

## A class of quasigroups associated with a cubic Pisot number

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### Abstract

In this paper idempotent medial quasigroups satisfying the identity  $(ab \cdot a)a = b$  are studied. An example are the complex numbers with multiplication defined by  $a \cdot b = (1 - q)a + qb$ , where  $q$  is a solution of  $q^3 - 2q^2 + q - 1 = 0$ . The positive root of this cubic equation can be viewed as a generalization of the golden ratio. It turns out that the quasigroups under consideration have many similar properties to the so-called golden section quasigroups.

### 1. Introduction

Let  $q \neq 0, 1$  be a complex number and define a binary operation on  $\mathbb{C}$  by  $a \cdot b = (1 - q)a + qb$ . It is known that  $(\mathbb{C}, \cdot)$  is an IM-quasigroup, i.e. satisfies the laws of *idempotency* and *mediality*:

$$a \cdot a = a, \tag{1}$$

$$ab \cdot cd = ac \cdot bd. \tag{2}$$

Immediate consequences are the identities known as *elasticity*, *left* and *right distributivity*:

$$ab \cdot a = a \cdot ba, \tag{3}$$

$$a \cdot bc = ab \cdot ac, \tag{4}$$

$$ab \cdot c = ac \cdot bc. \tag{5}$$

This quasigroup will be denoted by  $C(q)$ . For some special values of  $q$ , the quasigroup satisfies additional identities. If  $q = \frac{1+\sqrt{5}}{2}$  is the golden ratio,

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$C(q)$  is a representative example of the *golden section* or *GS-quasigroups*. GS-quasigroups were defined in [8] as idempotent quasigroups satisfying the (equivalent) identities  $a(ab \cdot c) \cdot c = b$ ,  $a \cdot (a \cdot bc)c = b$ ; see also [2], [3], [4] and [10]. An alternative definition would be as IM-quasigroups with the simpler identity  $a(ab \cdot b) = b$ . In this paper we study IM-quasigroups satisfying a similar identity:

$$(ab \cdot a)a = b. \quad (6)$$

Representative examples are the quasigroups  $C(q)$  with  $q$  a root of  $q^3 - 2q^2 + q - 1 = 0$ . Denote by  $r_{1,2} = \sqrt[3]{\frac{25 \pm \sqrt{69}}{2}}$ . The roots of this cubic equation are  $q_1 = \frac{1}{3}(2 + r_1 + r_2) \approx 1.755$  and  $q_{2,3} = \frac{1}{6}(4 - r_1 - r_2 \pm i\sqrt{3}(r_1 - r_2)) \approx 0.123 \pm 0.745i$ . The number  $q_1$  is a Pisot number, i.e. an algebraic integer greater than 1 whose algebraic conjugates  $q_{2,3}$  have absolute values less than 1. This number was considered in [5] as a generalization of the golden ratio and was called the *second upper golden ratio*. Therefore, we will refer to IM-quasigroups satisfying the identity (6) as *G<sub>2</sub>-quasigroups*.

In the context of [5], the *second lower golden ratio* was the positive root of  $p^3 - p - 1 = 0$ . This is the smallest Pisot number  $p_1 \approx 1.325$ ; note that  $q_1 = p_1^2$ . For more details about Pisot numbers see [1].

In this paper it is shown that G<sub>2</sub>-quasigroups have many properties similar to those of GS-quasigroups. For example, they allow a simple definition of parallelograms using an explicit formula for the fourth vertex. In the last section G<sub>2</sub>-quasigroups are characterized in terms of Abelian groups with a certain type of automorphism.

## 2. Basic properties and further identities

The following lemma will be used quite often.

**Lemma 2.1.** *In an IM-quasigroup, identity (6) is equivalent with either of the identities*

$$(a \cdot ba)a = b, \quad (7)$$

$$a(ba \cdot a) = b. \quad (8)$$

*Proof.* By using elasticity we get  $(ab \cdot a)a \stackrel{(3)}{=} (a \cdot ba)a \stackrel{(3)}{=} a(ba \cdot a)$ .  $\square$

Note that the equivalence holds even in a groupoid satisfying (1) and (2). Elasticity follows directly from idempotency and mediality, without using

solvability or cancellativity. Consequently, the definition of  $G_2$ -quasigroups can be relaxed to the identities alone.

**Proposition 2.2.** *Any groupoid satisfying (1), (2) and (6) is necessarily a quasigroup.*

*Proof.* Given  $a$  and  $b$  define  $x = ab \cdot a$  and  $y = ba \cdot a$ . From (6) and (8) we see that  $xa = b$  and  $ay = b$ , i.e. the groupoid is left and right solvable. Now assume  $ax_1 = ax_2$  and  $y_1a = y_2a$ . Then,  $x_1 \stackrel{(6)}{=} (ax_1 \cdot a)a = (ax_2 \cdot a)a \stackrel{(6)}{=} x_2$  and  $y_1 \stackrel{(8)}{=} a(y_1a \cdot a) = a(y_2a \cdot a) \stackrel{(8)}{=} y_2$ , so the groupoid is left and right cancellative.  $\square$

The next proposition is similar to [8, Theorem 5].

**Proposition 2.3.** *In a  $G_2$ -quasigroup, any two of the equalities  $ab = c$ ,  $ca = d$  and  $da = b$  imply the third.*

*Proof.* Denote the equalities by (i), (ii) and (iii), respectively. Then we have:

$$\begin{aligned} (i), (ii) \Rightarrow (iii) : \quad da &\stackrel{(ii)}{=} ca \cdot a \stackrel{(i)}{=} (ab \cdot a)a \stackrel{(6)}{=} b, \\ (i), (iii) \Rightarrow (ii) : \quad ca &\stackrel{(i)}{=} ab \cdot a \stackrel{(iii)}{=} (a \cdot da)a \stackrel{(7)}{=} d, \\ (ii), (iii) \Rightarrow (i) : \quad ab &\stackrel{(iii)}{=} a \cdot da \stackrel{(ii)}{=} a(ca \cdot a) \stackrel{(8)}{=} c. \end{aligned}$$

$\square$

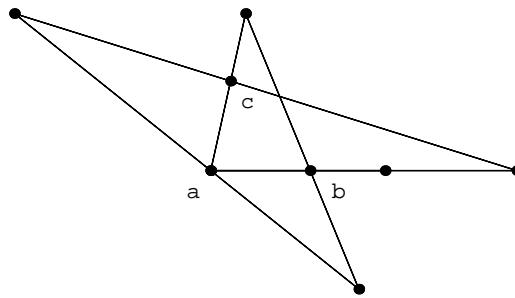


Figure 1: Identity (9) in the complex plane.

We list some more identities valid in  $G_2$ -quasigroups. They are accompanied by pictures illustrating the example of the complex plane with multiplication defined by  $a \cdot b = (1 - q_1)a + q_1b$ .

**Proposition 2.4.** *The following identity holds in any  $G_2$ -quasigroup:*

$$(a \cdot ab)c \cdot a = ac \cdot b. \quad (9)$$

*Proof.*  $(a \cdot ab)c \cdot a \stackrel{(5)}{=} (a \cdot ab)a \cdot ca \stackrel{(5)}{=} (a \cdot ab)(ca) \cdot (a \cdot ca) \stackrel{(3)}{=} (a \cdot ab)(ca) \cdot (ac \cdot a) \stackrel{(2)}{=} (ac)(ab \cdot a) \cdot (ac \cdot a) \stackrel{(4)}{=} ac \cdot (ab \cdot a)a \stackrel{(6)}{=} ac \cdot b. \quad \square$

**Proposition 2.5.** *The following identity holds in any  $G_2$ -quasigroup:*

$$(ab \cdot a)c \cdot b = (ab \cdot c)a. \quad (10)$$

*Proof.*  $(ab \cdot a)c \cdot b \stackrel{(5)}{=} (ab \cdot b)(ab) \cdot cb \stackrel{(3)}{=} (ab)(b \cdot ab) \cdot cb \stackrel{(2)}{=} (ab \cdot c) \cdot (b \cdot ab)b \stackrel{(7)}{=} (ab \cdot c)a. \quad \square$

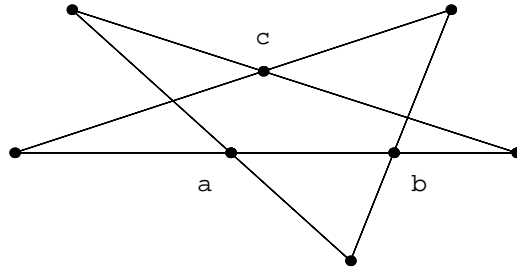


Figure 2: Identity (10) in the complex plane.

**Proposition 2.6.** *The following identity holds in any  $G_2$ -quasigroup:*

$$a \cdot (ba \cdot c)d = b(ac \cdot d). \quad (11)$$

*Proof.*  $a \cdot (ba \cdot c)d \stackrel{(5)}{=} a \cdot (ba \cdot d)(cd) \stackrel{(4)}{=} (a \cdot ba)(ad) \cdot (a \cdot cd) \stackrel{(2)}{=} (a \cdot ba)a \cdot (ad \cdot cd) \stackrel{(7)}{=} b(ad \cdot cd) \stackrel{(5)}{=} b(ac \cdot d). \quad \square$

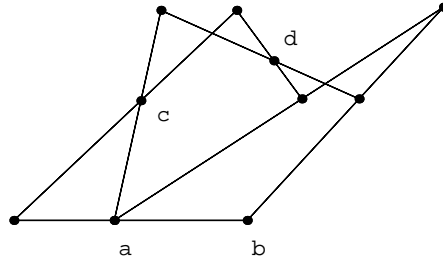


Figure 3: Identity (11) in the complex plane.

### 3. Parallelograms and other geometric concepts

The points  $a, b, c, d$  of a medial quasigroup are said to form a *parallelogram*, denoted by  $\text{Par}(a, b, c, d)$ , if there are points  $p, q$  such that  $pa = qb$  and  $pd = qc$ . In [7] it was proved that this relation satisfies the axioms of *parallelogram space*:

1. For any three points  $a, b, c$  there is a unique point  $d$  such that  $\text{Par}(a, b, c, d)$ .
2.  $\text{Par}(a, b, c, d)$  implies  $\text{Par}(e, f, g, h)$ , where  $(e, f, g, h)$  is any cyclic permutation of  $(a, b, c, d)$  or  $(d, c, b, a)$ .
3.  $\text{Par}(a, b, c, d)$  and  $\text{Par}(c, d, e, f)$  imply  $\text{Par}(a, b, f, e)$ .

In an IM-quasigroup, the unique point  $d$  of axiom 1 satisfies the following equation [9, Theorem 12]:

$$ab \cdot dc = ac. \tag{12}$$

This equation can be explicitly solved for  $d$  in GS-quasigroups:  $d = a \cdot b(ca \cdot a)$  [8, Theorem 6]. Here we prove a similar result for  $G_2$ -quasigroups.

**Proposition 3.1.** *In a  $G_2$ -quasigroup, for any  $a, b, c$  we have*

$$\text{Par}(a, b, c, (ba \cdot cb)b).$$

*Proof.* By substituting  $d = (ba \cdot cb)b$  into the equation (12) we get

$$ab \cdot [(ba \cdot cb)b \cdot c] = ac.$$

It suffices to show that this is a valid identity in any  $G_2$ -quasigroup:

$$\begin{aligned} ab \cdot [(ba \cdot cb)b \cdot c] &\stackrel{(5)}{=} ab \cdot [(ba \cdot c)(cb \cdot c) \cdot bc] \stackrel{(2)}{=} ab \cdot [(ba \cdot c)b \cdot (cb \cdot c)c] = \\ &\stackrel{(6)}{=} ab \cdot [(ba \cdot c)b \cdot b] \stackrel{(5)}{=} ab \cdot [(ba \cdot b) \cdot cb]b \stackrel{(5)}{=} ab \cdot [(ba \cdot b)b \cdot (cb \cdot b)] = \\ &\stackrel{(6)}{=} ab \cdot a(cb \cdot b) \stackrel{(4)}{=} a \cdot b(cb \cdot b) \stackrel{(8)}{=} ac. \end{aligned}$$

□

Now we have a direct definition of parallelograms in  $G_2$ -quasigroups, without using auxiliary points:

$$\text{Par}(a, b, c, d) \iff d = (ba \cdot cb)b. \quad (13)$$

Using the parallelogram relation geometric concepts such as midpoints, vectors and translations can be introduced. Of course, in the special case of the quasigroups  $C(q)$  the concepts agree with the usual definitions of plane geometry. Thus, geometric theorems can be proved by formal calculations in a quasigroup. We give an example particular to  $G_2$ -quasigroups (Theorem 3.4).

In any medial quasigroup,  $b$  is said to be the *midpoint* of the pair of points  $a, c$  if  $\text{Par}(a, b, c, b)$  holds. This is denoted by  $M(a, b, c)$ . The following proposition provides a characterization in  $G_2$ -quasigroups.

**Proposition 3.2.** *In a  $G_2$ -quasigroup,  $M(a, b, c)$  is equivalent with*

$$c = (ab \cdot ba)a. \quad (14)$$

*Proof.* By axiom 2 of parallelogram spaces,  $M(a, b, c)$  is equivalent with  $\text{Par}(b, a, b, c)$ , and the claim follows from (13). □

To facilitate notation, we introduce a new binary operation:

$$a * b = (ba \cdot a)b. \quad (15)$$

Starting from the quasigroup  $C(q_1)$ , this defines the binary operation in the quasigroup  $C(p_1)$ , i.e.  $a * b = (1 - p_1)a + p_1b$ . If  $ab = c$  (resp.  $a * b = c$ ), we say that  $b$  *divides the pair of points  $a, c$  in the second upper (resp. lower) golden ratio*. Here are some properties of the new binary operation. It is assumed that the original binary operation has higher priority than ‘\*’, e.g.  $a * bc$  means  $a * (bc)$ .

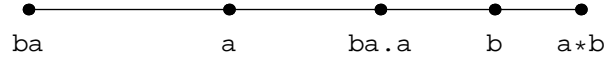


Figure 4: A new binary operation defined by (15).

**Lemma 3.3.** *The operation defined by (15) in a  $G_2$ -quasigroup satisfies the following identities:*

$$a * a = a, \tag{16}$$

$$ab * cd = (a * c)(b * d), \tag{17}$$

$$(a * (a * b)c) = b. \tag{18}$$

*Proof.* Idempotency of the new operation (16) follows directly from (1). Identity (17) follows by repeated application of mediality:

$$\begin{aligned} ab * cd &\stackrel{(15)}{=} (cd \cdot ab)(ab) \cdot cd \stackrel{(2)}{=} (ca \cdot db)(ab) \cdot cd \stackrel{(2)}{=} (ca \cdot a)(db \cdot b) \cdot cd = \\ &\stackrel{(2)}{=} (ca \cdot a)c \cdot (db \cdot b)d \stackrel{(15)}{=} (a * c)(b * d). \end{aligned}$$

Here is the proof of identity (18):

$$\begin{aligned} (a * (a * b)c)c &\stackrel{(15)}{=} \{[(ba \cdot a)b \cdot c]a \cdot a\}[(ba \cdot a)b \cdot c] \cdot c = \\ &\stackrel{(2)}{=} \{[(ba \cdot a)b \cdot c]a \cdot (ba \cdot a)b\}(ac) \cdot c = \\ &\stackrel{(2)}{=} \{[(ba \cdot a)b \cdot c](ba \cdot a) \cdot ab\}(ac) \cdot c = \\ &\stackrel{(2)}{=} \{[(ba \cdot a)b \cdot ba](ca) \cdot ab\}(ac) \cdot c = \\ &\stackrel{(5)}{=} \{[(ba \cdot b)(ab) \cdot ba](ca) \cdot ab\}(ac) \cdot c = \\ &\stackrel{(2)}{=} \{[(ba \cdot b)b \cdot (ab \cdot a)](ca) \cdot ab\}(ac) \cdot c = \\ &\stackrel{(6)}{=} \{[a(ab \cdot a) \cdot ca](ab) \cdot ac\}c \stackrel{(2)}{=} \{[ac \cdot (ab \cdot a)a](ab) \cdot ac\}c = \\ &\stackrel{(6)}{=} [(ac \cdot b)(ab) \cdot ac]c \stackrel{(5)}{=} [(ac \cdot a)b \cdot ac]c \stackrel{(2)}{=} [(ac \cdot a)a \cdot bc]c = \\ &\stackrel{(6)}{=} (c \cdot bc)c \stackrel{(7)}{=} b. \end{aligned}$$

□

Identity (17) could be called *mutual mediality* of the two binary operations. By identifying two factors various kinds of distributivities follow:

$a * bc = (a * b)(a * c)$ ,  $a(b * c) = ab * ac$  and their right counterparts. Identity (18) is an analogue of the defining identity for GS-quasigroups [8]. It is used in the proof of the following theorem.

**Theorem 3.4.** *In a  $G_2$ -quasigroup, suppose that  $a * e = c$ ,  $a * f = b$  and  $cg = f$ . Then,  $bg = e$ . Furthermore, suppose  $M(a, h, g)$  and  $h * g = d$ . Then,  $dh = a$  and  $M(b, d, c)$ .*

*Proof.* The first claim follows by substitution:

$$bg = (a * f)g = (a * cg)g = (a * (a * e)g)g \stackrel{(18)}{=} e.$$

If, in addition,  $M(a, h, g)$  and  $h * g = d$  hold, we get  $g = (ah \cdot ha)a$  by (14), and the remaining claims follow by tedious, but straightforward computations:

$$\begin{aligned} dh &= (h * g)h = [h * (ah \cdot ha)a]h \stackrel{(15)}{=} \{[(ah \cdot ha)a \cdot h]h \cdot (ah \cdot ha)a\}h = \\ &\stackrel{(2)}{=} \{[(ah \cdot ha)a \cdot h](ah \cdot ha) \cdot ha\}h \stackrel{(2)}{=} \{[(ah \cdot ha)a \cdot ah](h \cdot ha) \cdot ha\}h = \\ &\stackrel{(5)}{=} \{[(ah \cdot a)(ha \cdot a) \cdot ah](h \cdot ha) \cdot ha\}h = \\ &\stackrel{(2)}{=} \{[(ah \cdot a)a \cdot (ha \cdot a)h](h \cdot ha) \cdot ha\}h = \\ &\stackrel{(6)}{=} \{[h \cdot (ha \cdot a)h](h \cdot ha) \cdot ha\}h \stackrel{(4)}{=} \{h[(ha \cdot a)h \cdot ha] \cdot ha\}h = \\ &\stackrel{(5)}{=} \{h[(ha \cdot h)(ah) \cdot ha] \cdot ha\}h \stackrel{(2)}{=} \{h[(ha \cdot h)h \cdot (ah \cdot a)] \cdot ha\}h = \\ &\stackrel{(6)}{=} [h \cdot a(ah \cdot a)](ha) \cdot h \stackrel{(4)}{=} h[a(ah \cdot a) \cdot a] \cdot h \stackrel{(3)}{=} h[a \cdot (ah \cdot a)a] \cdot h = \\ &\stackrel{(6)}{=} (h \cdot ah)h \stackrel{(7)}{=} a. \end{aligned}$$

To prove  $M(b, d, c)$ , we utilize (14) once more:

$$\begin{aligned} (bd \cdot db)b &\stackrel{(4)}{=} (bd \cdot d)(bd \cdot b) \cdot b \stackrel{(5)}{=} (bd \cdot d)b \cdot (bd \cdot b)b \stackrel{(6)}{=} (bd \cdot d)b \cdot d = \\ &= (bd \cdot d)b \cdot (h * g) \stackrel{(15)}{=} (bd \cdot d)b \cdot (gh \cdot h)g = \\ &\stackrel{(2)}{=} (bd \cdot d)(gh \cdot h) \cdot bg \stackrel{(2)}{=} (bd \cdot gh)(dh) \cdot bg = \\ &\stackrel{(2)}{=} (bg \cdot dh)(dh) \cdot bg = (ea \cdot a)e \stackrel{(15)}{=} a * e = c. \end{aligned}$$

□

In the special case of the quasigroup  $C(q_1)$ , Theorem 3.4 proves:



**Corollary 3.5.** *Let  $ABC$  be a triangle and suppose the points  $E$  and  $F$  divide  $\overline{AC}$  and  $\overline{AB}$  in the second lower golden ratio, respectively. Then the cevians  $\overline{BE}$  and  $\overline{CF}$  intersect in a point  $G$  that divides them in the second upper golden ratio. Furthermore, the midpoint  $H$  of  $\overline{AG}$  divides the third cevian  $\overline{AD}$  in the second upper golden ratio.*

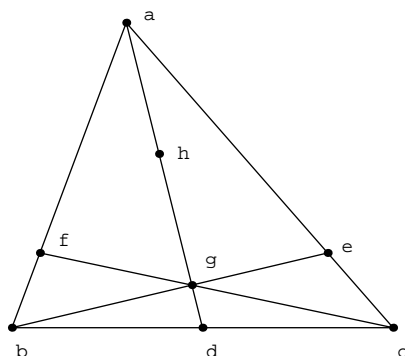


Figure 5: Geometric interpretation of Theorem 3.4.

The statement of Corollary 3.5 remains true if every instance of the second lower/upper golden ratio is replaced by the corresponding  $n$ -th golden ratio (for a definition see [5]). For  $n = 1$ , both the lower and the upper golden ratio are equal to  $\frac{1+\sqrt{5}}{2}$  and we get the geometric interpretation of [8, Theorem 15].

## 4. Representation theorems

Let  $(G, +)$  be an Abelian group with an automorphism  $\varphi$  such that the following equality holds for every  $x \in G$ :

$$\varphi^3(x) - 2\varphi^2(x) + \varphi(x) - x = 0. \quad (19)$$

Define another binary operation on  $G$  by the formula

$$a \cdot b = a + \varphi(b - a). \quad (20)$$

It is easy to verify that  $G$  is an IM-quasigroup with this new operation. Furthermore, the identity (6) follows from (19):

$$\begin{aligned}
(ab \cdot a)a &= ab \cdot a + \varphi(a) - \varphi(ab \cdot a) \\
&= ab + \varphi(a) - \varphi(ab) + \varphi(a) - \varphi(ab) - \varphi^2(a) + \varphi^2(ab) \\
&= 2\varphi(a) - \varphi^2(a) + ab - 2\varphi(ab) + \varphi^2(ab) \\
&= 2\varphi(a) - \varphi^2(a) + (id - 2\varphi + \varphi^2)(a + \varphi(b) - \varphi(a)) \\
&= [a - \varphi(a) + 2\varphi^2(a) - \varphi^3(a)] + [\varphi^3(b) - 2\varphi^2(b) + \varphi(b) - b] + b \\
&\stackrel{(19)}{=} b.
\end{aligned}$$

Therefore,  $(G, \cdot)$  is a  $G_2$ -quasigroup. The purpose of this section is to show that any  $G_2$ -quasigroup can be obtained in this way.

**Theorem 4.1.** *Let  $(G, \cdot)$  be a  $G_2$ -quasigroup. Choose an arbitrary  $o \in G$  and define a new binary operation on  $G$  by the formula*

$$a + b = (oa \cdot bo)o. \quad (21)$$

*Then,  $(G, +)$  is an Abelian group with neutral element  $o$ .*

*Proof.* We first prove associativity, commutativity and that  $o$  is the neutral element:

$$\begin{aligned}
(a + b) + c &\stackrel{(21)}{=} [o \cdot (oa \cdot bo)o](co) \cdot o \stackrel{(5)}{=} [o \cdot (oa \cdot bo)o]o \cdot (co \cdot o) = \\
&\stackrel{(7)}{=} (oa \cdot bo)(co \cdot o) \stackrel{(2)}{=} (ob \cdot ao)(co \cdot o) \stackrel{(2)}{=} (ob \cdot co)(ao \cdot o) = \\
&\stackrel{(7)}{=} [o \cdot (ob \cdot co)o]o \cdot (ao \cdot o) \stackrel{(5)}{=} [o \cdot (ob \cdot co)o](ao) \cdot o = \\
&\stackrel{(2)}{=} (oa)[(ob \cdot co)o \cdot o] \cdot o \stackrel{(21)}{=} a + (b + c), \\
a + b &\stackrel{(21)}{=} (oa \cdot bo)o \stackrel{(2)}{=} (ob \cdot ao)o \stackrel{(21)}{=} b + a, \\
a + o &\stackrel{(21)}{=} (oa \cdot oo)o \stackrel{(1)}{=} (oa \cdot o)o \stackrel{(6)}{=} a.
\end{aligned}$$

For any  $a \in G$  define  $-a = o \cdot (o \cdot oa)a$ . This is the inverse of  $a$ :

$$\begin{aligned}
a + (-a) &\stackrel{(21)}{=} \{oa \cdot [o \cdot (o \cdot oa)a]o\}o \stackrel{(5)}{=} (oa \cdot o)\{[o \cdot (o \cdot oa)a]o \cdot o\} = \\
&\stackrel{(6)}{=} (oa \cdot o) \cdot (o \cdot oa)a \stackrel{(2)}{=} (oa)(o \cdot oa) \cdot oa \stackrel{(7)}{=} o.
\end{aligned}$$

□

**Theorem 4.2.** *The mappings  $\varphi : x \mapsto ox$  and  $\psi : x \mapsto xo$  are automorphisms of the group  $(G, +)$  of Theorem 4.1 and satisfy the identity*

$$\psi(a) + \varphi(b) = ab. \tag{22}$$

*Proof.* The following shows that  $\varphi$  is an automorphism:

$$\begin{aligned} \varphi(a) + \varphi(b) &= oa + ob \stackrel{(21)}{=} (o \cdot oa)(ob \cdot o) \cdot o \stackrel{(3)}{=} (o \cdot oa)(o \cdot bo) \cdot o = \\ &\stackrel{(4)}{=} o(oa \cdot bo) \cdot o \stackrel{(3)}{=} o \cdot (oa \cdot bo)o \stackrel{(21)}{=} o(a + b) = \varphi(a + b). \end{aligned}$$

The proof that  $\psi$  is an automorphism is similar. Finally,

$$\begin{aligned} \psi(a) + \varphi(b) &= ao + ob \stackrel{(21)}{=} (o \cdot ao)(ob \cdot o) \cdot o \stackrel{(3)}{=} (o \cdot ao)(o \cdot bo) \cdot o = \\ &\stackrel{(4)}{=} o(ao \cdot bo) \cdot o \stackrel{(5)}{=} o(ab \cdot o) \cdot o \stackrel{(7)}{=} ab. \end{aligned}$$

□

**Theorem 4.3.** *Equations (19) and (20) are satisfied in the setting of the previous two theorems.*

*Proof.* As a special case of (22), we see that  $\psi(x) + \varphi(x) = xx \stackrel{(1)}{=} x$ , i.e.  $\psi(x) = x - \varphi(x)$ . Now equation (20) follows directly from (22):

$$ab = \psi(a) + \varphi(b) = a - \varphi(a) + \varphi(b) = a + \varphi(b - a).$$

To prove equation (19), note that

$$\psi^2(x) = \psi(x - \varphi(x)) = x - \varphi(x) - \varphi(x - \varphi(x)) = \varphi^2(x) - 2\varphi(x) + x.$$

Therefore,  $\varphi^3(x) - 2\varphi^2(x) + \varphi(x) = \varphi(\psi^2(x)) = o(xo \cdot o) \stackrel{(8)}{=} x$ . □

This is a direct proof of a  $G_2$ -version of Toyoda’s representation theorem for medial quasigroups [6].

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