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On primal ideals over semigroups

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Abstract

Let S be a commutative cancellation torsion-free additive semigroup with identity 0 and let $S \neq \{0\}$. This paper is devoted to study some properties of primal ideals and quasi-primary ideals of the semigroup S. First, a number of results concerning of these ideals are given. Second, we characterize primal ideals and quasi-primary ideals of a Prüfer semigroup and show that in such semigroup, the three concepts: primary, quasi-primary, and primal coincide.

1. Introduction

Throughout this paper S will be a commutative cancellation torsion-free additive semigroup with identity 0 and let $S \neq \{0\}$. We will study the structure of primal ideals and quasi-primary ideals of S. Our interest is motivated by the work [2].

Fuchs in [1] introduced the concept of a primal ideal, where a proper ideal I of S is said to be primal if the elements of S which are not prime to I form an ideal (see section 3). Fuchs and Mosteig proved in [2] that in a Prüfer domain of finite character every non-zero ideal is the intersection of a finite number of primal ideals, and moreover, the P-primal ideals form a semigroup under ideal multiplication. A similar result is established for decomposition into the intersection (even into the products) of quasi-primary ideals. The purpose of this paper is to explore some basic facts of these class of ideals of a semigroup. In the second section we characterize the semigroups in which every ideal is prime and prove that a semigroup is a group if and only if every its proper ideal is prime. We show also that every ideal over a Prüfer semigroup is quasi-primary and characterize primal

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ideals of a Prüfer semigroup. Connection between the primal ideals, the quasi-primary and the primary ideals of such semigroups are studied too.

Before we state some results let us introduce some notation and terminologies. Let S be a semigroup. Then $G = \{a-b : a, b \in S\}$ is a torsion-free ablian group with respect to the addition and S is a subsemigroup of G. G is called the *quotient group* of S. Any semigroup T between S and G is called an *oversemigroup* of S (see [3]).

By an *ideal* of S we mean a non-empty subset I of S such that for all $a \in I$ and for all $b \in S$ we have $a + b \in I$, that is, $I + S = I$. Thus for $x \in S$, $x + S = \{x + y : y \in S\}$ is the principal ideal generated by x. If I, J are ideals of S, then $I + J = (I + S) + (J + S) = (I + J) + S$ is an ideal of S too. For $a \in S$ and an ideal I of S, by $a + I$, we mean the sum $a + I = (a + S) + (I + S)$, which is an ideal of S. A proper ideal I of a semigroup S is called *maximal* if there does not exist an ideal J of S with $I \subset J \subset S$, where $\subset$ denotes the strict inclusion. An element $a \in S$ is called a *unit* if $a + b = 0$ for some $b \in S$. If $U(S)$ is the set of units in S and $0 \in U(S)$, then $U(S)$ is a subgroup of G and $M = S - U(S) \neq \emptyset$ is a maximal ideal of S. A *prime ideal* in a semigroup S is any proper ideal P of S such that for $a, b \in S$ $a + b \in P$ implies either $a \in P$ or $b \in P$. The maximal ideal is a prime ideal (see [3]).

Let I be an ideal of S. The set

$$\operatorname{rad}(I) = \{a \in S : na \in I \text{ for some positive integer } n\}$$

is an ideal of S. It is called the *radical* of I. A proper ideal I of S is *primary* if for $a, b \in S$ $a + b \in I$ implies either $a \in I$ or $b \in \operatorname{rad}(I)$. If I is primary, then $P = \operatorname{rad}(I)$ is a prime ideal of S and I is called a *P-primary ideal* of S. The set $\{a \in S : a + J \subseteq I\}$, where I, J are ideals, is denoted by $(I : J)$.

A non-empty subset T of a semigroup S is called an *additive system* of S if $a, b \in T$ implies $a + b \in T$ and $0 \in T$. $S_T = \{s - t : s \in S, t \in T\}$ is an oversemigroup of S which is called the *quotient semigroup* of S. If P is a prime ideal of S, then $T = S - P$ is an additive system of S. In this case the quotient semigroup S_T is denoted by S_P.

Throughout this paper we shall assume unless otherwise stated, that S is a semigroup with the maximal ideal $M = S - U(S) \neq \emptyset$.

Let S be a semigroup with quotient group G. We say that S is a *valuation semigroup* if $g \in S$ or $-g \in S$ for each $g \in G$, so its ideals are linearly ordered by inclusion (see [3, Lemma 4]). We say that S is a *Prüfer semigroup* if S_P is a valuation semigroup for every prime ideal P of S. An ideal of a semigroups S is *irreducible* if, for ideals J and K of S, $I = J \cap K$ implies that either $I = J$ or $I = K$.

2. Quasi-primary ideals

An ideal of S is called *quasi-primary* if its radical is a prime ideal of S.

Lemma 2.1. *Let I be an ideal of a semigroup S. Then:*

(i) *if I contains a unit of S, then $I = S$,*

(ii) *S is a subgroup of G if and only if S has exactly one ideal.*

Proof. (i) Let a be a unit of S such that $a \in I$. Then $a + b = 0$ for some $b \in S$, so $0 = a + b \in I + S = I$. If $z \in S$, then $z = 0 + z \in I + S = I$. Therefore $I = S$.

(ii) Let S be a subgroup of G and let I be an ideal of S. Then there exists $a \in I$ such that a is a unit of S; hence $I = S$ by (i). Conversely, it is enough to show that every element of S is a unit. Suppose that $c \in S$. Then $c + S \neq \emptyset$ is an ideal of S, so $c + S = S$; whence $c + d = 0$ for some $d \in S$. It is easy to see that S is a subgroup of G. □

Theorem 2.2. *Let S be a semigroup. Then S is a subgroup of G if and only if every proper ideal of S is prime.*

Proof. If S is a subgroup of G, then the result is clear. Conversely, let a be a non-zero and non-unit element of S. By assumption, $a + a + S = I$, where I is prime, and so $a + a \in I$ implies $a \in I$. Thus $a = a + 0 = a + a + b$ for some $b \in S$, and since S is a cancellation semigroup, we can cancel a to obtain $a + b = 0$, showing that a is unit, as required. □

Lemma 2.3. *Let I, J and K be ideals of a semigroup S. Then:*

(i) *$I = (I + S_M) \cap S$,*

(ii) *$K = I \cap J$ if and only if $K + S_M = (I + S_M) \cap (J + S_M)$.*

Proof. (i) Since $I \subseteq (I + S_M) \cap S$ is trivial, we will prove the reverse inclusion. Let $u \in (I + S_M) \cap S$. There exist $a \in I$ and $t \in S - M$ such that $u = a - t$, so $u + t = a \in I$ and $t + b = 0$ for some $b \in S$; hence $u = u + t + b \in I + S = I$, as required.

(ii) Suppose first that $K = I \cap J$. Clearly, $K + S_M \subseteq (I + S_M) \cap (J + S_M)$. For the reverse inclusion, assume that $z \in (I + S_M) \cap (J + S_M)$. Then there are elements $a \in I$, $b \in J$ and $t, u \in S - M$ such that $z = a - t = b - u$, so $a + u = (a - t) + u + t = (b - u) + u + t = b + t \in I \cap J$ since t, u are units of S; hence $z = a - t = (a + u) - (t + u) \in K + S_M$, as needed. The reverse implication follows from (i). □

Lemma 2.4. *For ideals I and J of a semigroup S the following statements hold:*

(i) $\mathrm{rad}(I + J) = \mathrm{rad}(I) \cap \mathrm{rad}(J) = \mathrm{rad}(I \cap J)$. *Moreover, $I + J = S$ if and only if* $\mathrm{rad}(I) + \mathrm{rad}(J) = S$.

(ii) *If N is an additive system of S, then $I + S_N = S_N$ if and only if $I \cap N \neq \emptyset$.*

(iii) *If N is an additive system of S, then* $\mathrm{rad}(I + S_N) = \mathrm{rad}(I) + S_N$.

Proof. (i) Is straightforward.

(ii) If $I + S_N = S_N$, then $0 \in I + S_N$, so $0 = a - t$ for some $a \in I$ and $t \in N$; hence $a = t \in I \cap N$. Conversely, assume that $u \in I \cap N$. As u is a unit of S_N, $I + S_N = S_N$ by Lemma 2.1.

(iii) Since $\mathrm{rad}(I) + S_N \subseteq \mathrm{rad}(I + S_N)$ is trivial, we will prove the reverse inclusion. Suppose that $z \in \mathrm{rad}(I + S_N)$. Then there exist a positive integer n such that $nz \in I + S_N$, so $nz = a - t$ for some $a \in I$, $t \in N$. As $n(z + t) = a + (n - 1)t \in I$, we get $z + t \in \mathrm{rad}(I)$. It follows that $z = z + t - t \in \mathrm{rad}(I) + S_N$, as required. □

Lemma 2.5. *Let I be an ideal of S with* $\mathrm{rad}(I) = M$. *Then I is M-primary.*

Proof. Since $I \subseteq M \neq S$, an ideal I is proper. Let $a, b \in S$ be such that $a + b \in I$ but $b \notin \mathrm{rad}(I)$. But M is maximal and $b \notin M$, so must be $M + (b + S) = S$. Then from Lemma 2.4 it follows $I + (b + S) = S$, i.e., $0 = c + (b + s)$ for some $c \in I$, $s \in S$. Therefore, we have $a = a + 0 = a + b + c + s \in I + S = I$, as needed. □

Proposition 2.6. *Let P be a prime ideal of a semigroup S, and let I be a quasi-primary ideal of S_P with a prime radical Q. Then $I \cap S$ is a quasi-primary ideal of S with a prime radical $Q \cap S$.*

Proof. Since Q is a prime ideal of S_P, $Q' = Q \cap S$ is a prime ideal of S with $Q' \subseteq P$ and $Q' + S_P = Q$ by [3, Proposition 2], so all that remains to be verified that Q' is the radical of $I \cap S$. Let $a \in \mathrm{rad}(I \cap S)$. Then $na \in I$ for some positive integer n; hence $a \in Q$. Thus, $a \in Q'$. Conversely, if $b \in Q'$, then $mb \in I \cap S$ for some positive integer m; so $b \in \mathrm{rad}(I \cap S)$, as required. $\square$

Proposition 2.7. *Let I be a quasi-primary ideal of a semigroup S with a prime radical P. Then $I + S_P$ is a primary ideal (so quasi-primary) of S_P. In particular, $(I + S_P) \cap S$ is a quasi-primary ideal of S.*

Proof. By Lemma 2.4 we have $\mathrm{rad}(I+S_P) = P+S_P$, so it is a maximal ideal of S_P by [3, Corollary 3]. Now Lemma 2.5 shows that $I + S_P$ is primary. The last claim follows from Proposition 2.6. $\square$

Proposition 2.8. *Every ideal of a valuation semigroup S is quasi-primary.*

Proof. Let I be an ideal of S with radical P. Let $a, b \in S$ such that $a+b \in P$. Then there exists a positive integer n such that $n(a + b) \in I$. Since S is a valuation semigroup, either $a+S \subseteq b+S$ or $b+S \subseteq a+S$. We may assume that $a + S \subseteq b + S$. Then there is an element $c \in S$ such that $a = b + c$, so $2na = na + nb + nc \in I + S = I$; hence $a \in P$. $\square$

Theorem 2.9. *Every ideal of a Prüfer semigroup S is quasi-primary.*

Proof. Let I be an ideal of S. By Theorem 2.8, the ideal $I + S_M$ of the valuation semigroup S_M is quasi-primary; hence Proposition 2.6 and Lemma 2.3 imply that $I = (I + S_M) \cap S$ is quasi-primary. $\square$

3. Primal ideals

An element $s \in S$ is called *prime to I* if $(r + s) \in I$ $(r \in S)$ implies that $r \in I$, that is, $(I : s) = (I : (s)) = I$. An ideal I of S is called *primal* if the elements of S that are not prime to I form an ideal (see [1]).

Lemma 3.1. *Let I be an ideal of a semigroup S and let P be the set of elements of S which are not prime to I. If P is an ideal of S, then P is prime.*

Proof. Let $a, b \in S - P$. Then $(I : a) = (I : b) = I$. If $s \in (I : a + b)$, then $a + b + s \in I$, whence $s + a \in (I : b) = I$. Therefore $s \in (I : a) = I$, consequently $(I : a + b) = I$. Thus $a + b \notin P$. $\square$

If I is a primal ideal of S, then, by Lemma 3.1, P is a prime ideal of S called the *adjoint prime ideal* of I. In this case we also say that I is a *P-primal ideal.*

Theorem 3.2. *For an ideal I of a semigroup S, the following statements are equivalent.*

(i) *I is primal with the adjoint prime ideal P,*

(ii) *If $a + b \in I$ and $b \notin I$, then $a \in P$ and conversely, for every $a \in P$ there exists an element $b \in S - I$ such that $a + b \in I$.*

Proof. (i) $\Rightarrow$ (ii) Let $a + b \in I$ with $b \notin I$. Then $b \in (I : a) - I$; hence $a \in P$. If $a \in P$, then $I \subset (I : a)$ because I is primal. So, there is an element x of $(I : a)$ which is not in I. Thus $a + x \in I$ and $x \notin I$.

(ii) $\Rightarrow$ (i) It is enough to show that $P + S \subseteq P$. Let $x + y \in P + S$ where $x \in P$, $y \in S$. Then there exists $c \notin I$ such that $x + c \in I$ by (ii), and hence $x + y + c \in I$ with $c \notin I$. Thus $x + y \in P$ by (ii). □

Lemma 3.3 *Let Q be a P-primary ideal of a semigroup S, and let $a \in S$.*

(i) *If $a \in Q$, then $(Q : a) = S$.*

(ii) *If $a \notin Q$, then $(Q : a)$ is P-primary.*

(iii) *If $a \notin P$, then $(Q : a) = Q$.*

Proof. The proof is straightforward. □

Proposition 3.4. *A P-primary ideal is primal.*

Proof. It is enough to show that the set of elements of S which are not prime to Q is just P. Suppose that s is such element of S which is not prime to Q. Then $Q \subset (Q : s)$. Hence there exists $a \in (Q : s)$ with $a \notin Q$ and $a + s \in Q$. Therefore, $s \in P$ because Q is primary. Conversely, if $s \notin P$, then $(Q : s) = Q$ by Lemma 3.3. □

Proposition 3.5. *Let I be a Q-primal ideal of a semigroup S, and let P be a prime ideal of S. Then:*

(i) *$I = (I + S_P) \cap S$ for $Q \subseteq P$,*

(ii) *$I \subset (I + S_P) \cap S$ for $Q \nsubseteq P$.*

Proof. (i) Clearly, $I \subseteq (I + S_P) \cap S$. For $x \in (I + S_P) \cap S$ we have $x = c - d \in S$ for some $c \in I$ and $d \notin P$. Therefore, $x + d = c \in I$. As $d \notin Q$, d is prime to I; hence $x \in I$.

(*ii*) Since $Q \nsubseteq P$, there is $y \in Q$ such that $y \notin P$. So $y + u \in I$ for some $u \notin I$ by Theorem 3.2. Then $u = (y + u) - y \in (I + S_P) \cap S$. But $u \notin I$, so $I \subset (I + S_P) \cap I$. □

Corollary 3.6. *Let I be a Q-primal ideal of a semigroup S, and let T be a quotient semigroup of S. Then either $I = (I + T) \cap S$ or $I \subset (I + T) \cap S$.*

Proof. By [3, Proposition 2], $T = S_P$ for some prime ideal P of S. The rest follows from Proposition 3.5. □

Proposition 3.7. *Let P be a prime ideal of a semigroup S, and let I be a Q-primal ideal of S_P. Then $I \cap S$ is a primal ideal of S with the adjoint prime ideal $Q \cap S$.*

Proof. As Q is prime ideal of S_P, by [3, Proposition 2], $Q' = Q \cap S$ is a prime ideal of S with $Q' \subseteq P$ and $Q' + S_P = Q$. To prove that Q' is exactly the set of elements non-prime to $I \cap S$ let $z \notin Q \cap S$. Then $z \notin Q$, so $(I :_{S_P} z) = I$. Thus $(I \cap S : z) = I \cap S$, whence z is prime to $I \cap S$. If $z \in Q \cap S$, then $z \in Q$, so there exists $u \in S_P$ with $z + u \in I$ and $u \notin I$ by Theorem 3.2. We can write $u = x - y$ for some $x \in S$, $y \in S - P$. If $x \in I$, then $x = u + y \in I$ with $y \notin Q$, so $u \in I$, a contradiction. So we can assume that $x \notin I$. Since $z + u \in I$ implies $z + x \in I \cap S$, we get $x \in (I \cap S : z)$. But $x \notin I$, so z is not prime to $I \cap S$. □

Corollary 3.8. *Let I be a Q-primal ideal of a quotient semigroup T of S. Then $I \cap S$ is a primal ideal of S with the adjoint prime ideal $Q \cap S$.*

Proof. Follows from [3, Proposition 2] and Proposition 3.7. □

Proposition 3.9. *Let I be an ideal of a semigroup S such that $(I : a) = P$ is a prime ideal of S for some $a \in S - I$. Then $(I + S_P) \cap S$ is a P-primal ideal of S.*

Proof. Let $J = (I + S_P) \cap S$. First, we show that $(J : a) = P$. If $t \in P = (I : a)$, then $t + a \in I \subseteq J$; hence $t \in (J : a)$. For the reverse inclusion, assume that $u \in (J : a)$, so $u + a = c - d \in J$ for some $c \in I$, $d \notin P$. Thus $u + a + d = c \in I$. Consequently $u + d \in (I : a) = P$. So, $u \in P$ since P is prime. As $P \neq S$, we get $a \notin J$. Therefore, in P no elements prime to J.

Let us show that every $b \notin P$ is prime to J. Clearly, $J \subseteq (J : b)$. To prove $(J : b) \subseteq J$, assume that $c \in (J : b)$, so $c + b = e - f \in I$ for some $e \in I$, $f \notin P$; hence $c = e - (b + f) \in J$ since $(b + f) \notin P$. Thus, $(J : b) \subseteq J$, which completes the proof. □

Lemma 3.10. *Every irreducible ideal of a semigroup S is primal.*

Proof. Let I be an irreducible ideal of S. Assume that P is the set of elements of S which are not prime to I. To prove that $P + S \subseteq P$ let $a+s \in P+S$ where $a \in P$, $s \in S$. Then $I \subset (I : a)$ because $a \in P$. Clearly, $I \subseteq (I : a) \cap (I : s) \subseteq (I : a + s)$. If $I = (I : a) \cap (I : s)$, then $I = (I : s)$ since I is irreducible. Let $t \in (I : a + s)$. Then $t + a \in (I : s) = I$, so $t \in (I : a)$; hence $I \subset (I : a) = (I : a+s)$. If $I \neq (I : a) \cap (I : s)$, then again $I \subset (I : a + s)$, that is, $a + s$ is not prime to I. Thus $a + s \in P$. □

Proposition 3.11. *An ideal I of a Prüfer semigroup is irreducible if and only if it is primal.*

Proof. By Lemma 3.10, it is sufficient to show that if I is P-primal, then I is irreducible. If $I = J \cap K$ for ideals J, K, then $I+S_M = (J+S_M)\cap(K+S_M)$ by Lemma 2.3. Since S_M is a valuation semigroup, either $I+S_M = J+S_M$ or $I + S_M = K + S_M$. Because M contains P then by Proposition 3.5 $I + S_M = J + S_M$ gives $I = (I + S_M) \cap S = (J + S_M) \cap S$. Hence $J \subseteq (J + S_M) \cap S = I$. The case $I + S_M = K + S_M$ is similar. So, I is irreducible. □

Proposition 3.12. *An ideal I of a valuation semigroup S is a primal ideal of S with the adjoint prime ideal $P = \{a \in S : (a + S) + I \subset I\}$.*

Proof. Let $I = J \cap K$ for ideals J, K of S. Then either $J \subseteq K$ or $K \subseteq J$ because S is a valuation semigroup. So either $I = J$ or $I = K$. Therefore, I is irreducible, and hence I is primal by Proposition 3.10. Let us show that P is an ideal of S. Let $a + s \in P + S$ where $a \in P$, $s \in S$. Then $(a+S)+I \subset I$; hence $(a+s)+S+I \subseteq (a+S)+I \subset I$, so $a+s \in P$. Thus, P is an ideal of S. To prove that P is prime let $x + y \in P$ with $x \notin P$. Then $(x+S)+I = I$ and $(y+S)+I = (x+y+S)+I \subset I$, whence $y \in P$.

To prove that P is the set of elements of S which are not prime to I consider $u \in P$. Then $(u + S) + I \subset I \subseteq (I : u)$. Suppose that $(I : u) = I$. If $v \in (I : u) = I$, then $u+v \in I$, so $v \in (u+S)+I$; hence $I = (u+S)+I$, a contradiction. □

Corollary 3.13. *Every ideal of a oversemigroup of a valuation semigroup is primal.*

Proof. This follows from [3, Lemma 4] and Proposition 2.12. □

Theorem 3.14. *Every ideal of a Prüfer semigroup is primal.*

Proof. If I is an ideal of a Prüfer semigroup S, then $I = (I + S_M) \cap S$ by Lemma 2.3, so, by Proposition 3.12, the ideal $I + S_M$ of S_M is primal. Proposition 3.7 completes the proof. □

Corollary 3.15 *An ideal of a Prüfer semigroup is primal* (*resp. quasi-primary*) *if and only if it is primary.*

Proof. Follows from Theorem 2.9 and Theorem 3.14. □

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