Quasigroups and Related Systems **13** (2005), 105 – 108

Automorphism group of Chein loops

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Abstract

In this paper we describe the automorphism group of Chein loops.

1. Introduction

First, we recall the definition of Chein loops (see [1]). Let G be a group and the element u be an indeterminate. Let $M(G,2) = G \cup Gu$ be the disjoint union of G and Gu and extend the operation on G to an operation (.) on $M(G,2)$ by the rules

$$g.(hu) = (hg)u, \quad (gu).h = (gh^{-1})u, \quad (gu).(hu) = h^{-1}g \quad \forall\, g,h \in G.$$

Then $M(G,2)$ is a Moufang loop, which is a group if and only if G is an abelian group. Moufang loops of this type are called *Chein loops*.

We mostly use standard notation. If G is a group then we consider the natural action of $AutG$ on G. This define a semidirect product $AutG \times G$ which is called the *Holomorph of* G and denoted by $HolG$. For $g \in G$ and $\varphi \in AutG$ we write g^{φ} for the image of g under φ.

The set

$$Stab_{AutG}(g) = \{\varphi \in AutG; g^{\varphi} = g\}$$

is a subgroup of $AutG$, called the *stabilizer* of g in $AutG$. For any $g,h \in G$ we write $[g,h] = g^{-1}h^{-1}gh$.

2000 Mathematics Subject Classification: 20N05, 20N15,
Keywords: loop, Chein loop, Mikheev group.
The first author is supported by FAPESP and CNPq, Brazil; the second by FAPESP, Brazil, Proc. 01/14811.4; the last by FAPESP, Brazil, Proc.04/07181-1.

2. The automorphisms

Consider $\psi \in Aut(G)$, we extend ψ to $a_\psi : M(G,2) \to M(G,2)$ as follows

$$a_\psi(gu^\lambda) = g^\psi u^\lambda, \quad \lambda = 0, 1.$$

Now consider an element $t \in G$ and let

$$d_t(gu) = g(tu) = (tg)u, \quad d_t(g) = g, \forall g \in G.$$

Lemma 1. *The set $A = \{a_\psi \,|\, \psi \in AutG\}$ is a subgroup of $AutM(G,2)$ isomorphic to $Aut(G)$ and the set $D = \{d_t \,|\, t \in G\}$ is a subgroup of $AutM(G,2)$ isomorphic to G. Moreover, $[A,D] = D$, $A \cap D = 1$ and the semidirect splitting extension AD is isomorphic to $Hol(G)$.*

Proof. By definition of the operation (.) in $M(G,2)$ we have

$$\left.\begin{array}{l} a_\psi(g.(hu)) = a_\psi((hg)u) = (hg)^\psi u \\ a_\psi(g).a_\psi(hu) = g^\psi.(h^\psi u) = (h^\psi g^\psi)u = (hg)^\psi u. \end{array}\right\} \quad (1)$$

$$\left.\begin{array}{l} d_t(g.(hu)) = d_t((hg)u) = (thg)u, \\ d_t(g).d_t(hu)) = g.((th)u) = (thg)u. \end{array}\right\} \quad (2)$$

Analogously, we get

$$\left.\begin{array}{l} a_\psi(gu.h) = a_\psi((gh^{-1})u) = (gh^{-1})^\psi u \\ a_\psi(gu).a_\psi(h) = g^\psi u.h^\psi = (g^\psi h^{-\psi})u = (gh^{-1})^\psi u. \end{array}\right\} \quad (3)$$

$$\left.\begin{array}{l} d_t(gu.h) = d_t((gh^{-1})u) = (tgh^{-1})u \\ d_t(gu).d_t(h)) = (tg)u.h = (tgh^{-1})u. \end{array}\right\} \quad (4)$$

Finally,

$$\left.\begin{array}{l} a_\psi(gu.hu) = a_\psi(h^{-1}g) = (h^{-1}g)^\psi \\ a_\psi(gu).a_\psi(hu) = g^\psi u.h^\psi u = (h^{-\psi}g^\psi = (h^{-1}g)^\psi. \end{array}\right\} \quad (5)$$

$$\left.\begin{array}{l} d_t(gu.hu) = d_t(h^{-1}g) = h^{-1}g \\ d_t(gu).d_t(hu)) = (tg)u.(th)u = (th)^{-1}tg = h^{-1}g. \end{array}\right\} \quad (6)$$

Hence a_ψ and d_t are automorphisms. It is easy to see that

$$a_\psi \circ a_\phi = a_{\psi\phi} \text{ and } d_t \circ d_h = d_{ht}, \quad {a_\psi}^{-1} = a_{\psi^{-1}}, \quad {d_t}^{-1} = d_{t^{-1}}$$

hence $A = \{a_\psi \,|\, \psi \in AutG\}$ is a subgroup of $AutM(G,2)$ isomorphic to $Aut(G)$ and the set $D = \{d_t \,|\, t \in G\}$ is a subgroup of $AutM(G,2)$ isomorphic to G.

We have $a_{\psi^{-1}} d_t a_\psi(h) = h,\ a_{\psi^{-1}} d_t a_\psi(hu) = t^{\psi^{-1}} h = d_{t^{\psi^{-1}}}(hu)$. Hence $a_{\psi^{-1}} d_t a_\psi = d_{t^{\psi^{-1}}}$. Therefore $AD \simeq Hol(G)$. □

Let G be a generalized dihedral group, i.e. a group such that there exists an abelian subgroup $G_0 \triangleleft G$ of index 2 and $G = G_0 \cup G_0 v$, where $v \notin G_0,\ v^2 = 1;\ vgv = g^{-1}, \forall g \in G_0$.

In the Chein loop $M(G,2)$ we have an abelian subgroup

$$K = \{1, u, v, w = uv = vu\}$$

and $M(G,2) = G_0 K$. For any $\phi \in AutK = S_3$ we can define an automorphism of $M(G,2)$, which we denote by the same letter ϕ :

$$\phi(gx) = gx^\phi \quad \forall x \in K,\ g \in G_0.$$

We have the following result.

Theorem 1. *Let G be a group. If G is not a dihedral group, then the automorphism group of the corresponding Chein loop $M(G,2)$ is $Hol(G)$. If $G = G_0 \cup G_0 v$ is a dihedral group and G_0 is not a group of period 2, then $AutM(G,2) = Hol(G)S_3$.*

Proof. If G is not a dihedral group then G is a characteristic subloop of $M(G,2)$. Indeed, if for some $\phi \in AutM(G,2)$ and $x \in G$ we have $y = x^\phi \notin G$, then $y^2 = 1$ and $ygy = g^{-1}$, for any $g \in G$.

Let $G_0 = \{h \in G \,|\, h^\phi \in G\}$, then G_0 is a subgroup of index 2 of G and $G^\phi = G_0 \cup G_0 y$ is a dihedral group, a contradiction, since G and G^ϕ are isomorphic.

Let $\phi \in AutM(G,2)$ and choose $a_\psi \in A$ such that $\psi(g) = \phi(g), \forall g \in G$. Then $\tau = \phi a_\psi^{-1} \in Stab_{AutM(G,2)} G$. It is clear that $Stab_{AutM(G,2)} G = D$ and $AutM(G,2) = AD = Hol(G)$.

Let $G = G_0 \cup G_0 v$ be a dihedral group and $N_0 = \{x \in M(G,2) \,|\, x^2 \neq 1\}$, $N = \{x \in M(G,2) \,|\, [x, N_0] = 1\}$. It is obvious that $N^\phi = N$, for any $\phi \in AutM(M,2)$, and $N = G_0$ if G is not of period 2. As above we have $AD \subset AutM(G,2)$. If $\phi \in AutM(G,2)$, then $u^\phi = ga,\ v^\phi = hb$, where $g, h \in G_0,\ a, b \in K$. Note that $a \neq b$. Indeed, if $a = b$, then $(uv)^\phi = ga\dot{h}a = gh^{-1} \in G_0$, but $uv \notin G_0$ and G_0 is a characteristic subloop, a contradiction. Then there exists $\psi \in S_3$ such that $u^\psi = a,\ v^\psi = b$ and $\phi\psi^{-1} \in AD$. This means that $AutM(G,2) = ADS_3 = Hol(G)S_3$. □

Remark 1. It is easy to see that $Hol(G) = \mathcal{W}(G_0)$ is a Mikheev group with triality with respect to the action of S_3 and the corresponding loop is G_0 (see [2]).

References

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Received April 26, 2005
Revised October 14, 2005

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