap*.

The next theorem was proved by Belousov, except that he forgot to exclude the trivial identities (i.e., $t = t$).

Theorem 4. (Belousov [1]) *Every Belousov quasigroup is almost commutative.*

A sequence $\alpha_1 \dots \alpha_n$ of zeros and ones is a *pattern*. It is a *normal pattern* if $\alpha_1 = \alpha_n = 1$.

Let π be a swap, $p = \alpha_1 \dots \alpha_n$ a pattern and st a term. We define :

$$\pi^{\alpha_1 \dots \alpha_n}(st) = \pi^{\alpha_1}(\pi^{\alpha_2 \dots \alpha_n}(s) \cdot \pi^{\alpha_2 \dots \alpha_n}(t)).$$

The relations $\pi^0 = Id$ $(Id(x) = x)$ and $\pi^1 = \pi$ are assumed.

Definition 2. Let π be a swap and p a pattern of length $n > 0$. The Belousov identity (B_p) is:

$$\prod_{i=1}^{2^n} x_i = \pi^p(\prod_{i=1}^{2^n} x_i). \tag{B_p}$$

This is a *normal Belousov identity* if p is a normal pattern.

We assume that (B_0) is also a normal Belousov identity.

Note that the identity (B_p) does not contain a single occurrence of π. It is used up while transforming various subterms st of $\prod_{i=1}^{2^n} x_i$ into ts.

Theorem 5. *Let p be a nontrivial normal pattern $\alpha_1 \dots \alpha_n$. A quasigroup satisfies the normal Belousov identity (B_p) iff it has a swap π satisfying:*

$$\pi(\prod_{i=1}^{2^{n-1}} y_i) = \pi^{0\alpha_2 \dots \alpha_{n-1}}(\prod_{i=1}^{2^{n-1}} \pi(y_i)). \tag{1}$$

Proof. Apply π to both sides of (B_p); then use $\pi^2(x) = x$; next push π inside the product on the right hand side of the equation; then pull out π back, preserving expressions $\pi(x_{2i-1}x_{2i})$; and finally substitute y_i for $x_{2i-1}x_{2i}$. We get (1).

All transformations are equivalent, so the theorem follows. □

Example 1. For $p = 1$ we get that a quasigroup is commutative if Id is a swap.

Example 2. For $p = 11$ we get the result of M. Polonijo [5] that a quasigroup satisfies (B_{11}) (or palindromic identity in the terminology of [5]) iff it has a swap π satisfying $\pi(xy) = \pi^{01}(x \cdot y) = \pi(x) \cdot \pi(y)$.

Example 3. For $p = 101$ we get that a quasigroup satisfies $\prod_{i=1}^{8} x_i = \pi^{101}(\prod_{i=1}^{8} x_i) = (x_6x_5 \cdot x_8x_7)(x_2x_1 \cdot x_4x_3)$ iff it has a swap π satisfying $\pi(xy \cdot uv) = \pi^{001}(xy \cdot uv) = \pi(x)\pi(y) \cdot \pi(u)\pi(v)$.

The last two examples suggest:

Corollary 1. A quasigroup satisfies $(B_{10\ldots01})$ (with $n \geqslant 0$ zeros) iff it has a swap π satisfying $\pi(\prod_{i=1}^{2^{n+1}} y_i) = \prod_{i=1}^{2^{n+1}} \pi(y_i)$.

Example 4. For $p = 111$ we get that a quasigroup satisfies $\prod_{i=1}^{8} x_i = \pi^{111}(\prod_{i=1}^{8} x_i) = (x_8x_7 \cdot x_6x_5)(x_4x_3 \cdot x_2x_1)$ iff it has a swap π satisfying $\pi(xy \cdot uv) = \pi^{011}(xy \cdot uv) = \pi(\pi(x)\pi(y)) \cdot \pi(\pi(u)\pi(v)) = \pi(y)\pi(x) \cdot \pi(v)\pi(u)$.

Another way of looking at Theorem 5 is:

Theorem 6. *The equational theory of B_p–quasigroups ($p = \alpha_1 \ldots \alpha_n, n > 0$) is equivalent to the equational theory of algebras $(S; \cdot, \backslash, /, \pi)$ with the quasigroup axioms: $x\backslash xy = y$, $x(x\backslash y) = y$, $xy/y = x$, $(x/y)y = x$, the swap axiom $\pi(xy) = yx$ and* (1).

The last axiom has a half as many variables as the identity (B_p).
In case of an equational theory with arbitrary nontrivial Belousov identities we can combine the Theorem 6 with the Theorem 3 to get the appropriate axiom (1).

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