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Ideals in AG-band and AG*-groupoid

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Abstract

We have shown that an ideal I of an AG-band is prime iff ideal (S) is totally ordered; it is prime iff it is strongly irreducible. The set of ideals of S form a semilattice structure. We have proved that if a belongs to the centre of S, then S is zero-simple if and only if $(Sa)S = S$, for every a in $S\backslash\{0\}$. Ideal structure in an AG*-groupoid S has also been investigated. It has been shown that if I is a minimal right ideal of S then Ia is a minimal left ideal of S, for all a in S. It has been shown also that every ideal of an AG*-groupoid S is prime if and only if it is idempotent and ideal (S) is totally ordered.

1. Introduction

A groupoid S is called an *Abel-Grassmann's groupoid*, abbreviated as an *AG-groupoid*, if its elements satisfy the left invertive law [4, 5], that is:

$$(ab)c = (cb)a \tag{1}$$

for all $a, b, c \in S$.

Several examples and interesting properties of AG-groupoids can be found in [5], [6], [7] and [8]. It has been shown in [5] that if an AG-groupoid contains a left identity then it is unique. It has been proved also that an AG-groupoid with right identity is a commutative monoid, that is, a semigroup with identity element.

It is also known [4] that in an AG-groupoid S, the *medial law*, that is,

$$(ab)(cd) = (ac)(bd) \tag{2}$$

for all $a, b, c, d \in S$ holds.

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2. AG-band

An AG-groupoid whose all elements are idempotents is called an *AG-band*. It is easy to see that in an AG-band S for any $a, b, c \in S$, $(ab)a = a(ba)$ and $(ab)c = (ac)(bc)$, $(ab)b = ba$.

Theorem 1. *If an AG-band S contains a left identity e then S becomes a semilattice with identity e.*

Proof. Let $x \in S$. Then

$$xe = (xx)e = (ex)x = xx = x$$

implies that x is the right identity for S and so by [5], the AG-band$S$ becomes a commutative monoid, that is, a semilattice with identity e. □

Due to Theorem 1, an AG-band with left identity becomes a semigroup with identity. So we cannot include automatically the left identity in an AG-band.

In an AG-band every congruence relation is trivially separative.

Theorem 2. *If S is an AG-band and a is a fixed element in S then*

$$H(a) = \{x \in S : xa = x\}$$

is a commutative subsemigroup with identity a.

Proof. Since $a \in H(a)$ we conclude that $H(a)$ is non-empty.

Let $x, y, z \in H(a)$, then

$$xy = (xa)(ya) = (xy)(aa) = (xy)a$$

implies that $H(a)$ is a groupoid.

Now

$$xy = (xa)y = (ya)x = yx$$

shows that $H(a)$ is commutative and so it becomes associative. Also

$$ax = (aa)x = (xa)a = xa = x,$$

imply that $H(a)$ is a commutative subsemigroup of idempotents with identity a in S. □

Example 1. Let $S = \{1,2,3,4,5,6\}$ and a binary operation be defined in S as follows:

$\cdot$	1	2	3	4	5	6
1	1	2	2	5	6	4
2	2	2	2	5	6	4
3	2	2	3	5	6	4
4	6	6	6	4	2	5
5	4	4	4	6	5	2
6	5	5	5	2	4	6

Then, as in [11] , $(S, \cdot)$ is an AG-band and $H(1) = \{1, 2\}$ is a semilattice with identity 1.

The following definitions are given in [10]. If S is an AG-groupoid and $A, B \subseteq S$, then A and B are called *right connected sets* if $AS \subseteq B$ and $BS \subseteq A$. Similarly, if S is an AG-groupoid and A, $B \subseteq S$, then A and B are called *left connected* if $SA \subseteq B$ and $SB \subseteq A$. Also A and B are called *connected sets* if they are both left and right connected.

A subset I of an AG-groupoid S is said to be *right* (*left*) *ideal* if $IS \subseteq I$ ($SI \subseteq I$). As usual I is said to be an *ideal* if it is both right and left ideal.

Proposition 1. *If A and B are left connected sets of an AG-band S and A is an ideal, then $S(A \cup B) \subseteq A$.*

Lemma 1. *If A and B are ideals of an AG-band S, then AB and BA are right and left connected sets.*

Proof. Using identity (1), we get

$$(AB)S = (SB)A \subseteq BA.$$

Similarly

$$(BA)S \subseteq AB.$$

This shows that AB and BA are right connected. Using identity (1), we get

$$S(BA) = (SS)(BA) = ((BA)S)S = ((SA)B)S \subseteq AB.$$

Also

$$S(AB) \subseteq BA.$$

This implies that AB and BA are left connected. □

Proposition 2. *A proper subset I of an AG-band S is a right ideal if and only if it is left.*

Proof. Let I be a right ideal of an AG-band S. Then $IS \subseteq S$, that is, $ix \in I$ for all $i \in I$ and $x \in S$. Hence

$$(xi) = (xx)i = (ix)x \in (IS)S \subseteq IS \subseteq I$$

shows that $SI \subseteq I$, that is, I is a left ideal of S. The converse can be proved similarly. □

It can easily be seen from Proposition 2, that $SI \subseteq IS$.

An ideal P of an AG-groupoid S is *prime* (*semiprime*) if for any other ideals A, B of S, $AB \subseteq P$ ($A^2 \subseteq P$) implies either $A \subseteq P$ or $B \subseteq P$ ($A \subseteq P$). A groupoid S is called *fully semiprime* if every ideal of S is semiprime. If S is an AG-band then trivially S is completely semiprime.

Lemma 2. *For every ideal I of an AG-band S we have*

$$\{x \in S \mid ax = x \text{ } for \text{ } a \in I\} \subseteq I \text{ } and \text{ } \{x \in S \mid ax = x \text{ } for \text{ } a \in I\} \subseteq I.$$

An AG-groupoid S is called *totally ordered* if for all ideals A, B of S either $A \subseteq B$ or $B \subseteq A$.

Theorem 3. *Every ideal of an AG-band S is prime if and only if the set of all ideals of S is totally ordered.*

Proof. Assume that every ideal of an AG-band S is prime. Let P, Q be the ideals of S. Then $PQ \subseteq P$ and $PQ \subseteq Q$ imply that $PQ \subseteq P \cap Q$. Since $P \cap Q$ is prime, so $P \subseteq P \cap Q$ or $Q \subseteq P \cap Q$ imply that $P \subseteq Q$ or $Q \subseteq P$. Hence the set of all ideals of S is totally ordered.

Conversely, let I, J and P be ideals of an AG-band S such that $IJ \subseteq P$. Being ideals of S they are totally ordered and that $I \subseteq J$. Thus P is prime. □

Theorem 4. *If I and J are ideals of an AG-band S then $IJ = I \cap J$.*

Proof. Let I and J be ideals of an AG-band S. Obviously, $IJ \subseteq I \cap J$. Since $I \cap J \subseteq I$, $I \cap J \subseteq J$, therefore $(I \cap J)^2 \subseteq IJ$. □

By Theorem 4, $IJ = JI$. Therefore the following Lemma is an easy consequence.

Lemma 3. *The set of ideals of an AG-band S form a semilattice structure.*

An ideal I of an AG-groupoid S is said to be *strongly irreducible* if and only if for ideals H and K of S, $H \cap K \subseteq I$ implies that $H \subseteq I$ or $K \subseteq I$. This leads to the following important theorem with a rather straight forward proof.

Theorem 5. *In an AG-band every ideal is strongly irreducible if and only if it is a prime ideal.*

An AG-groupoid S is (*left, right*) *simple*, if S contains no proper (left, right) ideals. Left simple, right simple and simple AG-bands coincide. The AG-band from Example 1 is not simple because $\{2, 4, 5, 6\}$ is a proper ideal of S.

An AG-groupoid S with zero is called *zero-simple* if $\{0\}$ and S are its only ideals and $S^2 \neq \{0\}$.

Example 2. Let $S = \{1, 2, 3, 4\}$ and the operation be defined on S as follows:

$\cdot$	1	2	3	4
1	1	4	2	3
2	3	2	4	1
3	4	1	3	2
4	2	3	1	4

Then, as in [11], $(S, \cdot)$ is a simple AG-band. If we adjoin 0 in S then it become a zero-simple AG-band.

Theorem 6. *If $aS = Sa$ for all non-zero a in an AG-band S, then S is zero-simple if and only if $(Sa)S = S$.*

Proof. Clearly $S^2 \neq \{0\}$ and $S^3 = S$. Now for any a in $S \backslash \{0\}$ the subset $(Sa)S$ of S is an ideal of S. Therefore either $(Sa)S = S$ or $(Sa)S = \{0\}$. If $(Sa)S = \{0\}$, then the set $I = \{x \in S : (Sx)S = \{0\}\}$ contains an element other than zero, and I becomes an ideal of S. As S is zero-simple so by definition $I = S$, that is, $(Sx)S = \{0\}$ for every x in S. This implies that $S^3 = \{0\}$. But this is a contradiction to the fact that $S = S^3$. Hence $(Sa)S = S$.

Conversely, assume that, $(Sa)S = S$ for every a in $S \backslash \{0\}$. Also if A is an ideal of S containing a, then $(SA)S \subseteq A$ implies $(Sa)S \subseteq A$. □

Corollary 1. *S is simple if and only if $(Sa)S = S$.*

Proof. If S is a simple AG-band, then $(Sa)S$ is an ideal of S and so $(Sa)S = S$. Conversely, if $(Sa)S = S$ for all $a \in S$, then we need to show that S is simple. Let A be an ideal of S and $a \in A$. Then $(SA)S \subseteq A$ implies that $(Sa)S \subseteq A$. Now, if $0 \in S$, then $(S0)S = \{0\} \neq S$. As $(Sa)S = S$ holds for all $a \in S$, it means that $0 \notin S$. Hence S without zero has no ideal except S itself. □

An ideal M in an AG-groupoid S with zero is called *zero-minimal* if it is minimal in the set of all non-zero ideals.

Proposition 3. *If M is a zero-minimal ideal of an AG-band S such that $aS = Sa$ for all non-zero $a \in S$, then M is a zero-simple AG-band.*

Proof. Clearly $M = M^3$ and if $a \in M\backslash\{0\}$, then $(Sa)S$ is an ideal of S contained in M. It is non-zero, since it contains a, and so $(Sa)S = M$. Thus using (2) and (1) we get

$$(Ma)M \subseteq (Sa)S = M = M^3 = (M((Sa)S))M \subseteq (Ma)M,$$

which implies $(Ma)M = M$. By Theorem 6, M is zero-simple. □

Proposition 4. *Let S be an AG-band without zero. If K is a minimal ideal of S, then K is a simple AG-band.*

Proof. Note that $0 \notin S$ implies $0 \notin K$. As K is uniquely minimum so it cannot contain any other ideal of S. Hence K is a simple AG-band. □

3. Ideals in an AG*-groupoid

An AG-groupoid S is called an *AG*-groupoid* if it satisfies one of the following equivalent weak associative laws [10]:

$$(ab)c = b(ac), \tag{3}$$

$$(ab)c = b(ca). \tag{4}$$

From (3) and (4), we obtain

$$b(ac) = b(ca) \tag{5}$$

for all $a, b, c \in S$.

If all elements of an AG*-groupoid S are idempotent, then $S = S^2$. This further implies that S is a commutative semigroup [10].

If S is an AG*-groupoid and $a = a^2$ (for a fixed element $a \in S$) then, as it is proved in [10], $aS = Sa$ and $(xa)y = x(ay)$ for any $x, y \in S$. If a belongs to $Sa = aS$, then $Sa = aS$ is a semilattice.

A non-associative left simple (right simple, simple) AG*-groupoid does not exist [9]. SA is a left ideal of an AG*-groupoid S for all subsets A of S.

Lemma 4. *If I is a right ideal of an AG*-groupoid S and J is a subset of S then IJ is a left ideal of S and it is a right ideal if $IJ = JI$, and $a(IJ)$ $\{(JI)a\}$ becomes a left (right) ideal of S.*

Proof. The proof is straight forward. □

By $\mathcal{K}$ we shall mean the set of all ideals of an AG*-groupoid S.

Proposition 5. *In any AG*-groupoid:*

(*i*) *$\mathcal{K}$ has associative powers,*

(*ii*) *$I^m I^n = I^{m+n}$, for all $I \in \mathcal{K}$,*

(*iii*) *$(I^m)^n = I^{mn}$, for all $I \in \mathcal{K}$ and all positive integers m, n,*

(*iv*) *$(AB)^n = A^n B^n$ for $n \geqslant 1$ and $(AB)^n = B^n A^n$ for $n \geqslant 2$, $\forall A, B \in \mathcal{K}$.*

Proof. The proof is obvious. □

Lemma 5. *If I is an ideal of an AG*-groupoid S then so is I^n for $n \geqslant 2$.*

Proof. Let I be a right ideal of an AG*-groupoid S and $x = ij \in I^2$ where $i, j \in I$. Using identity (3), we get

$$s(ij) = (is)j \subseteq II = I^2,$$
$$(ij)s = j(is) \subseteq II = I^2,$$

which shows that I^2 is an ideal of S. Now suppose that I^{n-1} is an ideal. Then using (1), (3), and Proposition 5(*ii*), we get

$$I^n S = (I^{n-1}I)S = (SI)I^{n-1} \subseteq II^{n-1} = I^n,$$
$$SI^n = S(I^{n-1}I) = (IS)I^{n-1} \subseteq I^n,$$

which completes the proof. □

Lemma 6. *If I is an ideal of an AG^*-groupoid S and $a = a^2$, then aI^2 is an ideal of S.*

Proof. Using Proposition 5(iv) and identity (3), we get $I^2a = aI^2$. Then it is not difficult to see that aI^2 is an ideal. □

An ideal I of an AG-groupoid S is called *minimal* if and only if it does not contain any ideal of S other than itself.

Theorem 7. *If I is a minimal right ideal of an AG^*-groupoid S then for all $a \in S$ Ia is a minimal left ideal of S.*

Proof. Let I be the minimal right ideal of an AG*-groupoid S and $x = ia \in Ia$, where $i \in I$. Then using identity (3) we get $sx = s(ia) = (is)a \in Ia$ which shows that Ia is a left ideal of S. Let H be a non-empty left ideal of S properly contained in Ia. Define $H' = \{r \in I : ra \in H\}$. If $y \in H'$, then $ya \in H$, and so $(ys)a = s(ya) \in SH \subseteq H$, imply that H' is a right ideal of S properly contained in I. This is a contradiction to the minimality of I. Hence Ia is a minimal left ideal of S. □

Theorem 8. *If I is a minimal left ideal of an AG^*-groupoid S then aI $(a^2 = a)$ is a minimal right ideal of S.*

Proof. Let $ai \in aI$ where I is a minimal left ideal of an AG*-groupoid S. Then using identities (3) and (2) we get

$$ia = i(aa) = (ai)a = (ai)(aa) = (aa)(ia) = a(ia) = (aa)i = ai.$$

Also $(ai)s = (ia)s = a(is) \in aI$, shows that aI is a right ideal of S. Let H be a non-empty right ideal of S properly contained in aI. Define $H' = \{r : ar \in I\}$. Then $a(sy) = (sy)a = (ay)s \in HS \subseteq H$ imply that H' is a left ideal of S properly contained in I. But this is a contradiction to the minimality of I. Hence aI is a minimal left ideal of S. □

Theorem 9. *Every ideal of an AG^*-groupoid S is prime if and only if it is idempotent and the set of all ideals of S is totally ordered.*

Proof. Let every ideal of S be prime. Assume that I is any ideal of S. Then I^2 is an ideal of S by Lemma 5. Also $I^2 \subseteq I$ implies that $I \subseteq I^2$ or $I = I^2$. If P and Q are any ideals of S then, $PS \subseteq P$ and $SQ \subseteq Q$ implies that $PQ \subseteq P$ and $PQ \subseteq Q$, and so $PQ \subseteq P \cap Q$. Since intersection of two prime ideals is prime. So, $P \subseteq P \cap Q$ or $Q \subseteq P \cap Q$. This implies that $P \subseteq Q$ or $Q \subseteq P$. Hence the set of all ideals of S is totally ordered.

Conversely, assume that every ideal of S is idempotent and the set of all ideals of S is totally ordered. Let I, J and P be any ideals of S such that $IJ \subseteq P$ with $I \subseteq J$. Then $I = I^2 = II \subseteq IJ \subseteq P$, implies that every ideal of S is prime. $\square$

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