Quasigroups and Related Systems **14** (2006), 195 − 206

A graphical technique to obtain homomorphic images of $\Delta(2,3,11)$

Qaiser Mushtaq and Munir Ahmed

Abstract

In this paper we have developed a technique by which a suitably created fragment of a coset diagram for the action of $PSL(2,Z)$ or $PGL(2,Z)$ on projective lines over Galois fields F_p, $p \equiv \pm 1 (\mathrm{mod}\ 11)$, can be used to obtain a family of permutation groups $\Delta(2,3,11) = < x, y : x^2 = y^3 = (xy)^{11} = 1 >$.

1. Introduction

It is well known that the modular group $PSL(2,Z)$ is generated by the linear fractional transformations $x : z \to -1/z$ and $y : z \to (z-1)/z$, satisfying the relations $x^2 = y^3 = 1$. The extended modular group $PGL(2,Z)$ is generated by the linear fractional transformations $x : z \to -1/z$, $y : z \to (z-1)/z$, and $t : z \to 1/z$, such that $x^2 = y^3 = t^2 = (xt)^2 = (yt)^2 = 1$.

Let $q = p^m$ where $m > 0$ and p is a prime number. A number $\omega \in F_q$ is said to be a non-zero square in F_q if $\omega \equiv a^2 \pmod p$ for some non-zero element a in F_q. The projective lines over a finite field F_q, $F_q \cup \{\infty\}$, is denoted by $PL(F_q)$.

The group $PGL(2,q)$ is a group consisting of all the transformations $z \to (az+b)/(cz+d)$, where a, b, c, $d \in F_q$ and $ad - bc \neq 0$. The group $PSL(2,q)$ is a group containing transformations $z \to (az+b)/(cz+d)$ where a, b, c, $d \in F_q$ and $ad - bc$ is a non-zero square in F_q.

Let $\Delta(l,m,n)$ denote the triangle group $< x, y : x^l = y^m = (xy)^k = 1 >$. The triangle group $\Delta(l,m,k)$ is infinite for $k \geqslant 6$. For $k \leqslant 5$, $\Delta(2,3,k)$ is trivial, S_3, A_4, S_4, and A_5 respectively. The group $\Delta(2,3,6)$ is an extension by the cyclic group C_6 of a free abelian group of rank 2. For $k = 7$, the triangle group $\Delta(2,3,k)$ is a Hurwitz group [1]. The group $\Delta(2,3,k)$,

2000 Mathematics Subject Classification: 05C38, 15A15, 05A15
Keywords: Diagram, group, homomorphic image.

when $k = 8, 9$ and 10 are known to be less interesting. There is relatively less information available about $\Delta(2,3,11)$. We therefore consider $\Delta(2,3,11)$ and use coset diagrams for the actions of $PGL(2,Z)$ on $PL(F_p)$, $p \equiv \pm 1 (mod\ 11)$ and see for what values of p these actions evolve triangle groups $\Delta(2,3,11)$ as subgroups of S_{p+1}.

A coset diagram for $PGL(2,Z)$ consists of a set of small triangles and a set of edges. The three cycles of y are denoted by small triangles whose vertices are permuted counter-clockwise by y and any two vertices which are interchanged by x are joined by an edge. The action of t is represented by reflection about a vertical line of axis in the case of $PGL(2,Z)$. The fixed points of x and y are denoted by heavy dots.

Let $PSL(2,Z)$ act on a space Ω. If an element $(xy)^{n_1}\left(xy^{-1}\right)^{n_2}\cdots\left(xy^{-1}\right)^{n_l}$ of $PSL(2,Z)$ fixes an element of Ω, then the patch of the coset diagram is called a *circuit*. We denote it by $(n_1, n_2, ..., n_l)$. For a given sequence of positive integers $(n_1, n_2, n_3, ..., n_{2k})$ the circuit of the type

$$(n_1, n_2, n_3, \ldots, n_{2k'}, n_1, n_2, n_3, \ldots, n_{2k'}, \ldots, n_1, n_2, n_3, \ldots, n_{2k'}),$$

where k' divides k, is said to be a *periodic circuit of length* $2k'$. A trivial circuit consists of a path followed by its own inverse. A portion of a coset diagram is called a *fragment* of a coset diagram. First of all we construct a fragment composed of two connected, non-trivial circuits such that neither of them is periodic and more than two vertices in the fragment are fixed by $(xy)^{11}$. Corresponding to two circuits we have two words (elements of $PSL(2,Z)$), yielding a polynomial $f(z)$ in $Z[z]$ as in [4]. A homomorphism $\alpha : PGL(2,Z) \to PGL(2,q)$ is called *non-degenerate* if x, y, t do not belong to $Ker(\alpha)$. Of course α gives rise to an action of $PGL(2,Z)$ on $PL(F_q)$. Two non-degenerate homomorphisms α and β are called *conjugate* if there exists an inner automorphism ρ on $PGL(2,q)$ such that $\alpha = \rho\beta$. In [5], these actions, or conjugacy classes, have been parameterized with the elements $\theta \in F_q$. Corresponding to each root $\theta (\neq 0, 3)$ of $f(\theta) = 0$ in F_p where $p \equiv \pm 1 (mod\ k)$, we obtain a conjugacy class of actions of $PGL(2,Z)$ on $PL(F_p)$ each action evolving $\Delta(2,3,k)$. By $D(\theta, q)$ we mean a coset diagram of the conjugacy class corresponding to parameter $\theta \in F_q$.

We need the following results proved in [4] and [5].

Theorem 1. [4] *Given a fragment γ, where γ is a non-simple fragment consisting of two connected, non-trivial circuits such that neither of them is periodic, there exists a polynomial $F(z)$ in $Z[z]$ such that*

(*i*) *if the fragment γ occurs in $D(\theta, q)$, then $F(\theta) = 0$,* *(ii) if* $F(\theta) = 0$ *then the fragment, or a homomorphic image of it occurs in* $D(\theta, q)$ *or in* $D(\theta, \overline{q})$, *where* $D(\theta, \overline{q})$ *denotes the diagram with the vertices from the complement* $PL(F_{q^2}) \setminus PL(F_q)$.

Theorem 2. [5] *The conjugacy classes of a non-degenerate homomorphisms of* $PGL(2, Z)$ *into* $PGL(2, q)$ *are in one-to-one correspondence with the elements* $\theta \neq 0$, 3 *of* F_q. *under the correspondence which maps each class to its parameter.*

2. Appropriate fragments

By an *appropriate fragment* we shall mean a fragment composed of two non-trivial, connected circuits C_1 and C_2 such that neither of them is periodic and at least three vertices of this fragment are fixed by $(xy)^{11}$.

By Theorems 1 and 2, we can find conjugacy classes of non-degenerate homomorphisms corresponding to the elements $\theta(\neq 0, 3)$ in some finite field F_p, $p \equiv \pm 1(mod\ 11)$ obtained from the condition in the form of a polynomial. Each conjugacy class corresponds to a diagram. These coset diagrams will be such that every vertex in these diagrams will be a fixed point of $(\overline{x}\,\overline{y})^{11}$, and so by a well known fact that no non-trivial linear fractional transformation in $PGL(2, q)$ can fix more than two vertices in F_q, it will depict the triangle group $\alpha(\Delta(2, 3, 11))$.

Theorem 3. *Let* γ *be an appropriate fragment of a coset diagram for* $PGL(2, Z)$ *with at least one of the three vertices as the common vertex of* C_1 *and* C_2. *Then there exists a coset diagram* $D(\theta, p)$ *containing* γ, *or its homomorphic image, representing* $\alpha(\Delta(2, 3, 11))$.

Proof. Consider γ which is composed of two non-periodic circuits C_1 and C_2. Let w_1 and w_2 be two elements of $PSL(2, Z)$ induced by the circuits C_1 and C_2 respectively. That is $w_1 = xyxyxyyxyxy$ and $w_2 = xyyxyyxyxyy$. We can represent w_1 and w_2 as matrices $W_1 = XYXYXYYXYXY$ and $W_2 = XYYXYYXYXYY$ which are elements of $SL(2, Z)$, where X and Y are the matrices representing the elements x and y (of orders 2 and 3 respectively) of $PSL(2, Z)$. According to Mushtaq [4], we can express W_1 and W_2 as linear combinations of I, X, Y and XY, that is,

$$W_1 = \lambda_0 I + \lambda_1 X + \lambda_2 Y + \lambda_3 XY$$

and

$$W_2 = \mu_0 I + \mu_1 X + \mu_2 Y + \mu_3 XY.$$

We take $X = \begin{bmatrix} a & kc \\ c & -a \end{bmatrix}$, $Y = \begin{bmatrix} d & kf \\ f & -d-1 \end{bmatrix}$, with $trace(X) = 0$ and $\det(X) = \Delta$. Then the characteristic equation of X is $X^2 + \Delta I = O$, and since $trace(Y) = -1$ and $\det(Y) = 1$, the characteristic equation of Y is $Y^2+Y+I = O$. Thus the characteristic equation of XY is $(XY)^2-r(XY)+\Delta I = O$, where $r = trace(XY)$ and $\Delta = \det(XY)$. Also, $\Delta = -(a^2 + kc^2)$ and $d^2 + d + kf^2 + 1 = 0$. Using these equations, we obtain

$$\begin{aligned}(XY)^n &= \left\{\binom{n-1}{0}r^{n-1} - \binom{n-2}{1}r^{n-3}\triangle + ...\right\}XY \\ &\quad -\triangle\left\{\binom{n-2}{0}r^{n-2} - \binom{n-3}{1}r^{n-4}\triangle + ...\right\}I.\end{aligned}$$

After a suitable manipulation of the above equations, we get $XYX = rX + \triangle I + \triangle Y$, $YXY = rY + X$ and $YX = rI - X - XY$. Of course

$(XY)^3 = (r^2 - \Delta)XY - r\Delta I$,
$(XY)^4 = (r^3 - 2r\Delta)XY - (r^2\Delta - \Delta^2)I$,
$(XY^2)^2 = rXY + rX - \Delta I$,
$(XY^2)^3 = (r^2 - \Delta)XY + (r^2 - \Delta)X - r\Delta I$, and
$(XY^2)^4 = (r^3 - 2r\Delta)XY + (r^3 - 2r\Delta)X - (r^2\Delta - \Delta^2)I$.

Now,

$$\begin{aligned}W_1 &= XYXYXY^2XYXY \\ &= (XY)^3Y(XY)^2 \\ &= [(r^2-\Delta)XY - r\Delta I]Y(rXY - \Delta I) \\ &= [(r^2-\Delta)XY^2 - r\Delta Y][rXY - \Delta I] \\ &= [(r^2-\Delta)(-XY - X) - r\Delta Y][rXY - \Delta I] \\ &= [(-r^2+\Delta)XY + (-r^2+\Delta)X - r\Delta Y][rXY - \Delta I] \\ &= [(-r^3+r\Delta)(XY)^2 + (-r^3+r\Delta)X^2Y - r^2\Delta YXY + (r^2\Delta - \Delta^2)XY \\ &\quad +(r^2\Delta - \Delta^2)X + r\Delta^2 Y] \\ &= [(-r^3+r\Delta)(rXY - \Delta I) + (-r^3+r\Delta)(-\Delta I)Y - r^2\Delta(rY + X)+ \\ &\quad (r^2\Delta - \Delta^2)XY + (r^2\Delta - \Delta^2)X + r\Delta^2 Y] \\ &= [(-r^4+r^2\Delta)XY + (r^3\Delta - r\Delta^2)I + (r^3\Delta - r\Delta^2)Y - r^3\Delta Y - r^2\Delta X \\ &\quad +(r^2\Delta - \Delta^2)XY + (r^2\Delta - \Delta^2)X + r\Delta^2 Y] \\ &= [(r^3\Delta - r\Delta^2)I - \Delta^2 X + 0Y + (-r^4 + 2r^2\Delta - \Delta^2)XY\end{aligned}$$

and

$$\begin{aligned}W_2 &= XYYXYYXYXYY \\ &= \left(XY^2\right)^2\left(XY\right)^2 Y \\ &= [rXY + rX - \Delta I][rXY - \Delta I]Y\end{aligned}$$

$= [rXY + rX - \Delta I][rXY^2 - \Delta Y]$
$= [rXY + rX - \Delta I][r(-XY - X) - \Delta Y]$
$= [rXY + rX - \Delta I][-rXY - rX - \Delta Y]$
$= [-r^2 (XY)^2 - r^2XYX - r\Delta XY^2 - r^2XY - r^2X^2 - r\Delta XY + r\Delta XY +$
$r\Delta X + \Delta^2 Y]$
$= [-r^2 (rXY - \Delta I) - r^2(rX + \Delta Y + \Delta I) + r\Delta(XY + X) + r^2\Delta Y +$
$r^2\Delta I - r\Delta XY + r\Delta XY + r\Delta X + \Delta^2 Y]$
$= r^2\Delta I + (-r^3 + 2r\Delta)X + \Delta^2 Y + (-r^3 + r\Delta)XY.$

Using equations

$$W_1 = \lambda_0 I + \lambda_1 X + \lambda_2 Y + \lambda_3 XY,$$
$$W_2 = \mu_0 I + \mu_1 X + \mu_2 Y + \mu_3 XY,$$

we obtain $\lambda_1 = -\Delta^2$, $\lambda_2 = 0$ and $\lambda_3 = (-r^4 + 2r^2\Delta - \Delta^2)$, $\mu_1 = (-r^3 + 2r\Delta)$, $\mu_2 = \Delta^2$ and $\mu_3 = (-r^3 + r\Delta)$.

Now substituting these values in the equation

$$(\lambda_2\mu_3 - \mu_2\lambda_3)^2 + \triangle (\lambda_3\mu_1 - \mu_3\lambda_1)^2 + (\lambda_1\mu_2 - \mu_1\lambda_2)^2$$
$$+ r (\lambda_2\mu_3 - \mu_2\lambda_3) (\lambda_3\mu_1 - \mu_3\lambda_1) + (\lambda_2\mu_3 - \mu_2\lambda_3) (\lambda_1\mu_2 - \mu_1\lambda_2) = 0$$

we get:

$$[0 - \Delta^2(-r^4 + 2r^2\Delta - \Delta^2)]^2 + \Delta[(-r^3 + 2r\Delta)(-r^4 + 2r^2\Delta - \Delta^2)$$
$$+\Delta^2(-r^3 + r\Delta)]^2 + [-\Delta^2.\Delta^2 - 0]^2 + r[0 - \Delta^2(-r^4 + 2r^2\Delta - \Delta^2)]$$
$$[(-r^3 + 2r\Delta)(-r^4 + 2r^2\Delta - \Delta^2) + \Delta^2(-r^3 + r\Delta)]+$$
$$[0 - \Delta^2(-r^4 + 2r^2\Delta - \Delta^2)][-\Delta^4 - 0] = 0,$$

$$[\Delta^4(r^4 - 2r^2\Delta + \Delta^2)^2 + \Delta(r^7 - 2r^5\Delta + \Delta^2 r^3 - 2r^5\Delta + 4r^3\Delta^2 - 2r\Delta^3$$
$$-r^3\Delta^2 + r\Delta^3) + \Delta^8 + r\Delta^2(r^4 - 2r^2\Delta + \Delta^2)(r^7 - 2r^5\Delta + r^3\Delta^2 - 2r^5\Delta$$
$$+4r^3\Delta^2 - 2r\Delta^3 + r^3\Delta^2 + r\Delta^3) + \Delta^6[-r^4 + 2r^2\Delta - \Delta^2]] = 0,$$

$$\Delta^4[\Delta^2\theta^2 - 2\Delta^2\theta + \Delta^2]^2 + \Delta[r^7 - 4r^5\Delta + 4r^3\Delta^2 - r\Delta^3]^2 + \Delta^8$$
$$+r\Delta^2[\Delta^2\theta^2 - 2\Delta^2\theta + \Delta^2][r^7 - 4r^5\Delta + 4r^3\Delta^2 - r\Delta^3]$$
$$+\Delta^6[-\Delta^2\theta^2 + 2\Delta^2\theta - \Delta^2] = 0.$$

That is,

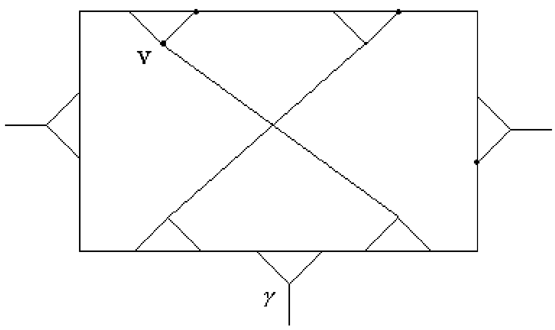
$$\Delta^8[(\theta^2 - 2\theta + 1)^2 + \theta(\theta^3 - 4\theta^2 + 4\theta - 1)^2 + 1+$$
$$\theta(\theta^2 - 2\theta + 1)(\theta^3 - 4\theta^2 + 4\theta - 1) + (-\theta^2 + 2\theta - 1)] = 0.$$

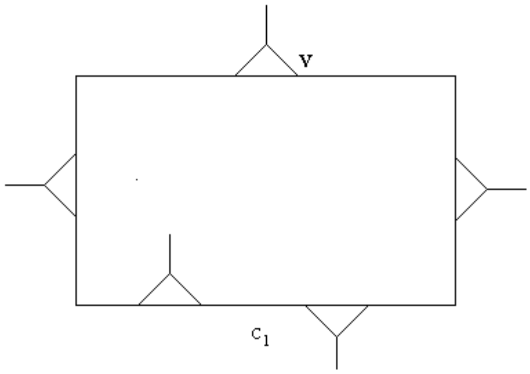
By Theorem 1 we obtain a polynomial $f(\theta) = \theta^7 - 7\theta^6 + 18\theta^5 - 20\theta^4 + 7\theta^3 + 3\theta^2 - 2\theta + 1$. If we let $f(\theta) = 0$, then $f(\theta_i) = 0$ where $\theta_i \in F_p$ and

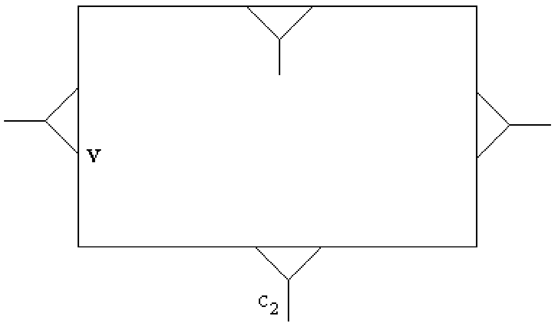
$p \equiv \pm 1 (\text{mod } 11)$, then according to our Theorem 2, $D(\theta_i, p)$ is such that it corresponds to a conjugacy class of non-degenerate homomorphisms α from $PGL(2, Z)$ into $PGL(2, p)$. This depicts an action of $PGL(2, Z)$ on $PL(F_p)$ and the diagram depicting the action is such that every vertex in the diagram is fixed by the element $(\overline{x}\,\overline{y})^{11}$. Since no non-trivial linear fractional transformation can fix more than two vertices in $PL(F_p)$, thus $(\overline{x}\,\overline{y})^{11} = 1$ and so the coset diagram represents the homomorphic image of the triangle group $\Delta(2, 3, 11)$, that is, $\alpha(\Delta(2, 3, 11))$. □

Theorem 4. *There exists only two coset diagrams* $D(19, 67)$ *and* $D(125, 199)$ *for the action of* $PGL(2, Z)$ *on* $PL(F_p)$ *depicting* $\alpha(\Delta(2, 3, 11))$, *where* $2 \leqslant p \leqslant 1033$, *and p is a prime congruent to* ± 1 *(mod* 11).

Proof. In order to obtain the required coset diagram first of all we take the following fragment γ which is composed of two non-trivial, and non-periodic circuits C_1 and C_2 with the vertex $\mathbf{v}$ of γ as the common vertex of C_1 and C_2 as shown in the fragment. Note that the fragment is required to contain at least three vertices, namely $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}$ which are fixed by $(\overline{x}\,\overline{y})^{11}$. Let $w_1 = xyxyxyyxyxy$ and $w_2 = xyyxyyxyxyy$ be the elements induced by the circuits C_1 and C_2 respectively. Notice that w_1 and w_2 are the elements of $PSL(2, Z)$ and represent the matrices $W_1 = XYXYXYYXYXY$ and $W_2 = XYYXYYXYXYY$ belonging to $GL(2, Z)$, where X and Y are the matrices representing x and y of $PGL(2, Z)$, so $(185, 185)\,(0, \infty)(3, 198)(88, 156)$.







As in [4] ,

$$W_1 = XYXYXY^2XYXY$$
$$= [(r^3\Delta - r\Delta^2)I - \Delta^2 X + 0Y + (-r^4 + 2r^2\Delta - \Delta^2)XY$$

and

$$W_2 = XYYXYYXYXYY$$
$$= r^2\Delta I + (-r^3 + 2r\Delta)X + \Delta^2 Y + (-r^3 + r\Delta)XY$$

and by using Theorem 1, we can obtain a polynomial $f(\theta) = \theta^7 - 7\theta^6 + 18\theta^5 - 20\theta^4 + 7\theta^3 + 3\theta^2 - 2\theta + 1$. If we convert this polynomial into an equation $f(\theta) = 0$, and solve it in the field F_{67}, we obtain 19, 60 and 61 as its roots. By using theorem 2 for $\theta = 19$, we obtain the matrices $X = \begin{bmatrix} 9 & 38 \\ 19 & -9 \end{bmatrix}$, $Y = \begin{bmatrix} 0 & 20 \\ 10 & -1 \end{bmatrix}$ and $T = \begin{bmatrix} 0 & -2 \\ 1 & 0 \end{bmatrix}$. Therefore the corresponding transformations are, $\overline{x} : z \mapsto \frac{9z+38}{19z-9}$, $\overline{y} : z \mapsto \frac{20}{10z-1}$ and

$\bar{t} : z \mapsto \frac{-2}{z}$. So,

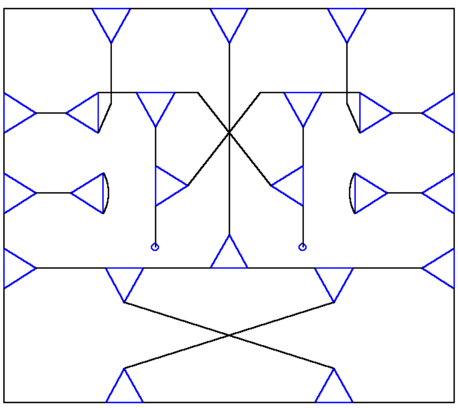
$$\begin{aligned}\overline{x} \quad : \quad & (33,0)(1,65)(62,2)(3,53)(4,\infty)(22,5)(6,13)(7,10)(8,42)(21,9)\\ & (11,64)(12,23)(14,46)(15,30)(39,16)(26,17)(18,34)(19,32)(20,47)\\ & (24,25)(31,27)(28,55)(61,29)(35,37)(36,59)(38,40)(41,57)(43,56)\\ & (44,48)(45,60)(49,58)(50,51)(52,63)(54,66),\end{aligned}$$

$$\begin{aligned}\overline{y} \quad : \quad & (0,47,\infty)(32,49,1)(65,15,46)(48,2,61)(45,66,53)(3,3,3)(44,44,44)\\ & (28,14,4)(33,19,43)(34,5,51)(42,13,63)(31,25,6)(22,16,41)(10,9,7)\\ & (38,37,40)(24,64,8)(50,23,39)(26,35,11)(12,21,36)(55,17,58)\\ & (30,59,56)(18,60,62)(54,29,52)(27,20,57),\end{aligned}$$

and

$$\begin{aligned}\overline{t} \quad : \quad & (0,\infty)\,(1,65)\,(2,66)\,(3,44)\,(4,33)\,(5,13)\,(6,22)\,(7,38)\,(8,50)\,(9,37)\\ & (10,40)\,(11,12)\,(14,19)\,(15,49)\,(16,25)\,(17,59)\,(18,52)\,(20,20)\,(21,35)\\ & (23,64)\,(24,39)\,(26,36)\,(27,57)\,(28,43)\,(29,60)\,(30,58)\,(31,41)\,(32,46)\\ & (33,4)\,(34,63)\,(55,56)\,(42,51)\,(54,62)\,(48,53)\,(61,45)\,(47,47)\,.\end{aligned}$$

Thus we have a coset diagram $D(19,67)$ in which each vertex of the diagram is fixed by $(\overline{x}\,\overline{y})^{11}$, and we have $(\overline{x}\,\overline{y})^{11} = 1$. Thus the diagram $D(19,67)$ is a representation of the triangle group $\alpha(\Delta\,(2,3,11))$.



Now solving the equation $f(\theta) = \theta^7 - 7\theta^6 + 18\theta^5 - 20\theta^4 + 7\theta^3 + 3\theta^2 - 2\theta + 1 = 0$ in F_{199}, we obtain 125, 159, and 193 as its roots.

For instance, if we consider $\theta = 125$, we obtain $X = \begin{bmatrix} 121 & 124 \\ 174 & -121 \end{bmatrix}$, $Y = \begin{bmatrix} 0 & 14 \\ 71 & -1 \end{bmatrix}$ and $T = \begin{bmatrix} 0 & -3 \\ 1 & 0 \end{bmatrix}$ as before. The corresponding transformations are: $\overline{x} : z \mapsto \frac{121z+124}{174z-121}$, $\overline{y} : z \mapsto \frac{14}{71z-1}$, and $\overline{t} : z \mapsto \frac{-3}{z}$. Thus,

$$\begin{aligned}
\overline{x} \; : \; & (22,0)(106,1)(2,141)(96,3)(99,4)(5,48)(6,49)(7,70)(8,177) \\
& (9,119)(10,101)(31,11)(18,12)(13,60)(14,185)(165,15)(16,69) \\
& (113,17)(35,19)(93,20)(104,21)(43,23)(181,24)(59,25)(91,26) \\
& (27,\infty)(28,162)(194,29)(72,30)(32,54)(33,149)(160,34)(42,36) \\
& (140,37)(184,38)(39,88)(40,68)(41,193)(44,152)(45,134)(46,76) \\
& (183,47)(154,50)(51,157)(52,112)(53,147)(110,55)(114,56) \\
& (131,57)(58,179)(61,189)(62,173)(63,180)(64,192)(65,151) \\
& (66,107)(67,105)(71,116)(73,190)(195,74)(92,75)(77,169) \\
& (135,78)(79,87)(80,191)(81,129)(82,138)(83,168)(84,176) \\
& (85,170)(90,86)(89,103)(94,130)(95,108)(97,100)(98,127) \\
& (188,102)(109,133)(111,121)(115,171)(117,128)(118,175) \\
& (120,144)(122,196)(123,159)(124,172)(125,136)(126,155) \\
& (132,142)(137,182)(139,197)(143,198)(145,158)(146,187) \\
& (148,186)(150,164)(153,156)(161,178)(163,167)(166,174),
\end{aligned}$$

$$\begin{aligned}
\overline{y} \; : \; & (0,185,\infty)(1,40,188)(145,184,196)(137,87,2)(98,48,183) \\
& (3,47,186)(138,182,198)(166,136,4)(49,19,181)(106,5,63) \\
& (180,79,122)(86,6,30)(179,99,155)(7,57,157)(128,178,28) \\
& (172,61,8)(124,13,177)(9,78,119)(107,176,66)(46,10,25) \\
& (175,139,160)(131,11,24)(174,54,161)(69,36,12)(149,116,173) \\
& (171,14,192)(55,26,15)(159,130,170)(123,16,20)(169,62,165) \\
& (167,33,17)(152,18,168)(74,43,21)(142,111,164)(22,83,158) \\
& (102,163,27)(70,64,23)(121,115,162)(88,41,29)(144,97,156) \\
& (146,153,31)(32,39,154)(112,109,34)(76,73,151)(60,35,195)
\end{aligned}$$

$$(150,125,189)(82,56,37)(129,103,148)(134,117,38)(68,51,147)$$
$$(96,114,42)(71,89,143)(127,72,44)(113,58,141)(45,81,132)$$
$$(104,140,53)(50,84,197)(101,135,187)(120,52,190)(133,65,194)$$
$$(90,67,59)(118,95,126)(75,85,193)(100,110,191)(105,92,77)$$
$$(93,80,108)(91,91,91)(94,94,94),$$

and

$$\bar{t}\ :\ (2,98)\,(4,49)\,(5,79)\,(6,99)\,(7,28)\,(8,124)\,(9,66)\,(10,139)\,(11,54)$$
$$(12,149)\,(13,61)\,(14,14)\,(15,159)\,(16,62)\,(17,152)\,(18,33)\,(19,136)$$
$$(20,169)\,(21,142)\,(22,27)\,(23,121)\,(24,174)\,(25,175)\,(26,130)$$
$$(29,144)\,(30,179)\,(31,32)\,(34,76)\,(35,125)\,(36,116)\,(37,129)$$
$$(38,68)\,(39,153)\,(40,184)\,(41,97)\,(42,71)\,(43,111)\,(44,113)$$
$$(45,53)\,(46,160)\,(47,182)\,(48,87)\,(50,187)\,(51,117)\,(52,65)$$
$$(55,170)\,(56,103)\,(57,178)\,(58,72)\,(59,118)\,(60,189)\,(63,180)$$
$$(64,115)\,(67,95)\,(69,173)\,(70,162)\,(73,109)\,(74,164)\,(75,191)$$
$$(77,93)\,(78,176)\,(80,92)\,(81,140)\,(82,148)\,(83,163)\,(84,135)$$
$$(85,110)\,(86,155)\,(89,114)\,(90,126)\,(91,94)\,(96,143)\,(100,193)$$
$$(101,197)\,(102,158)\,(104,132)\,(105,108)\,(106,122)\,(107,119)$$
$$(112,151)\,(120,194)\,(123,165)\,(127,141)\,(128,157)\,(131,161)$$
$$(133,190)\,(134,147)\,(137,183)\,(138,186)\,(145,188)\,(146,154)$$
$$(150,195)\,(166,181)\,(1,196)\,(167,168)\,(171,192)\,(172,177)$$
$$(185,185)\,(0,\infty)(3,198)(88,156).$$

Thus we have the coset diagram $D(125,199)$ (see the next page) in which each vertex is fixed by $(\overline{x}\,\overline{y})^{11}$. We have therefore $(\overline{x}\,\overline{y})^{11}=1$.

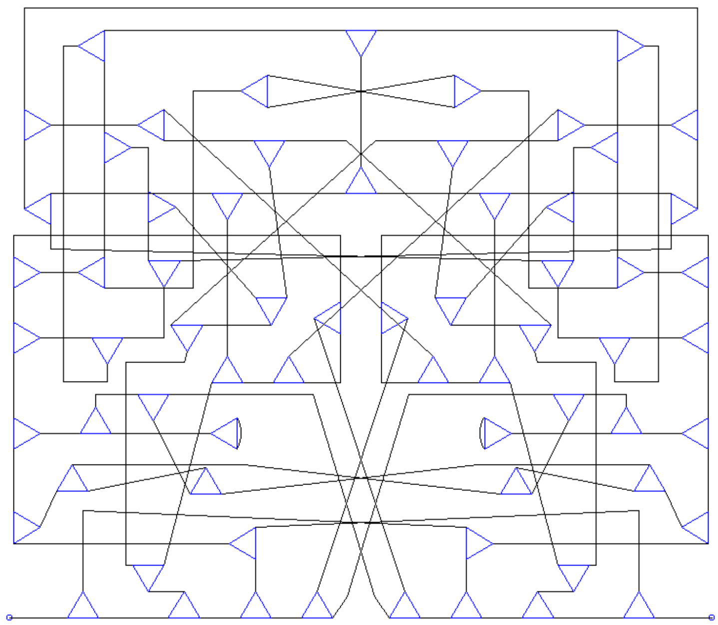
Thus the diagram $D(125,199)$ is a representation of the triangle group $\alpha(\Delta\,(2,3,11))$. □

Corollary 5. *For prime p, $2\leqslant p\leqslant 1033$ such that $p\equiv\pm1(mod\ 11)$,*
 (i) the action of $PGL(2,Z)$ on $PL\,(F_p)$ is transitive,
 (ii) the diagram of $\alpha(\Delta\,(2,3,11))$ is connected.

Proof. (i) Consider the action of $PGL(2,Z)$ on $PL\,(F_{67})$. Of course, by Theorem 3, there is only one orbit $\Omega=\{\infty,0,1,2,\ldots,66\}$ under this action. Thus the action of $PGL(2,Z)$ on $PL\,(F_{67})$ is transitive.

A similar argument shows that the action of $PGL(2,Z)$ on $PL(F_{199})$ is transitive.

(ii) The coset diagrams given in theorem 3 are the connected diagrams of $\alpha(\Delta(2,3,11))$. □



References

[1] **M. D. E. Conder:** *Hurwitz groups: a brief survey*, Bull. Amer. Math. Soc. **23** (1990), 359 – 370.

[2] **B. Fine and G. Rosenberger:** *A note on generalized triangle groups*, Abh. Math. Sem. Univ. Hamburg **56** (1986), 233 – 244.

[3] **Q. Mushtaq and F. Shaheen:** *Coset diagrams for a homomorphic image of* $\triangle(2,3,6)$, Ars Combinatoria **23A** (1987), 187 – 193.

[4] **Q. Mushtaq:** *A condition for the existence of a fragment of a coset diagram*, Quart. J. Math. (Oxford) **2(39)** (1988), 81 – 95.

[5] **Q. Mushtaq:** *Parametrization of all homomorphisms from* $PGL(2,Z)$ *into* $PGL(2,q)$, Comm. algebra **20** (1992), 1023 – 1040.

[6] **J. S. Rose:** *A course on group theory*, Cambridge University Press, Cambridge, 1978.

Received October 17, 2005

Department of Mathematics
Quaid-i-Azam University
Islamabad 45320
Pakistan
e-mail: qmushtaq@apollo.net.pk