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On the prime graph of $L_2(q)$ where $q = p^\alpha < 100$

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Abstract

Let G be a finite group. We construct the prime graph of G as follows: the vertices of this graph are the prime divisors of $|G|$ and two vertices p and q are joined by an edge if and only if G contains an element of order pq. The prime graph of G is denoted by $\Gamma(G)$.

Mina Hagie (Comm. Algebra, 2003) determined finite groups G such that $\Gamma(G) = \Gamma(S)$, where S is a sporadic simple group. In this paper we determine finite groups G such that $\Gamma(G) = \Gamma(L_2(q))$ for some $q < 100$.

1. Introduction

Let G be a finite group. We denote by $\pi(G)$ the set of all prime divisors of $|G|$. If $|\pi(G)| = n$, then G is called a K_n*−group*.

The *prime graph* (Gruenberg-Kegel graph) $\Gamma(G)$ of a group G is the graph whose vertex set is $\pi(G)$, and two distinct primes p and q are joined by an edge (we write $p \sim q$) if and only if G contains an element of order pq. Let $t(G)$ be the number of connected components of $\Gamma(G)$ and let π_1, π_2, ..., $\pi_{t(G)}$ be the connected components of $\Gamma(G)$. If $2 \in \pi(G)$, then we always suppose $2 \in \pi_1$. Also the set of orders of the elements of G is denoted by $\pi_e(G)$. Obviously $\pi_e(G)$ is partially ordered by divisibility. Therefore it is uniquely determined by $\mu(G)$, the subset of its maximal elements. We know that $\mu(L_2(q)) = \{p, (q-1)/d, (q+1)/d\}$ and $\mu(PGL(2, q)) = \{p, (q-1), (q+1)\}$ where $q = p^\alpha$ and $d = (2, q-1)$. Also we know that the prime graph components of $L_2(q)$ are cliques (i.e., complete subgraphs).

The structure of finite groups G with disconnected prime graph has been determined by Gruenberg and Kegel (1975) and they have been described

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in [9, 13, 16]. It has been proved that $t(G) \leqslant 6$ [9, 13, 16], and we know that the diameter of $\Gamma(G)$ is at most 5 (see[14]).

Mina Hagie in [8] determined finite groups G such that $\Gamma(G) = \Gamma(S)$ where S is a sporadic simple group. Also in [12] finite groups were determined which have the same prime graph as a CIT simple group. In this paper we determine finite groups G such that $\Gamma(G) = \Gamma(L_2(q))$, where $q < 100$ is a prime power. Throughout this paper we denote by (a, b), the greatest common divisor of a and b.

2. Preliminary results

Lemma 2.1. ([5]) *Let G be a finite group, H a subgroup of G and N a normal subgroup of G. Then*

(1) *if p and q are joined in $\Gamma(H)$, then p and q are joined in $\Gamma(G)$;*

(2) *if p and q are joined in $\Gamma(G/N)$, then p and q are joined in $\Gamma(G)$.*

In fact if $xN \in G/N$ has order pq, then there is a power of x which has order pq.

Lemma 2.2. ([1]) *If G is a simple K_3-group, then G is isomorphic to one of the following groups: A_5, A_6, $L_2(7)$, $L_2(8)$, $L_2(17)$, $L_3(3)$, $U_3(3)$ and $U_4(2)$.*

Lemma 2.3. ([1]) *If G is a simple K_4-group, then G is isomorphic to one of the following groups:*
A_7, A_8, A_9, A_{10}, M_{11}, M_{12}, J_2, $L_3(4)$, $L_3(5)$, $L_3(7)$, $L_3(8)$, $L_3(17)$, $L_4(3)$, $O_5(4)$, $O_5(5)$, $O_5(7)$, $O_5(9)$, $O_7(2)$, $O_8^+(2)$, $G_2(3)$, ${}^3D_4(2)$, ${}^2F_4(2)'$, $Sz(8)$, $Sz(32)$, $U_3(4)$, $U_3(5)$, $U_3(7)$, $U_3(8)$, $U_3(9)$, $U_4(3)$, $U_5(2)$, $L_2(q)$,
where q is a prime power satisfying $q(q^2-1) = (2, q-1)2^{\alpha_1}3^{\alpha_2}p^{\alpha_3}r^{\alpha_4}$, $\alpha_i \in \mathbb{N}$ $(1 \leqslant i \leqslant 4)$ and 2, 3, p, r are distinct primes.

The next lemma is an immediate consequence of Theorem A in [16]:

Lemma 2.4. *If G is a finite group whose prime graph is disconnected, then one of the following holds: G is a Frobenius group or a $2-$Frobenius group; or G has a normal series $1 \trianglelefteq N \trianglelefteq M \trianglelefteq G$ such that G/M and N are π_1-groups, N is a nilpotent π_1-group and M/N is a non-abelian simple group.*

Corollary 2.1. [16] *If G is a solvable group with disconnected prime graph, then $t(G) = 2$ and G is either Frobenius or $2-$Frobenius group and G has* *exactly two components, one of which consists of the primes dividing the lower Frobenius complement.*

The next lemma follows from [2] and the structure of Frobenius complements:

Lemma 2.5. *Let G be a Frobenius group of even order and let H, K be Frobenius complement and Frobenius kernel of G, respectively. Then $t(G) = 2$ and the prime graph components of G are $\pi(H)$, $\pi(K)$ and G has one of the following structures:*

- (a) $2 \in \pi(K)$ *and all Sylow subgroups of H are cyclic;*
- (b) $2 \in \pi(H)$, *K is an abelian group, H is a solvable group, the Sylow subgroups of odd order of H are cyclic groups and the $2-$Sylow subgroups of H are cyclic or generalized quaternion groups;*
- (c) $2 \in \pi(H)$, *K is an abelian group and there exists $H_0 \leqslant H$ such that $|H : H_0| \leqslant 2$, $H_0 = Z \times SL(2,5)$, $\pi(Z) \cap \{2,3,5\} = \emptyset$ and the Sylow subgroups of Z are cyclic.*

Also the next lemma follows from [2] and the properties of Frobenius groups:

Lemma 2.6. *Let G be a $2-$Frobenius group of even order, i.e. G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, such that K and G/H are Frobenius groups with kernels H and K/H, respectively. Then*

- (a) $t(G) = 2$, $\pi_1 = \pi(G/K) \cup \pi(H)$ *and* $\pi_2 = \pi(K/H)$;
- (b) *G/K and K/H are cyclic,* $|G/K| \mid (|K/H|-1)$ *and* $G/K \leqslant Aut(K/H)$;
- (c) *H is nilpotent and G is a solvable group.*

Lemma 2.7. *Let L be a finite group with $t(L) = 3$. If G is a finite group such that $\Gamma(G) = \Gamma(L)$, then G has a normal series $1 \trianglelefteq N \trianglelefteq M \trianglelefteq G$ such that G/M and N are π_1-groups, N is a nilpotent π_1-group and M/N is a non-abelian simple group, where $t(M/N) \geqslant 3$. Also $|G/M| \mid |Out(M/N)|$.*

Proof. The first part of theorem follows from the above lemmas. Since $t(G) = 3$, it follows that $t(G/N) \geqslant 3$. Moreover, we have $Z(G/N) = 1$. For any $xN \in G/N$ and $xN \notin M/N$, xN induces an automorphism of M/N and this automorphism is trivial if and only if $xN \in Z(G/N)$. Therefore $G/M \leqslant Out(M/N)$ and since $Z(G/N) = 1$, the result follows. □

Lemma 2.8. ([7]) *The equation $p^m - q^n = 1$, where p and q are prime numbers and $m, n > 1$, has only one solution, namely $3^2 - 2^3 = 1$.*

Lemma 2.9. ([7]) *With the exceptions of the relations* $(239)^2-2(13)^4=-1$ *and* $(3)^5-2(11)^2=1$ *every solution of the equation* $p^m-2q^n=\pm1$*;* p*,* q *prime;* $m,n\geqslant 1$*, has exponents* $m=n=2$*.*

Lemma 2.10. (Zsigmondy Theorem) ([17]) *Let* p *be a prime and* n *be a positive integer. Then one of the following holds:*

- (i) *there is a prime* p' *such that* $p'|(p^n-1)$ *but* $p'\nmid(p^m-1)$ *for every* $1\leqslant m<n$*;*
- (ii) $p=2$*,* $n=1$ *or 6;*
- (iii) p *is a Mersenne prime and* $n=2$*.*

Lemma 2.11. ([15, Proposition 3.2]) *Let* G *be a finite group and* H *a normal subgroup of* G*. Suppose* G/H *is isomorphic to* $PSL(2,q)$*,* q *odd and* $q>5$*, and that an element* t *of order* 3 *in* $G\backslash H$ *has no fixed points on* H*. Then* $H=1$*.*

3. Main results

In this section we determine finite groups G satisfying $\Gamma(G)=\Gamma(L_2(q))$, where $q<100$ is a prime power.

Theorem 3.1. *Let* $L=L_2(q)$ *where* $q<100$*. If* G *is a non-abelian simple group such that* $\Gamma(G)$ *is a subgraph of* $\Gamma(L)$ *and* $\pi_i(L)\subseteq\pi_i(G)$ *for* $2\leqslant i\leqslant 3$*, then* G *is one of the groups in the* 2nd *column of Table* 1*.*

In the table, X is one of the following non-abelian simple groups: $L_2(q)$ such that $q=p^\alpha$ is a prime power and $q\neq 7^2$, $16\leqslant q<100$.

Proof. By assumptions we have $\pi(G)\subseteq\pi(L)$. We consider three steps:

Step 1. If $|\pi(L)|=3$, then $L\cong L_2(5)$, $L_2(7)$, $L_2(8)$, $L_2(9)$ or $L_2(17)$, by Lemma 2.2. Also G is a simple K_3-group, since G is a non-abelian simple group. Now by using the atlas of finite groups [6], it follows that the result holds.

Table 1.

L	G	L	G
$L_2(5)$	$L_2(5)$, $L_2(9)$	$L_2(11)$	$L_2(11)$, M_{11}
$L_2(7)$	$L_2(7)$, $L_2(8)$	$L_2(13)$	$L_2(13)$, $G_2(3)$
$L_2(8)$	$L_2(7)$, $L_2(8)$	$L_2(49)$	A_7, $L_2(49)$, $L_3(4)$, $U_4(3)$
$L_2(9)$	$L_2(5)$, $L_2(9)$	X	X

Step 2. If $|\pi(L)|$=4, then L is isomorphic to one of the following groups: $L_2(11)$, $L_2(13)$, $L_2(16)$, $L_2(19)$, $L_2(23)$, $L_2(25)$, $L_2(27)$, $L_2(31)$, $L_2(32)$, $L_2(37)$, $L_2(47)$, $L_2(49)$, $L_2(53)$, $L_2(73)$, $L_2(81)$ and $L_2(97)$, by Lemma 2.3. Since G is a non-abelian simple group and $\Gamma(G)$ is a subgraph of $\Gamma(L)$, it follows that G is a simple K_3-group or a simple K_4-group. For each L, there exists a prime number p in $\pi_i(L)$, for $2 \leqslant i \leqslant 3$, which is not in $\pi(G)$, for every simple K_3-group G [6]. So G is a simple K_4-group. Then G is one of the groups listed in Lemma 2.3. Since the proofs of these cases are similar, we do only one of them, namely $L_2(11)$.

Let $L = L_2(11)$ and G be a simple K_4-group such that $\Gamma(G)$ is a subgraph of $\Gamma(L)$. Since G and L are K_4-groups, it follows that $\pi(G) = \pi(L)$. Therefore $\pi(G) = \{2, 3, 5, 11\}$. Hence by using Lemma 2.3 and [6], it follows that $G \cong M_{11}, M_{12}, U_5(2)$ or $L_2(q)$ where q is a prime power satisfying $q(q^2-1) = (2, q-1)2^{\alpha_1}3^{\alpha_2}5^{\alpha_3}11^{\alpha_4}$, where $\alpha_i \in \mathbb{N}$ $(1 \leqslant i \leqslant 4)$. We know that $2 \sim 5$ in M_{12} and $3 \sim 5$ in $U_5(2)$, but $2 \not\sim 5$ and $3 \not\sim 5$ in $\Gamma(L)$. Hence $G \cong M_{11}$ or $G \cong L_2(q)$ where q is a prime power satisfying $q(q^2-1) = (2, q-1)2^{\alpha_1}3^{\alpha_2}5^{\alpha_3}11^{\alpha_4}$, where $\alpha_i \in \mathbb{N}$ $(1 \leqslant i \leqslant 4)$.

If $\Gamma(G) = \Gamma(L_2(q))$ is a subgraph of $\Gamma(L)$, it follows that $\Gamma(L_2(11)) = \Gamma(L_2(q))$, since the components of $L_2(q)$ are cliques. Now we prove that q=11. We know that $\mu(L_2(11)) = \{5, 6, 11\}$. Note that $\{p\}$ is a prime graph component of $G \cong L_2(q)$ where $q = p^\alpha$, and so $p \not\sim p'$ for every prime number $p' \neq p$. As $\Gamma(L_2(11)) = \Gamma(L_2(q))$, it follows that $\Gamma(L_2(q))$ has the same components as $\Gamma(L_2(11))$. Also $2 \sim 3$ in $\Gamma(L_2(11))$ and hence $q \neq 2^\alpha$ and $q \neq 3^\alpha$ where $\alpha \in \mathbb{N}$. Therefore $q = 5^\alpha$ or $q = 11^\beta$ for some $\alpha, \beta \in \mathbb{N}$.

If $q = 5^\alpha$, then $4 \mid (q-1)$ and so $\mu(L_2(q)) = \{5, (5^\alpha-1)/2, (5^\alpha+1)/2\}$. Also 2 divides $(5^\alpha-1)/2$ and hence $2 \in \pi((5^\alpha-1)/2)$. Therefore $(5^\alpha+1)/2 = 11^k$ for some $k > 0$. Then $5^\alpha - 2\cdot 11^k = -1$ and as this diophantine equation has no solution, by Lemma 2.9, we have a contradiction. If $q = 11^\beta$, then we consider two cases: if β is even, then $4 \mid (11^\beta - 1)$ and so $2 \mid (11^\beta-1)/2$ which implies that $(11^\beta+1)/2 = 5^k$, for some $k > 0$. Again by using Lemma 2.9 we get a contradiction. If β is odd, then $2 \mid (11^\beta+1)/2$ and hence $(11^\beta-1)/2 = 5^k$, for some $k > 0$. Now by using Lemma 2.9, it follows that $\beta = k$=1. Therefore $G \cong L_2(11)$ and the result follows.

The proof of the other cases are similar and we omit them for convenience.

Step 3. Let $|\pi(L)| = 5$. Now by using [6], we can see that L is isomorphic to one of groups $L_2(29)$, $L_2(41)$, $L_2(43)$, $L_2(59)$, $L_2(61)$, $L_2(64)$, $L_2(67)$, $L_2(71)$, $L_2(79)$, $L_2(83)$ or $L_2(89)$. In Steps 1 and 2 we use Lemmas 2.2 and

2.3. But in this step we have no result about simple K_5-group. Therefore we use the following method to get the result. Since $\Gamma(L)$ has three components and $\pi_i(L) \subseteq \pi_i(G)$ for $2 \leqslant i \leqslant 3$, it follows that $\Gamma(G)$ has at least three components, by Lemma 2.1. Now by using the table of non-abelian simple groups with at least three components (see [10]), we consider all possibilities. Again the proof of these cases are similar and for convenience we do one of them, namely $L_2(29)$.

Let $L = L_2(29)$. We know that $\mu(L_2(29)) = \{29, 14, 15\}$. If $G \cong A_p$ where p and $p-2$ are prime numbers, then we get a contradiction, since $2, 3 \in \pi_1(A_p)$ and $\Gamma(G)$ is a subgraph of $\Gamma(L_2(29))$. If $G \cong A_1(q)$ where $4 \mid (q+1)$, then $q = 29^k$ or $(q-1)/2 = 29^k$ for some $k \in \mathbb{N}$. Since $4 \nmid (29^k + 1)$, thus $q \neq 29^k$. So $(q-1)/2 = 29^k$. Then the third component of $\Gamma(G)$ is $\pi(q) = \{3, 5\}$, which is a contradiction, since q is a prime power. If $G \cong A_1(q)$ where $4 \mid (q-1)$, then $q = 29^k$ or $(q+1)/2 = 29^k$. First let $q = 29^k$ and $k > 1$. Then $q - 1 = 29^k - 1$ has a prime divisor p where $p \notin \{2, 7\}$, by Zsigmondy theorem, which is a contradiction. If k=1, then $G \cong L_2(29)$. If $(q+1)/2 = 29^k$, then $\pi(q) = \{3, 5\}$, which is a contradiction. If $G \cong A_1(q)$ where $4 \mid q$, then $q - 1 = 29^k$ or $q + 1 = 29^k$ and these diophantine equations have no solution by Lemma 2.8, a contradiction. If $G \cong {}^2B_2(q)$ where $q = 2^{2n+1} > 2$, then $q-1$ is equal to a power of 3, 5, 7, 29 or $q-1 = 3^\alpha 5^\beta$ for some, α, $\beta \in \mathbb{N}$. The equation $q-1 = 7^\alpha$ has only one solution, namely $\alpha = n = 1$. Since $29 \notin \pi(Sz(8))$, we get a contradiction. Also the diophantine equations $q - 1 = 3^\alpha$, $q - 1 = 5^\beta$ or $q - 1 = 29^\gamma$ have no solution by Lemma 2.8. If $q - 1 = 3^\alpha 5^\beta$, then $3 \mid (2^2 - 1)$, $5 \mid (2^4 - 1)$ and so $q - 1$ has a prime divisor, except 3, 5 for every $n > 2$ by Zsigmondy theorem, which is a contradiction. Also $29 \notin \pi(Sz(32))$ and so $n \neq 2$. Therefore this case is impossible. Since the cases ${}^2D_p(3)$ where $p = 2^n + 1$, $n \geqslant 2$, ${}^2D_{p+1}(2)$ where $p = 2^n - 1$, $n \geqslant 2$, $G_2(q)$ where $3 \mid q$ and ${}^2G_2(q)$ where $q = 3^{2n+1}$ have similar proofs, we consider only one of them, namely ${}^2D_p(3)$. If $G \cong {}^2D_p(3)$ where $p = 2^n + 1$, $n \geqslant 2$, then $2, 3 \in \pi_1({}^2D_p(3))$. Since $\Gamma(G)$ is a subgraph of $\Gamma(L_2(29))$ and $2 \not\sim 3$ in $\Gamma(L_2(29))$, we get a contradiction.

If $G \cong F_4(q)$ such that $2 \mid q$, $q > 2$, then $\pi_1(G)$ contains at least three prime numbers, by Zsigmondy theorem. Since $\Gamma(G)$ is a subgraph of $\Gamma(L_2(29))$, this gives a contradiction.

By the same method we can show that $G \ncong {}^2F_4(q)$ where $q = 2^{2n+1} > 2$.

Since $\pi_i(L) \subseteq \pi_i(G)$ for i=2,3, it follows that G is not isomorphic to the following groups: $A_2(2)$, $A_2(4)$, ${}^2A_5(2)$, $E_7(2)$, $E_7(3)$, ${}^2E_6(2)$, M_{11}, M_{22},

M_{23}, M_{24}, J_1, J_3, J_4, HS, Sz, ON, Ly, Co_2, F_{23}, F'_{24}, M, B or Th.

If $G \cong E_8(q)$, then $\pi_1(G)$ contains at least three prime numbers by Zsigmondy theorem, which is a contradiction. Now the proof of this theorem is complete. □

Corollary 3.1. *Let $L= L_2(q)$, where $q < 100$ and G be a finite simple group such that $|G| = |L|$. Then G is isomorphic to L.*

Proof. Straightforward from Theorem 3.1. □

Theorem 3.2. *Let $L= L_2(q)$, where $q < 100$ and G be a finite group satisfying $\Gamma(G) = \Gamma(L)$. Then G is one of the groups in 2nd column of Table 2 ($\overline{G}$ means $G/O_\pi(G)$).*

Proof. Since $t(L) \geqslant 3$, we can apply Lemma 2.4. Also note that G is neither a Frobenius group nor a 2-Frobenius group by Lemmas 2.5 and 2.6. Therefore G has a normal series $1 \trianglelefteq N \trianglelefteq M \trianglelefteq G$ such that G/M and N are π_1-groups, N is a nilpotent π_1-group and M/N is a non-abelian simple group, such that M/N satisfies the following conditions:

$$\begin{cases} (1) \quad \Gamma(M/N) \text{ is a subgraph of } \Gamma(G); \\ (2) \quad \pi_i(L) \subseteq \pi_i(G) \subseteq \pi(L) \text{ for } i = 2,\ 3; \\ (3) \quad \Gamma(G) = \Gamma(L). \end{cases} \qquad (*)$$

Case I. Since $L_2(5)$ and $L_2(9)$ have the same prime graph, we only consider one of them. So let $L = L_2(5)$. By (∗) and Theorem 3.1, it follows that $M/N \cong L_2(5)$ or $M/N \cong L_2(9)$. First let $M/N \cong L_2(5)$. We note that $Out(L_2(5)) \cong Z_2$ [6]. Therefore $G/M \leqslant Out(L_2(5)) \cong Z_2$, by Lemma 2.7. If $G/M \cong Z_2$, then since $L_2(5).2$ has an element of order 6, it follows that $\Gamma(L_2(5).2)$ is not a subgraph of $\Gamma(L)$. Thus $G = M$ and $G/O_\pi(G) \cong L_2(5)$ Where $\pi \subseteq \{2\}$. Let $M/N \cong L_2(9)$. We know that $Out(L_2(9)) \cong Z_2 \times Z_2$ and there exists three involutions in $Z_2 \times Z_2$.

By using the notations of atlas $L_2(9).2_1$ and $L_2(9).2_2$ have elements of order 6 and 10, respectively [6]. Thus $\Gamma(L_2(9).2_1)$ and $\Gamma(L_2(9).2_2)$ are not subgraphs of $\Gamma(L)$, and $G = M$. By the atlas of finite groups, $\Gamma(L_2(9)) = \Gamma(L_2(9).2_3)$. So $G/N \cong L_2(9)$ or $G/N \cong L_2(9).2_3$. If $2 \in \pi(N)$, then let $P \in Syl_2(N)$ and $Q \in Syl_3(G)$. Since N is a nilpotent group, P *char* N and $N \trianglelefteq G$, we conclude that $P \trianglelefteq G$. Also $2 \not\sim 3$ in $\Gamma(L)$, so Q acts fixed point freely on P. Hence QP is a Frobenius group, with kernel P and complement Frobenius Q. Therefore Q is cyclic. This is a contradiction since $L_2(9)$ has no element of order 9. Hence we have $N{=}1$ and $G \cong L_2(9)$ or $L_2(9).2_3$.

Table 2.

L	G	L	G
$L_2(5)$	$\overline{G} \cong L_2(5)$, $\pi \subseteq \{2\}$	$L_2(41)$	$L_2(41)$
	$L_2(9)$, $L_2(9).2_3$	$L_2(43)$	$L_2(43)$
$L_2(7)$	$L_2(7)$ or $\overline{G} \cong L_2(8)$	$L_2(47)$	$L_2(47)$
	$\pi \subseteq \{2\}$	$L_2(49)$	$L_2(49)$, $L_2(49).2_3$,
$L_2(8)$	$L_2(7)$ or $\overline{G} \cong L_2(8)$		$\overline{G} \cong L_3(4)$, $L_3(4).2'_2$
	$\pi \subseteq \{2\}$		$L_3(4).2''_3$,
$L_2(9)$	$\overline{G} \cong L_2(5)$, $\pi \subseteq \{2\}$		$L_3(4).2''_2$, $L_3(4).2'_3$,
	$L_2(9)$, $L_2(9).2_3$		$U_4(3)$, $U_4(3).2_3$, A_7
$L_2(11)$	$L_2(11)$ or M_{11}		$\pi \subseteq \{2,3\}$
$L_2(13)$	$\overline{G} \cong L_2(13)$ or $G_2(3)$	$L_2(53)$	$L_2(53)$
	$\pi \subseteq \{2,3\}$	$L_2(59)$	$L_2(59)$
$L_2(16)$	$\overline{G} \cong L_2(16)$, $\pi \subseteq \{2\}$	$L_2(61)$	$\overline{G} \cong L_2(61)$, $\pi \subseteq \{2,3,5\}$
$L_2(17)$	$L_2(17)$	$L_2(64)$	$\overline{G} \cong L_2(64))$, $\pi \subseteq \{2\}$
$L_2(19)$	$L_2(19)$	$L_2(67)$	$L_2(67)$
$L_2(23)$	$L_2(23)$	$L_2(71)$	$L_2(71)$
$L_2(25)$	$L_2(25)$, $L_2(25).2_3$	$L_2(73)$	$\overline{G} \cong L_2(73)$, $\pi \subseteq \{2,3\}$
$L_2(27)$	$L_2(27)$	$L_2(79)$	$L_2(79)$
$L_2(29)$	$L_2(29)$	$L_2(81)$	$L_2(81)$, $L_2(81).2_3$
$L_2(31)$	$L_2(31)$	$L_2(83)$	$L_2(83)$
$L_2(32)$	$\overline{G} \cong L_2(32)$, $\pi \subseteq \{2\}$	$L_2(89)$	$L_2(89)$
$L_2(37)$	$\overline{G} \cong L_2(37)$, $\pi \subseteq \{2,3\}$	$L_2(97)$	$\overline{G} \cong L_2(97)$, $\pi \subseteq \{2,3\}$

Case II. Since $L_2(7)$ and $L_2(8)$ have the same graph, we only consider one of them. So let $L = L_2(7)$. By $(*)$ and Theorem 3.1 it follows that $M/N \cong L_2(7)$ or $M/N \cong L_2(8)$. First let $M/N \cong L_2(7)$. Therefore $G/M \leqslant Out(L_2(7)) \cong Z_2$ by Lemma 2.7. Since $L_2(7).2$ has an element of order 6, $\Gamma(L_2(7).2)$ is not a subgraph of $\Gamma(L)$, thus $G = M$. We know that N is a $2-$group. If $2 \in \pi(N)$, then M has a solvable $\{2,3,7\}-$subgroup H, since $L_2(7)$ contains a 7:3 subgroup [6]. Since there exist no edge between 2, 3 and 7 in $\Gamma(L)$, it follows that $t(H){=}\ 3$, a contradiction since $t(H) \leqslant 2$, by Remark 2.1. Therefore $N = 1$ and $G = L_2(7)$. Let $M/N \cong L_2(8)$. As $L_2(8).3$ has an element of order 6 and $Out(L_2(8)) \cong Z_3$, then $\Gamma(L_2(8).3)$ is not a subgraph of $\Gamma(L)$. Therefore $G = M$ and $G/O_\pi(G) \cong L_2(8)$ where $\pi \subseteq \{2\}$.

Case III. $L = L_2(11)$. By $(*)$ and Theorem 3.1, it follows that $M/N \cong L_2(11)$ or $M/N \cong M_{11}$. We consider both cases simultaneously. Since $Out(L_2(11)) \cong Z_2$, $Out(M_{11}){=}1$ and $L_2(11).2$ has an element of order 10, in each case it follows that $G = M$. We know that N is a $\{2,3\}-$group. If $2 \in \pi(N)$, then M has a solvable $\{2,5,11\}-$subgroup H, since $L_2(11)$ and M_{11} have a 11:5 subgroup. Then $\Gamma(L)$ yields $t(H){=}3$, which is a contradiction, since $t(H) \leqslant 2$, by Remark 2.1. Similarly $3 \notin \pi(N)$. Hence $N{=}1$ and $G \cong L_2(11)$ or M_{11}.

Case IV. Since $L_2(13)$, $L_2(16)$, $L_2(17)$, $L_2(29)$, $L_2(32)$, $L_2(37)$, $L_2(41)$, $L_2(53)$, $L_2(61)$, $L_2(73)$, $L_2(89)$ and $L_2(97)$ have similar proofs, we consider only one of them. So let $L = L_2(13)$. By $(*)$ and Theorem 3.1, it follows that $M/N \cong L_2(13)$ or $M/N \cong G_2(3)$. Let $M/N \cong L_2(13)$. Since $Out(L_2(13)) \cong Z_2$ and $L_2(13).2$ has an element of order 14, it follows that $\Gamma(L_2(13).2)$ is not a subgraph of $\Gamma(L)$. Thus $G = M$ and $G/O_\pi(G) \cong L_2(13)$ where $\pi \subseteq \{2,3\}$. By the same method easily we can show that, if $M/N \cong G_2(3)$, then $G/O_\pi(G) \cong G_2(3)$, where $\pi \subseteq \{2,3\}$.

Case V. Since $L_2(19)$, $L_2(23)$, $L_2(27)$, $L_2(31)$, $L_2(43)$, $L_2(47)$, $L_2(59)$, $L_2(67)$, $L_2(71)$, $L_2(79)$ and $L_2(83)$ have similar proofs, we only consider a few of them. Let $L = L_2(19)$. Similar to the last cases, $M/N \cong L_2(19)$. Since $Out(L_2(19)) \cong Z_2$ and $L_2(19).2$ has an element of order 6, it follows that $\Gamma(L_2(19).2)$ is not a subgraph of $\Gamma(L)$, and so $G = M$. We know that $L_2(19)$ has a 19:9 subgroup [6]. If $2 \in \pi(N)$, then M has a solvable $\{2,3,19\}-$subgroup H and $\Gamma(L)$ yields $t(H){=}3$, a contradiction since $t(H) \leqslant 2$. It follows that $2 \notin \pi(N)$. Similarly, if $5 \in \pi(N)$, then M has a solvable $\{3,5,19\}-$subgroup H. Hence $\Gamma(L)$ yields $t(H){=}3$, a contradiction. This yields $5 \notin \pi(N)$. Now $N{=}1$ and $G \cong L_2(19)$. Let $L = L_2(43)$. We know that $Out(L_2(43)) \cong Z_2$, $L_2(43).2 \cong PGL(2,43)$ and $PGL(2,43)$ has an element of order 6, so $\Gamma(L_2(43).2)$ is not a subgraph of $\Gamma(L)$, and $G = M$. Since $L_2(43)$ contains a $43:21$ subgroup, then $N = 1$ and $G \cong L_2(43)$.

Case VI. $L = L_2(25)$. In this case we have $M/N \cong L_2(25)$. We note that $Out(L_2(25)) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ and by using the notations of the atlas of finite groups we know that $L_2(25).2_1$ and $L_2(25).2_2$ have element of order 26 and 10, respectively [6]. Thus $\Gamma(L_2(25).2_1)$ and $\Gamma(L_2(25).2_2)$ are not subgraphs of $\Gamma(L)$, and in this case $G = M$. By using the atlas of finite groups, $\Gamma(L_2(25)) = \Gamma(L_2(25).2_3)$. So $G/N \cong L_2(25)$ or $G/N \cong L_2(25).2_3$. If $3 \in \pi(N)$, then let $P \in Syl_3(N)$ and $Q \in Syl_5(G)$. We know that N is nilpotent and P *char* N, $N \lhd G$. Therefore $P \unlhd G$. Since $3 \not\sim 5$ in $\Gamma(L)$, so QP is a Frobenius group, with kernel P and complement Frobenius Q. Therefore Q is cyclic. This is a contradiction since $L_2(25)$ has no element of order 25. By the same method we can show that $2 \notin \pi(N)$. Therefore $N = 1$ and $G \cong L_2(25)$ or $G \cong L_2(25).2_3$.

Case VII. $L = L_2(49)$. In this case we have $M/N \cong L_2(49), A_7, L_3(4)$ or $U_4(3)$. First let $M/N \cong L_2(49)$. We know that $Out(L_2(49)) \cong Z_2 \times Z_2$ and by using the notations of atlas, $L_2(49).2_1$ and $L_2(49).2_2$ have elements of order 10 and 14, respectively. Therefore $\Gamma(L_2(49).2_1)$ and $\Gamma(L_2(49).2_2)$ are not subgraphs of $\Gamma(L)$, and so in this case $G = M$. But $\Gamma(L_2(q)) = \Gamma(L_2(49).2_3)$, so $G/N \cong L_2(49)$ or $G/N \cong L_2(49).2_3$. If $2 \in \pi(N)$, then let $P \in Syl_2(N)$ and $Q \in Syl_7(G)$. Since $2 \not\sim 7$ in $\Gamma(L)$, so QP is a Frobenius group. Therefore Q is cyclic. This is a contradiction, since $L_2(49)$ has no element of order 49. By the same method we can show that $3 \notin \pi(N)$. Therefore $N = 1$ and $G \cong L_2(49)$, $G \cong L_2(49).2_3$. Let $M/N \cong A_7$. Since $Out(A_7) \cong \mathbb{Z}_2$ and $A_7.2$ has an element of order 10, it follows that $\Gamma(A_7.2)$ is not a subgraph of $\Gamma(L)$, and so $G = M$. Hence $G/O_\pi(G) \cong A_7$ where $\pi \subseteq \{2,3\}$. Let $M/N \cong L_3(4)$. We know that $Out(L_3(4)) \cong \mathbb{Z}_2 \times S_3$. Similar to the last cases it follows that by the notations in the atlas of finite groups, $G/O_\pi(G) \cong L_3(4)$, $L_3(4).2_2'$, $L_3(4).2_2''$, $L_3(4).2_3'$ or $L_3(4).2_3''$ where $\pi \subseteq \{2,3\}$. Let $M/N \cong U_4(3)$. We note that $Out(U_4(3)) \cong D_8$. Then similarly we conclude that by the notations of the atlas $G/N \cong U_4(3)$ or $G/N \cong U_4(3).2_3$, since $2 \nsim 5$ in $\Gamma(L)$. Hence $G/O_\pi(G) \cong U_4(3)$ or $U_4(3).2_3$ where $\pi \subseteq \{2,3\}$. Since $U_4(3).2_1$, $U_4(3).2_2$, $U_4(3).2_3$ and $U_4(3).4$ have element of order 10, then $\Gamma(U_4(3).2_1)$, $\Gamma(U_4(3).2_2)$, $\Gamma(U_4(3).2_3)$ and $\Gamma(U_4(3).4)$ are not subgraphs of $\Gamma(L)$, and $G = M$.

Case VIII. $L = L_2(64)$. By assumption we have $M/N \cong L_2(64)$. We note that $Out(L_2(64)) \cong \mathbb{Z}_2 \times \mathbb{Z}_3$. Since $L_2(64).2$ and $L_2(64).3$ have elements of order 6, thus $\Gamma(L_2(64).2)$ and $\Gamma(L_2(64).3)$ are not subgraphs of $\Gamma(L)$. Therefore $G = M$ and $G/O_\pi(G) \cong L_2(64)$ where $\pi \subseteq \{2\}$.

Case IX. $L = L_2(81)$. By (∗), it follows from Theorem 3.1 that $M/N \cong L_2(81)$. Since $Out(L_2(81)) \cong Z_2 \times Z_4$ and $L_2(81).2_1$ and $L_2(81).2_2$ have element of order 82 and 6, respectively, thus $\Gamma(L_2(81).2_1)$ and $\Gamma(L_2(81).2_2)$ are not subgraphs of $\Gamma(L)$, and in these cases $G = M$. But $\Gamma(L_2(81) = \Gamma(L_2(81).2_3$. If $2 \in \pi(N)$, then let $P \in Syl_2(N)$ and $Q \in Syl_3(G)$. Therefore $P \trianglelefteq G$. Since $2 \not\sim 3$ in $\Gamma(L)$, so QP is a Frobenius group, with kernel P and complement Frobenius Q. Therefore Q is cyclic. This is a contradiction, since $L_2(81)$ has no element of order 3^4. By the same method we can show $5 \notin \pi(N)$. Therefore N=1 and $G \cong L_2(81)$ or $L_2(81).2_3$. □

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