

Left regular and intra-regular ordered semigroups in terms of fuzzy subsets

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Abstract

In this paper we extend some results concerning ideals of left regular and intra-regular ordered semigroups to fuzzy ordered semigroups. A theory of fuzzy sets on ordered groupoids and ordered semigroups can be developed. Some results on ordered groupoids-semigroups have been already given by the same authors. The aim of writing this paper is to show the way we pass from the theory of ordered semigroups based on ideals or from the theory of *poe*-semigroups (i.e. ordered semigroups having a greatest element "e") based on ideal elements to the theory of ordered semigroups based on fuzzy ideals. Then we also have the way we pass from the theory of semigroups based on ideals to the theory of semigroups based on fuzzy ideals.

1. Introduction

Given a set S , a fuzzy subset of S (or a fuzzy set in S) is, by definition, an arbitrary mapping $f : S \rightarrow [0, 1]$ where $[0, 1]$ is the usual closed interval of real numbers. If the set S bears some structure, one may distinguish some fuzzy subsets of S in terms of that additional structure. This important concept of the fuzzy set was first introduced by Zadeh in [28]. Since then, many papers on fuzzy sets appeared showing the importance of the concept and its applications to logic, set theory, group theory, groupoids, real analysis, measure theory, topology, etc. The concept of a fuzzy set introduced by Zadeh, was applied in [2] to generalize some of the basic concepts of general topology. Rosenfeld [26] was the first who considered the case when S is a

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groupoid. He gave the definition of a fuzzy subgroupoid and the fuzzy left (right, two-sided) ideal of S and justified these definitions by showing that a (conventional) subset A of a groupoid S is a (conventional) subgroupoid or a left (right, two-sided) ideal of S if the characteristic function

$$f_A : S \rightarrow [0, 1] \mid a \rightarrow f_A(x) := \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

is, respectively, a fuzzy subgroupoid or a fuzzy left (right, two-sided) ideal of S . Kuroki has been first studied the fuzzy sets on semigroups [19–24]. See also Liu's paper [25] where "fuzzy" analogous of several further important notions, e.g. those of bi-ideals or interior ideals have been defined and justified in a similar fashion. Fuzzy sets on semigroups have been also considered by Kehayopulu, Xie and Tsingelis in [18] and by Kehayopulu and Tsingelis in [10–12,15]. Fuzzy pseudo-symmetric ideals of semigroups and their radicals have been studied by K. P. Shum, Chen Degang and Wu Congxin in [27]. A theory of fuzzy sets on ordered groupoids and ordered semigroups can be developed. We endow S with the structure of an ordered groupoid or semigroup and define "fuzzy" analogous for several notions that have been proved to be useful in the theory of ordered semigroups. Following the terminology given by Zadeh, if S is an ordered groupoid (resp. ordered semigroup), a fuzzy subset of S (or a fuzzy set in S) is any mapping of S into the real closed interval $[0,1]$. Based on the terminology given by Zadeh, fuzzy sets in ordered groupoids have been first considered by Kehayopulu and Tsingelis in [13,16,17]. Moreover, each ordered groupoid can be embedded into an ordered groupoid having a greatest element in terms of fuzzy sets [16]. The aim of writing this paper is to show the way we pass from the theory of ordered semigroups based on ideals to the theory of *poe*-semigroups based on ideal elements, and then to the theory of ordered semigroups based on fuzzy ideals. Then we have the way we pass from the theory of semigroups -without order- based on ideals to the theory of semigroups based on fuzzy ideals. The paper serves as an example to have an easy comparison among the theory of ordered semigroups (or semigroups) based on ideals, the theory of *poe*-semigroups (i.e. ordered semigroups having a greatest element " e ") based on ideal elements and the theory of ordered semigroups based on fuzzy ideals.

Croisot [4] connects the matter of decompositions of a semigroup S with two other sets of conditions on S , regularity and semiprime conditions. Croisot uses the term "inversif" instead of "regular". The following decomposition theorems are well known. A semigroup S is a disjoint union

of left simple subsemigroups (equivalently, S is left regular) if and only if every left ideal of S is semiprime. It might be also noted that a semigroup S is left regular if and only if it is just a union of left simple subsemigroups of S . The results remain true if we replace the word "left" by "right" [3]. A semigroup S is a union of groups, equivalently, a union of disjoint groups if and only if it is both left regular and right regular [3]. A semigroup S is intra-regular if and only if it is a union of simple semigroups (cf. [3]). The characterizations mentioned above can be expressed by means of Green's relations as well. For details we refer to [3]. It has been proved in [7] that a poe -semigroup S is left regular if and only if every left ideal element of S is semiprime, equivalently, if every left ideal of S is semiprime. Moreover, a poe -semigroup S is left regular if and only if it is a union of left simple subsemigroups of S . Exactly as in semigroups, the left regularity of poe -semigroups can be expressed in terms of Green's relations as well (cf. [7]). Furthermore, an ordered semigroup S is left regular if and only if every left ideal of S is semiprime, equivalently, if S is a union of left regular subsemigroups of S . In addition, an ordered semigroup S is left regular if and only if it is a complete semilattice of left regular and simple semigroups. For details we refer to [14]. The following structure theorem is known as well: An ordered semigroup S is intra-regular if and only if it is a semilattice of simple semigroups, equivalently, if S is a union of simple subsemigroups of S [9]. Moreover, an ordered semigroups S is intra-regular if and only if the ideals of S are semiprime (cf. [9; Remark 3]). In addition, a poe -semigroup is a semilattice of simple semigroups if and only if it is a semilattice of simple poe -semigroups.

In the present paper we first give some further characterizations of left (resp. right) regular and intra-regular ordered semigroups in terms of right ideals and semiprime subsets, then we characterize the left regular, right regular and intra-regular poe -semigroups in terms of left ideal elements, right ideal elements and semiprime ideal elements. Finally we characterize the left regular, right regular and intra-regular ordered semigroups in terms of fuzzy left, fuzzy right ideals and fuzzy semiprime subsets.

By a poe -groupoid we mean an ordered groupoid (po -groupoid [1]) S having a greatest element " e " (i.e. $e \geq a \forall a \in S$). A \vee -semigroup is a semigroup at the same time a semilattice under \vee such that $a(b \vee c) = ab \vee ac$ and $(a \vee b)c = ac \vee bc$ for all $a, b, c \in S$ [1]. A poe -semigroup or \vee -semigroup is a po -semigroup or \vee -semigroup having a greatest element " e ".

2. A characterization of left regular and intra-regular ordered semigroups in terms of semiprime left ideals

If (S, \cdot, \leq) is an ordered groupoid and $H \subseteq S$, we denote $[H]$ the subset of S defined as follows:

$$[H] = \{t \in S \mid t \leq h \text{ for some } h \in H\}.$$

If (S, \cdot, \leq) is an ordered groupoid, a non-empty subset A of S is called a *left* (resp. *right*) ideal of S if 1) $SA \subseteq A$ (resp. $AS \subseteq A$) and 2) If $a \in A$ and $S \ni b \leq a$, then $b \in A$. A is called an *ideal* of S if it is both a left and a right ideal of S [6]. For an ordered semigroup S , we denote by $L(a)$, $R(a)$, $I(a)$ the left ideal, right ideal, and the ideal of S , respectively, generated by a ($a \in S$). For each $a \in S$, we have $L(a) = (a \cup Sa]$, $R(a) = (a \cup aS]$, and $I(a) = (a \cup Sa \cup aS \cup SaS]$ [6].

An ordered semigroup (S, \cdot, \leq) is called *left regular* if for every $a \in S$ there exists $x \in S$ such that $a \leq xa^2$. Equivalent definitions: 1) $A \subseteq (SA^2]$ for every $A \subseteq S$. 2) $a \in (Sa^2]$ for every $a \in S$. An ordered semigroup (S, \cdot, \leq) is called *right regular* if for every $a \in S$ there exists $x \in S$ such that $a \leq a^2x$. Equivalent definitions: 1) $A \subseteq (A^2S]$ for every $A \subseteq S$. 2) $a \in (a^2S]$ for every $a \in S$ [8]. S is called *intra-regular* if for every $a \in S$ there exist $x, y \in S$ such that $a \leq xa^2y$. Equivalent Definitions: 1) $A \subseteq (SA^2S]$ for every $A \subseteq S$. 2) $a \in (Sa^2S]$ for every $a \in S$ [9].

A subset T of an ordered groupoid S is called *semiprime* if $a \in S$, $a^2 \in T$ imply $a \in T$. Equivalent Definition: $A \subseteq S$, $A^2 \subseteq T$ imply $A \subseteq T$ [7].

Proposition 2.1. *For an ordered semigroup (S, \cdot, \leq) the following are equivalent:*

- 1) S is left regular.
- 2) Every left ideal of S is semiprime.
- 3) $L(a)$ is a semiprime left ideal of S for every $a \in S$.
- 4) $L(a^2)$ is a semiprime left ideal of S for every $a \in S$.

Proof. 1) \implies 2). Cf. [14]. It can be independently proved as follows: Let L be a left ideal of S and $a \in S$, $a^2 \in L$. Since S is left regular, there exists $x \in S$ such that $a \leq xa^2$. Since $a^2 \in L$ and L is a left ideal of S , we have $xa^2 \in SL \subseteq L$. Since $S \ni a \leq xa^2 \in L$ and L is a left ideal of S , we have $a \in L$.

2 \implies 3) \implies 4). It is obvious.

4) \implies 1). Let $a \in S$. Since $a^2 \in L(a^2)$ and $L(a^2)$ is semiprime, we have $a \in L(a^2) = (a^2 \cup Sa^2]$. Then $a \leq a^2$ or $a \leq xa^2$ for some $x \in S$. If $a \leq a^2$, then $a \leq aa \leq aa^2$. Thus S is left regular. \square

In a similar way we prove the following two propositions.

Proposition 2.2. *For an ordered semigroup (S, \cdot, \leq) the following are equivalent:*

- 1) S is right regular.
- 2) Every right ideal of S is semiprime.
- 3) $R(a)$ is a semiprime right ideal of S for every $a \in S$.
- 4) $R(a^2)$ is a semiprime right ideal of S for every $a \in S$. □

Proposition 2.3. *For an ordered semigroup (S, \cdot, \leq) the following are equivalent:*

- 1) S is intra-regular.
- 2) Every ideal of S is semiprime.
- 3) $I(a)$ is a semiprime ideal of S for every $a \in S$.
- 4) $I(a^2)$ is a semiprime ideal of S for every $a \in S$. □

3. A characterization of left regular and intra-regular poe-semigroups in terms of left ideal elements

An element a of an ordered groupoid S is called a *left* (resp. *right*) *ideal element* if $xa \leq a$ (resp. $ax \leq x$) for all $x \in S$ [1]. An element which is both a left and a right ideal element is called an *ideal element*. One can easily see that in poe-groupoids, an element a is a left (resp. right) ideal element if and only if $ea \leq a$ (resp. $ae \leq a$) [5]. We denote by $l(a)$, $r(a)$, $i(a)$ the left ideal element, right ideal element and the ideal element of S , respectively, generated by a ($a \in S$). For a $\forall e$ -semigroup S , we have $l(a) = ea \vee a$, $r(a) = ae \vee a$ and $i(a) = a \vee ea \vee ae \vee eae$ for all $a \in S$ (cf. also [5]).

Let (S, \cdot, \leq) be an ordered semigroup. Suppose that S has a greatest element "e", that is S is a poe-semigroup. Then, one can easily see that S is left (resp. right) regular if and only if $a \leq ea^2$ (resp. $a \leq a^2e$) for every $a \in S$. S is intra-regular if and only if $a \leq ea^2e$ for every $a \in S$. For further information concerning the left, right ideal elements and the left, right regularity and intra-regularity (in poe-semigroups) we refer to [5]. An element t of an ordered groupoid S is said to be *semiprime* if for every $a \in S$ such that $a^2 \leq t$, we have $a \leq t$ [5].

Proposition 3.1. *Let (S, \cdot, \leq) be a poe-semigroup. We consider the statements:*

- 1) S is left regular.
- 2) Every left ideal element of S is semiprime.

3) $l(a)$ is a semiprime left ideal element of S for every $a \in S$.

4) $l(a^2)$ is a semiprime left ideal element of S for every $a \in S$.

Then 1) \implies 2). In particular, if S is a $\vee e$ -semigroup, then the properties 1) – 4) are equivalent.

Proof. 1) \implies 2). Let a be a left ideal element of S and $t \in S$ such that $t^2 \leq a$. Then, since S is left regular, we have $t \leq et^2 \leq ea \leq a$.

Let now S be a $\vee e$ -semigroup. Then

2) \implies 3) \implies 4). It is obvious.

4) \implies 1). Let $a \in S$. Since $a^2 \leq l(a^2)$ and $l(a^2)$ is a semiprime element of S , by 4), we have $a \leq l(a^2) = a^2 \vee ea^2$. Then $a^2 \leq a^3 \vee ea^3 \leq ea^2$, so $a \leq ea^2$, and S is left regular. \square

In a similar way the following two propositions hold true.

Proposition 3.2. *Let (S, \cdot, \leq) be a poe-semigroup. We consider the statements:*

1) S is right regular.

2) Every right ideal element of S is semiprime.

3) $r(a)$ is a semiprime right ideal element of S for every $a \in S$.

4) $r(a^2)$ is a semiprime right ideal element of S for every $a \in S$.

Then 1) \implies 2). In particular, if S is a $\vee e$ -semigroup, then the properties 1) – 4) are equivalent. \square

Proposition 3.3. *Let (S, \cdot, \leq) be a poe-semigroup. We consider the statements:*

1) S is intra-regular.

2) Every ideal element of S is semiprime.

3) $i(a)$ is a semiprime ideal element of S for every $a \in S$.

4) $i(a^2)$ is a semiprime ideal element of S for every $a \in S$.

Then 1) \implies 2). In particular, if S is a $\vee e$ -semigroup, then the properties 1) – 4) are equivalent. \square

For the equivalence 1) \iff 2) cf. also [5].

4. A characterization of left regular and intra-regular ordered semigroups in terms of fuzzy subsets

If S is a groupoid or an ordered groupoid and $A \subseteq S$, the fuzzy subset f_A of S is the characteristic function of A defined as follows:

$$f_A : S \rightarrow [0, 1] \mid a \rightarrow f_A(x) := \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A. \end{cases}$$

Let (S, \cdot, \leq) be an ordered groupoid. A fuzzy subset f of S is called a *fuzzy left ideal* of S if 1) $f(xy) \geq f(y)$ for every $x, y \in S$ and 2) If $x \leq y$, then $f(x) \geq f(y)$.

A fuzzy subset f of S is called a *fuzzy right ideal* of S if 1) $f(xy) \geq f(x)$ for every $x, y \in S$ and 2) If $x \leq y$, then $f(x) \geq f(y)$ [13].

A fuzzy subset f of S is called a *fuzzy ideal* of S if it is both a fuzzy right and a fuzzy left ideal of S . Equivalent Definition: 1) $f(xy) \geq \max\{f(x), f(y)\}$ for every $x, y \in S$ and 2) If $x \leq y$, then $f(x) \geq f(y)$ [13].

Definition 4.1. Let S be a groupoid or an ordered groupoid. A fuzzy subset f of S is called *semiprime* if $f(a) \geq f(a^2)$ for every $a \in S$.

For an equivalent definition of the semiprime fuzzy subsets we refer to [18].

Remark 4.2. If f is a semiprime fuzzy left ideal of S , then $f(a) = f(a^2)$ for every $a \in S$. In fact, let $a \in S$. Since f is a fuzzy left ideal of S , we have $f(xy) \geq f(y)$ for each $x, y \in S$, so $f(a^2) \geq f(a)$. Since f is semiprime, we have $f(a) \geq f(a^2)$, hence we have $f(a) = f(a^2)$. Similarly, if f is a semiprime fuzzy right ideal of S , then $f(a) = f(a^2)$ for every $a \in S$.

Lemma 4.3. [13] *A non-empty subset L of an ordered groupoid (S, \cdot, \leq) is a left ideal of S if and only if the characteristic function f_L is a fuzzy left ideal of S .*

Lemma 4.4. *A non-empty subset R of an ordered groupoid (S, \cdot, \leq) is a right ideal of S if and only if the characteristic function f_R is a fuzzy right ideal of S . \square*

Lemma 4.5. (Cf. also [18]) *A non-empty subset A of a groupoid (S, \cdot) or an ordered groupoid (S, \cdot, \leq) is a semiprime subset of S if and only if the fuzzy subset f_A of S is semiprime. \square*

Proposition 4.6. *An ordered semigroup (S, \cdot, \leq) is left (resp. right) regular if and only if the fuzzy left (resp. fuzzy right) ideals of S are semiprime.*

Proof. Let S be left regular, f a fuzzy left ideal of S and $a \in S$. Since S is left regular, there exists $x \in S$ such that $a \leq xa^2$. Then, since f is a fuzzy left ideal of S , we have $f(a) \geq f(xa^2) \geq f(a^2)$. Thus f is semiprime.

Conversely, let $a \in S$. Since $L(a^2)$ is a left ideal of S , by Lemma 4.3, the characteristic function $f_{L(a^2)}$ is a fuzzy left ideal of S . By hypothesis, $f_{L(a^2)}$ is semiprime. By Lemma 4.5, $L(a^2)$ is a semiprime left ideal of S . Then, by Proposition 2.1, S is left regular.

In a similar way we prove S is right regular if and only if the fuzzy right ideals of S are semiprime. \square

Remark 4.7. Each of the following two conditions also characterizes the left regular ordered semigroups.

- 1) $f_{L(a)}$ is a semiprime fuzzy left ideal of S for every $a \in S$.
- 2) $f_{L(a^2)}$ is a semiprime fuzzy left ideal of S for every $a \in S$.

Remark 4.8. Each of the following two conditions also characterizes the right regular ordered semigroups.

- 1) $f_{R(a)}$ is a semiprime fuzzy right ideal of S for every $a \in S$.
- 2) $f_{R(a^2)}$ is a semiprime fuzzy right ideal of S for every $a \in S$.

In a similar way we have the following:

Proposition 4.9. *An ordered semigroup S is intra-regular if and only if the fuzzy ideals of S are semiprime.* \square

Remark 4.10. Each of the following two conditions also characterizes the intra-regular ordered semigroups.

- 1) $f_{I(a)}$ is a semiprime fuzzy ideal of S for every $a \in S$.
- 2) $f_{I(a^2)}$ is a semiprime fuzzy ideal of S for every $a \in S$. \square

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