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On central loops and the central square property

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Abstract

The representation sets of a central square C-loop are investigated. Isotopes of central square C-loops of exponent 4 are shown to be both C-loops and A-loops.

1. Introduction

C-loops are one of the least studied loops. Few publications that have considered C-loops include Fenyves [10], [11], Beg [3], [4], Phillips et. al. [17], [19], [15], [14], Chein [7] and Solarin et. al. [2], [23], [21], [20]. The difficulty in studying them is as a result of the nature of their identities when compared with other Bol-Moufang identities (the element occurring twice on both sides has no other element separating it from itself). Latest publications on the study of C-loops which has attracted fresh interest on the structure include [17], [19], and [15].

LC-loops, RC-loops and *C-loops* are loops that satisfies the identities

$$(xx)(yz) = (x(xy))z, \quad (zy)(xx) = z((yx)x), \quad x(y(yz)) = ((xy)y)z,$$

respectively. Fenyves' work in [11] was completed in [17]. Fenyves proved that LC-loops and RC-loops are defined by three equivalent identities. In [17] and [18], it was shown that LC-loops and RC-loops are defined by four equivalent identities. Solarin [21] named the fourth identities the *left middle* (LM) and *right middle* (RM) *identities* and loops that obey them are called LM*-loops* and RM*-loops*, respectively. These terminologies were also used in [22]. Their basic properties are found in [19], [11] and [9].

Definition 1.1. A set Π of permutations on a set L is the *representation* of a loop $(L, \cdot)$ if and only if

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(i) $I \in \Pi$ (identity mapping),

(ii) Π is transitive on L (i.e., for all $x, y \in L$, there exists a unique $\pi \in \Pi$ such that $x\pi = y$),

(iii) if $\alpha, \beta \in \Pi$ and $\alpha\beta^{-1}$ fixes one element of L, then $\alpha = \beta$.

The left (right) representation of a loop L is denoted by $\Pi_\lambda(L)$ (resp. $\Pi_\rho(L)$) or Π_λ (resp. Π_ρ) and is defined as the set of all left (right) translation maps on the loop i.e., if L is a loop, then $\Pi_\lambda = \{L_x : L \to L \mid x \in L\}$ and $\Pi_\rho = \{R_x : L \to L \mid x \in L\}$, where $R_x : L \to L$ and $L_x : L \to L$ are defined as $yR_x = yx$ and $yL_x = xy$ are bijections.

Definition 1.2. Let $(L, \cdot)$ be a loop. The *left nucleus* of L is the set

$$N_\lambda(L, \cdot) = \{a \in L : ax \cdot y = a \cdot xy \ \forall\ x, y \in L\}.$$

The *right nucleus* of L is the set

$$N_\rho(L, \cdot) = \{a \in L : y \cdot xa = yx \cdot a \ \forall\ x, y \in L\}.$$

The *middle nucleus* of L is the set

$$N_\mu(L, \cdot) = \{a \in L : ya \cdot x = y \cdot ax \ \forall\ x, y \in L\}.$$

The *nucleus* of L is the set

$$N(L, \cdot) = N_\lambda(L, \cdot) \cap N_\rho(L, \cdot) \cap N_\mu(L, \cdot).$$

The *centrum* of L is the set

$$C(L, \cdot) = \{a \in L : ax = xa \ \forall\ x \in L\}.$$

The *center* of L is the set

$$Z(L, \cdot) = N(L, \cdot) \cap C(L, \cdot).$$

L is said to be a *centrum square loop* if $x^2 \in C(L, \cdot)$ for all $x \in L$. L is said to be a *central square loop* if $x^2 \in Z(L, \cdot)$ for all $x \in L$. L is said to be *left alternative* if for all $x, y \in L$, $x \cdot xy = x^2 y$ and is said to right alternative if for all $x, y \in L$, $yx \cdot x = yx^2$. Thus, L is said to be *alternative* if it is both left and right alternative. The triple (U, V, W) such that $U, V, W \in SYM(L, \cdot)$ is called an *autotopism* of L if and only if

$$xU \cdot yV = (x \cdot y)W \quad \forall\ x, y \in L.$$

$SYM(L,\cdot)$ is called the *permutation group* of the loop $(L,\cdot)$. The group of autotopisms of L is denoted by $AUT(L)$. Let $(L,\cdot)$ and $(G,\circ)$ be two distinct loops.

The triple $(U,V,W):(L,\cdot)\to(G,\circ)$ such that $U,V,W:L\to G$ are bijections is called a *loop isotopism* if and only if

$$xU\circ yV=(x\cdot y)W \quad \forall\, x,y\in L.$$

In [13], the three identities stated in [11] were used to study finite central loops and the isotopes of central loops. It was shown that in a finite RC(LC)-loop L, $\alpha\beta^2\in\Pi_\rho(L)\big(\Pi_\lambda(L)\big)$ for all $\alpha,\beta\in\Pi_\rho(L)\big(\Pi_\lambda(L)\big)$ while in a C-loop L, $\alpha^2\beta\in\Pi_\rho(L)\big(\Pi_\lambda(L)\big)$ for all $\alpha,\beta\in\Pi_\rho(L)\big(\Pi_\lambda(L)\big)$. A C-loop is both an LC-loop and an RC-loop [11], hence it satisfies the formal. Here, it will be shown that LC-loops and RC-loops satisfy the later formula.

Also in [13], under triples of the form (A,B,B), (A,B,A), alternative centrum square loop isotopes of centrum square C-loops were shown to be C-loops.

It is shown that a finite loop is a central square central loop if and only if its left and right representations are closed relative to some left and right translations. Central square C-loops of exponent 4 are groups, hence their isotopes are both C-loops and A-loops.

For other definitions see [5], [22] and [16].

2. Preliminaries

Definition 2.1. (cf. [16]) Let $(L,\cdot)$ be a loop and $U,V,W\in SYM(L,\cdot)$. If $(U,V,W)\in AUT(L)$ for some U,V,W, then U is called an *autotopism*. If there exists $V\in SYM(L,\cdot)$ such that $xU\cdot y=x\cdot yV$ for all $x,y\in L$, then U is called *μ-regular*, while $U'=V$ is called its *adjoint*.

The set of autotopic bijections in a loop $(L,\cdot)$ is denoted by $\Sigma(L,\cdot)$, the set of all μ-regular bijections by $\Phi(L)$, the set of all adjoints by $\Phi^*(L)$.

Theorem 2.1. ([16]) *Groups of autotopisms of isotopic quasigroups are isomorphic.* □

Theorem 2.2. ([16]) *The set of all μ-regular bijections of a quasigroup $(Q,\cdot)$ is a subgroup of the group $\Sigma(Q,\cdot)$ of all autotopic bijections of $(Q,\cdot)$.* □

Corollary 2.1. ([16]) *If two quasigroups Q and Q' are isotopic, then the corresponding groups Φ and Φ' [Φ^* and Φ'^*] are isomorphic.* □

Definition 2.2. A loop $(L,\cdot)$ is called a *left inverse property loop* or *right inverse property loop* (L.I.P.L. or R.I.P.L.) if and only if it satisfies the left inverse property (resp. right inverse property): $x^\lambda(xy)=y$ (resp. $(yx)x^\rho=y$. Hence, it is called an *inverse property loop* (I.P.L.) if and only if it has the inverse property (I.P.) i.e., it has a left inverse property (L.I.P.) and right inverse property (R.I.P.).

Most of our results and proofs, are written in dual form relative to RC-loops and LC-loops. That is, a statement like 'LC(RC)-loop... A(B)' where 'A' and 'B' are some equations or expressions means that 'A' is for LC-loops and 'B' is for RC-loops.

3. Finite central loops

Lemma 3.1. *Let L be a loop. L is an LC(RC)-loop if and only if $\beta\in\Pi_\rho$ (Π_λ) implies $\alpha\beta\in\Pi_\rho$ (Π_λ) for some $\alpha\in\Pi_\rho$ (Π_λ).*

Proof. L is an LC-loop if and only if $x\cdot(y\cdot yz)=(x\cdot yy)z$ for all $x,y,z\in L$. L is an RC-loop if and only if $(zy\cdot y)x=z(yy\cdot x)$ for all $x,y,z\in L$. Thus, L is an LC-loop if and only if $xR_{y\cdot yz}=xR_{y^2}R_z$ if and only if $R_{y^2}R_z=R_{y\cdot yz}$ for all $y,z\in L$ and L is an RC-loop if and only if $xL_{zy\cdot y}=xL_{y^2}L_z$ if and only if $L_{zy\cdot y}=L_{y^2}L_z$. With $\alpha=R_{y^2}$ (L_{y^2}) and $\beta=R_z(L_z)$, $\alpha\beta\in\Pi_\rho$ (Π_λ). □

Lemma 3.2. *A loop L is an LC(RC)-loop if and only if $\alpha^2\beta=\beta\alpha^2$ for all $\alpha\in\Pi_\lambda$ (Π_ρ) and $\beta\in\Pi_\rho$ (Π_λ).*

Proof. L is an LC-loop if and only if $x(x\cdot yz)=(x\cdot xy)z$ while L is an RC-loop if and only if $(zy\cdot x)x=z(yx\cdot x)$. Thus, when L is an LC-loop, $yR_zL_x^2=yL_x^2R_z$ if and only if $R_zL_x^2=L_x^2R_z$, while when L is an RC-loop, $yL_zR_x^2=yR_x^2L_z$ if and only if $L_zR_x^2=R_x^2L_z$. Thus, replacing L_x (R_x) and R_z (L_z) respectively by α and β, We obtain our result. The converse statement can be proved analogously. □

Theorem 3.1. *A loop L is an LC(RC)-loop if and only if $\alpha,\beta\in\Pi_\lambda$ (Π_ρ) implies $\alpha^2\beta\in\Pi_\lambda$ (Π_ρ).*

Proof. L is an LC-loop if and only if $x \cdot (y \cdot yz) = (x \cdot yy)z$ for all $x, y, z \in L$ while L is an RC-loop if and only if $(zy \cdot y)x = z(yy \cdot x)$ for all $x, y, z \in L$. Thus when L is an LC-loop, $zL_{x \cdot yy} = zL_y^2 L_x$ if and only if $L_y^2 L_x = L_{x \cdot yy}$ while when L is an RC-loop, $zR_y^2 R_x = zR_{yy \cdot x}$ if and only if $R_y^2 R_x = R_{yy \cdot x}$. Replacing $L_y(R_y)$ and $L_x(R_x)$ with α and β respectively, we have $\alpha^2 \beta \in \Pi_\lambda(\Pi_\rho)$ when L is an LC(RC)-loop. The converse follows by reversing the procedure. □

Theorem 3.2. *Let L be an LC(RC)-loop. L is centrum square if and only if $\alpha \in \Pi_\rho$ (Π_λ) implies $\alpha\beta \in \Pi_\rho$ (Π_λ) for some $\beta \in \Pi_\rho(\Pi_\lambda)$.*

Proof. By Lemma 3.1, $R_{y^2} R_z = R_{y \cdot yz}$ $(L_{y^2} L_z = L_{zy \cdot y})$. Using Lemma 3.2, if L is centrum square, $R_{y^2} = L_{y^2}$ $(L_y^2 = R_{y^2})$. So, when L is an LC-loop, $R_{y^2} R_z = L_y^2 R_z = R_z L_y^2 = R_z R_{y^2} = R_{y \cdot yz}$, while when L is an RC-loop, $L_{y^2} L_z = R_y^2 L_z = L_z R_{y^2} = L_z L_{y^2} = L_{zy \cdot y}$. Let $\alpha = R_z$ (L_z) and $\beta = R_{y^2}$ (L_{y^2}), then $\alpha\beta \in \Pi_\rho$ (Π_λ) for some $\beta \in \Pi_\rho$ (Π_λ).

Conversely, if $\alpha\beta \in \Pi_\rho$ (Π_λ) for some $\beta \in \Pi_\rho$ (Π_λ) such that $\alpha = R_z$ (L_z) and $\beta = R_{y^2}$ (L_{y^2}) then $R_z R_{y^2} = R_{y \cdot yz}$ $(L_z L_{y^2} = L_{zy \cdot y})$. By Lemma 3.1, $R_{y^2} R_z = R_{y \cdot yz}$ $(L_{zy \cdot y} = L_{y^2} L_z)$, thus $R_z R_{y^2} = R_{y^2} R_z$ $(L_z L_{y^2} = L_{y^2} L_z)$ if and only if $xz \cdot y^2 = xy^2 \cdot z$ $(y^2 \cdot zx = z \cdot y^2 x)$. Let $x = e$, then $zy^2 = y^2 z$ $(y^2 z = zy^2)$ implies L is centrum square. □

Corollary 3.1. *Let L be a loop. L is a centrum square LC(RC)-loop if and only if*

1. $\alpha\beta \in \Pi_\rho$ (Π_λ) *for all* $\alpha \in \Pi_\rho$ (Π_λ) *and for some* $\beta \in \Pi_\rho$ (Π_λ),
2. $\alpha\beta \in \Pi_\rho$ (Π_λ) *for all* $\beta \in \Pi_\rho$ (Π_λ) *and for some* $\alpha \in \Pi_\rho$ (Π_λ).

Proof. This follows from Lemma 3.1 and Theorem 3.2. □

4. Isotopes of central loops

In [23] is concluded that central loops are not CC-loops. This means that the study of the isotopic invariance of C-loops will be trivial. This is, because if C-loops are CC-loops, then commutative C-loops would be groups since commutative CC-loops are groups. But from the constructions in [19], it follows that there are commutative C-loops which are not groups. The conclusion in [23] is based on the fact that the authors considered a loop of units in a central algebra.

Theorem 4.1. *A loop L is an LC(RC)-loop if and only if $(R_{y^2}, L_y^{-2}, I) \in AUT(L)$ (resp. $(R_y^2, L_{y^2}^{-1}, I) \in AUT(L)$) for all $y \in L$.*

Proof. According to [19], L is an LC-loop if and only if $x\cdot(y\cdot yz)=(x\cdot yy)z$ for all $x,y,z\in L$, while L is an RC-loop if and only if $(zy\cdot y)x=z(yy\cdot x)$ for all $x,y,z\in L$. $x\cdot(y\cdot yz)=(x\cdot yy)z$ if and only if $x\cdot zL_y^2=xR_{y^2}\cdot z$ if and only if $(R_{y^2},L_y^{-2},I)\in AUT(L)$ for all $y\in L$, while $(zy\cdot y)x=z(yy\cdot x)$ if and only if $zR^2\cdot x=z\cdot xL_{y^2}$ if and only if $(R_y^2,L_{y^2}^{-1},I)\in AUT(L)$ for all $y\in L$. □

Corollary 4.1. *Let $(L,\cdot)$ be an LC(RC)-loop, then $(R_{y^2}L_x^2,L_y^{-2},L_x^2)$ (resp. $(R_y^2,L_{y^2}^{-1}R_x^2,R_x^2)$) belongs to $AUT(L)$ for all $x,y\in L$.*

Proof. In an LC-loop L, $(L_x^2,I,L_x^2)\in AUT(L)$ while in an RC-loop L we have $(I,R_x^2,R_x^2)\in AUT(L)$. Thus, by Theorem 4.1, for any LC-loop, $(R_{y^2},L_y^{-2},I)(L_x^2,I,L_x^2)=(R_{y^2}L_x^2,L_y^{-2},L_x^2)\in AUT(L)$ and for any RC-loop, $(R_y^2,L_{y^2}^{-1},I)(I,R_x^2,R_x^2)=(R_y^2,L_{y^2}^{-1}R_x^2,R_x^2)\in AUT(L)$. □

Theorem 4.2. *A loop L is a C-loop if and only if L is a right (left) alternative LC(RC)-loop.*

Proof. If $(L,\cdot)$ is an LC(RC)-loop, then by Theorem 4.1, (R_{y^2},L_y^{-2},I) (resp. $(R_y^2,L_{y^2}^{-1},I)$) $\in AUT(L)$ for all $y\in L$. If L has the right (left) alternative property, then $(R_y^2,L_y^{-2},I)\in AUT(L)$ for all $y\in L$ if and only if L is a C-loop. □

Lemma 4.1. *A loop L is an LC(RC,C)-loop if and only if $R_{y^2}\in\Phi(L)$ (resp. R_y^2, $R_y^2\in\Phi(L)$) and $(R_{y^2})^*=L_y^2\in\Phi^*(L)$ (resp. $(R_y^2)^*=L_{y^2}\in\Phi^*(L)$, $(R_y^2)^*=L_y^2\in\Phi^*(L)$) for all $y\in L$.*

Proof. This can be deduced from Theorem 4.1. □

Theorem 4.3. *Let $(G,\cdot)$ and $(H,\circ)$ be two distinct loops. If G is a central square LC(RC)-loop, H an alternative central square loop and the triple $\alpha=(A,B,B)$ (resp. $\alpha=(A,B,A)$) is an isotopism of G onto H, then H is a C-loop.*

Proof. G is a LC(RC)-loop if and only if R_{y^2} $(R_y^2)\in\Phi(G)$ and $(R_{y^2})^*=L_y^2$ (resp. $(R_y^2)^*=L_{y^2}$) $\in\Phi^*(G)$ for all $x\in G$. Using the idea of [6], $L'_{xA}=B^{-1}L_xB$ and $R'_{xB}=A^{-1}R_xA$ for all $x\in G$. Using Corollary 2.1, for the case when G is an LC-loop: let $h:\Phi(G)\to\Phi(H)$ and $h^*:\Phi^*(G)\to\Phi^*(H)$ be defined as $h(U)=B^{-1}UB$ for all $U\in\Phi(G)$ and $h^*(V)=B^{-1}VB$ for all $V\in\Phi^*(G)$. This mappings are isomorphisms. Using the hypothesis, $h(R_{y^2})=h(L_{y^2})=h(L_y^2)=B^{-1}L_y^2B=$

$B^{-1}L_yBB^{-1}L_yB = L'_{yA}L'_{yA} = L'^2_{yA} = L'_{(yA)^2} = R'_{(yA)^2} = R'^2_{(yA)} \in \Phi(H)$. $h^*[(R_{y^2})^*] = h^*(L_y^2) = B^{-1}L_y^2B = B^{-1}L_yL_yB = B^{-1}L_yBB^{-1}L_yB = L'_{yA}L'_{yA} = L'^2_{yA} \in \Phi^*(H)$. So, $R'^2_y \in \Phi(H)$ and $(R'^2_y)^* = L'^2_y \in \Phi^*(H)$ for all $y \in H$ if and only if H is a C-loop.

For the case of RC-loops, using h and h^* as above, but now defined as: $h(U) = A^{-1}UA$ for all $U \in \Phi(G)$ and $h^*(V) = A^{-1}VA$ for all $V \in \Phi^*(G)$. This mappings are still isomorphisms. Using the hypotheses, $h(R_y^2) = A^{-1}R_y^2A = A^{-1}R_yAA^{-1}R_yA = R'_{yB}R'_{yB} = R'^2_{yB} \in \Phi(H)$. $h^*[(R_y^2)^*] = h^*(L_{y^2}) = h^*(R_{y^2}) = A^{-1}R_y^2A = A^{-1}R_yR_yB = B^{-1}R_yBB^{-1}R_yB = R'_{yA}R'_{yA} = R'^2_{yA} = R'_{(yA)^2} = L'_{(yA)^2} = L'^2_{yA} \in \Phi^*(H)$. So, $R'^2_y \in \Phi(H)$ and $(R'^2_y)^* = L'^2_y \in \Phi^*(H)$ if and only if H is a C-loop. □

Corollary 4.2. *Let $(G,\cdot)$ and $(H,\circ)$ be two distinct loops. If G is a central square left (right) RC(LC)-loop, H an alternative central square loop and the triple $\alpha = (A,B,B)$ (resp. $\alpha = (A,B,A)$) is an isotopism of G onto H, then H is a C-loop.*

Proof. By Theorem 4.2, G is a C-loop in each case. The rest of the proof follows by Theorem 4.3. □

Remark 4.1. Corollary 4.2 was proved in [13].

5. Central square C-loops of exponent 4

For a loop $(L,\cdot)$, the bijection $J : L \to L$ is defined by $xJ = x^{-1}$.

Theorem 5.1. *If for a C-loop $(L,\cdot)$ (I, L_z^2, JL_z^2J) or (R_z^2, I, JR_z^2J) lies in $AUT(L)$, then L is a loop of exponent 4.*

Proof. If $(I, L_z^2, JL_z^2J) \in AUT(L)$ for all $z \in L$, then: $x \cdot yL_z^2 = (xy)JL_z^2J$ for all $x,y,z \in L$ implies $x \cdot z^2y = xy \cdot z^{-2}$, whence $z^2y \cdot z^2 = y$. Then $y^4 = e$. Hence L is a C-loop of exponent 4.

If $(R_z^2, I, JR_z^2J) \in AUT(L)$ for all $z \in L$, then: $xR_z^2 \cdot y = (xy)JR_z^2J$ for all $x,y,z \in L \longrightarrow (xz^2)\cdot y = [(xy)^{-1}z^2]^{-1} \longrightarrow (xz^2)\cdot y = z^{-2}(xy) \longrightarrow (xz^2)\cdot y = z^{-2}x \cdot y \longrightarrow xz^2 = z^{-2}x \longrightarrow z^4 = e$. Hence L is a C-loop of exponent 4. □

Theorem 5.2. *If in a C-loop L for all $z \in L$ (I, L_z^2, JL_z^2J) or (R_z^2, I, JR_z^2J) is in $AUT(L)$, then L is a central square C-loop of exponent 4.*

Proof. If $(I, L_z^2, JL_z^2 J) \in AUT(L)$ for all $z \in L$, then $x \cdot yL_z^2 = (xy)JL_z^2 J$ for all $x, y, z \in L$, whence $x \cdot z^2 y = xy \cdot z^{-2}$.

If $(R_z^2, I, JR_z^2 J) \in AUT(L)$ for all $z \in L$, then $xR_z^2 \cdot y = (xy)JR_z^2 J$ for all $x, y, z \in L$, whence $xz^2 \cdot y = z^{-2} \cdot xy$.

So, in both these cases we have $x{\cdot}z^2 y = xz^2{\cdot}y \longleftrightarrow xy{\cdot}z^{-2} = z^{-2}{\cdot}xy$. For $t = xy$, we get $tz^{-2} = z^{-2}t \longleftrightarrow z^2 t^{-1} = t^{-1} z^2$, which implies $z^2 \in C(L, \cdot)$ for all $z \in L$.

Since C-loops arenuclear square (cf. [19]), we have $z^2 \in Z(L, \cdot)$. Hence L is a central square C-loop. By Theorem 5.1, $x^4 = e$. □

Remark 5.1. In [19], C-loops of exponent 2 were found. In [19] and [11] i is proved that C-loops are naturally nuclear square. Our Theorem 5.2 gives some conditions under which a C-loop can be naturally central square.

Theorem 5.3. *If $A = (U, V, W) \in AUT(L)$ for a C-loop $(L, \cdot)$, then $A_\rho = (V, U, JWJ) \notin AUT(L)$, but $A_\mu = (W, JVJ, U), A_\lambda = (JUJ, W, V)$ are in $AUT(L)$.*

Proof. The fact that $A_\mu, A_\lambda \in AUT(L)$ has been shown in [5] and [16] for an I.P.L. L. Let L be a C-loop. Since C-loops are inverse property loops, $A_\mu = (W, JVJ, U)$, $A_\lambda = (JUJ, W, V) \in AUT(L)$. A C-loop is both an RC-loop and an LC-loop. So, $(I, R_x^2, R_x^2), (L_x^2, I, L_x^2) \in AUT(L, \cdot)$ for all $x \in L$. Thus, if $A_\rho \in AUT(L)$ when $A = (I, R_x^2, R_x^2)$ and $A = (L_x^2, I, L_x^2)$, $A_\rho = (I, L_x^2, JL_x^2 J) \in AUT(L)$ and $A_\rho = (R_x^2, I, JR_x^2 J) \in AUT(L)$ hence by Theorem 5.1 and Theorem 5.2, all C-loops are central square and of exponent 4 (in fact it will soon be seen in Theorem 5.4 that central square C-loops of exponent 4 are groups), which is false. So, $A_\rho = (V, U, JWJ) \notin AUT(L)$. □

Corollary 5.1. *If $(I, L_z^2, JL_z^2 J) \in AUT(L)$, and $(R_z^2, I, JR_z^2 J) \in AUT(L)$ for all $z \in L$, where $(L, \cdot)$ is a C-loop, then*

1. *L is flexible,*
2. *$(xy)^2 = (yx)^2$ for all $x, y \in L$,*
3. *$x \mapsto x^3$ is an anti-automorphism.*

Proof. This is a consequence of Theorem 5.2, Lemma 5.1 and Corollary 5.2 of [15]. □

Theorem 5.4. *A central square C-loop of exponent 4 is a group.*

Proof. To prove this, it shall be shown that $R(x, y) = I$ for all $x, y \in L$.

Using Corollary 5.1 we see that for any $w \in L$ will be $wR(x, y) = wR_xR_yR_{xy}^{-1} = (wx)y{\cdot}(xy)^{-1} = (wx)(x^2yx^2){\cdot}(xy)^{-1} = (wx^3)(yx^2){\cdot}(xy)^{-1} = (w^2(w^3x^3))(yx^2){\cdot}(xy)^{-1} = (w^2(xw)^3)(yx^2){\cdot}(xy)^{-1} = w^2(xw)^3{\cdot}(yx^2)(xy)^{-1} = w^2(xw)^3{\cdot}[y{\cdot}x^2(xy)^{-1}] = w^2(xw)^3{\cdot}[y{\cdot}x^2(y^{-1}x^{-1})] = w^2(xw)^3{\cdot}[y(y^{-1}x^{-1}{\cdot}x^2)] = w^2(xw)^3 \cdot [y(y^{-1}x)] = w^2(xw)^3 \cdot x = w^2(w^3x^3) \cdot x = w^2 \cdot (w^3x^3)x = w^2 \cdot (w^3x^{-1})x = w^2w^3 = w^5 = w \longleftrightarrow R(x, y) = I \longleftrightarrow R_xR_yR_{xy}^{-1} = I \longleftrightarrow R_xR_y = R_{xy} \longleftrightarrow zR_xR_y = zR_{xy} \longleftrightarrow zx \cdot y = z \cdot xy \longleftrightarrow L$ is a group. □

Corollary 5.2. *If* $(I, L_z^2, JL_z^2J) \in AUT(L)$ *and* $(R_z^2, I, JR_z^2J) \in AUT(L)$ *for all* $z \in L$*, where* L *is a C-loop, then* L *is a group.*

Proof. This follows from Theorem 5.2 and Theorem 5.4. □

Remark 5.2. Central square C-loops of exponent 4 are A-loops. □

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