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Factorization of simple groups involving the alternating group

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Abstract

In this paper we will find the structure of the finite simple groups G with two subgroups A and B such that $G = AB$, where A is a non-abelian simple group and B is isomorphic to the alternating group on seven letters.

1. Introduction

Let A and B be subgroups of a group G. If $G = AB$, then G is called a *factorizable group*. We also say G is the product of the two subgroups A and B, or G is a factorizable group. Since we always have the identity $G = AG$, hence in this paper we assume both factors A and B are proper subgroups of G and we say $G = AB$ a non-trivial factorization of G. If $G \cong A \times B$, then we call G a factorizable group as well. In [1] page 13 the question of finding all the factorizable groups is raised. Of course not all groups are factorizable, for example by [14] the Conway's simple group Co_2 of order $2^{18}.3^6.5^3.7.11.23$ is not a factorizable group. Similarly an infinite group whose proper subgroups are finite does not have a proper factorization. Therefore we always search for a special kind of factorization.

A factorization $G = AB$ is called *maximal* if both factors A and B are maximal subgroups of G. In [14] the authors found all the maximal factorization of all the finite simple groups and their automorphism groups. This special kind of factorization is useful because every factorization of a finite group is contained in a maximal factorization. In [2] simple groups G with factorization $G = AB$ and with the additional condition $(|A|, |B|) = 1$

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are determined. In this case we also have $A \cap B = 1$ the trivial group. A factorization $G = AB$ with the condition $A \cap B = 1$ is called an exact factorization. In [19] the authors found all the exact factorizations of the alternating and the symmetric groups. But later in [17] all the factorizations of the alternating and the symmetric groups were found where both factors are simple groups. Recently an interesting application of exact factorization is given in [9]. The authors show that an exact factorization of a finite group leads to the construction of a biperfect Hopf algebra, and then they find such a factorization for the Mathieu group M_{24}. This factorization is of the form $M_{24} = AB$, where $A \cong M_{23}$ and $B \cong 2^4 : \mathbb{A}_7$, both perfect groups.

Here we quote some results concerning the involvement of the alternating or symmetric groups in a factorization. In [13] all finite groups $G = AB$, $A \cong B \cong \mathbb{A}_5$ are classified and in [16] factorizable groups where one factor is a non-abelian simple group and the other factor is isomorphic to the alternating group on 5 letters are classified. In [18] factorizations of finite groups are classified in the case where one factor of a factorizable group is simple and the other factor is almost simple. In [5] all finite groups $G = AB$, where $A \cong \mathbb{A}_6$ and B is isomorphic to the symmetric group on $n \geqslant 5$ letters are determined. Also in [6] we determined the structure of a finite factorizable group with one factor a simple group and the other factor isomorphic to the symmetric group on 6 letters. In [7] we determined the structure of factorizable groups $G = AB$ where $A \cong \mathbb{A}_7$ and $B \cong \mathbb{S}_n$. Motivated by this paper here we will find the structure of the finite simple factorizable groups $G = AB$ such that A is a non-abelian simple group and $B \cong \mathbb{A}_7$, the symmetric group on seven letters. Through the paper all groups are assumed to be finite. Notations for the simple groups is taken from [4].

2. Preliminary results

In the following we quote two Lemmas from [18] which are useful when dealing with factorizable groups.

Lemma 1. *Let A and B be subgroups of a group G. The following statements are equivalent.*

(*a*) *$G = AB$.*

(*b*) *A acts transitively on the coset space $\Omega(G : B)$ of right coset of B in G.*

(*c*) *B acts transitively on the coset space $\Omega(G : A)$ of right coset of A in G.*

(d) $(\pi_A, \pi_B) = 1$, *where* π_A *and* π_B *are the permutation characters of* G *on* $\Omega(G : A)$ *and* $\Omega(G : B)$ *respectively.*

Lemma 2. *Let* G *be a permutation group on a set* Ω *of size* n. *Suppose the action of* G *on* Ω *is k-homogeneous,* $1 \leqslant k \leqslant n$. *If a subgroup* H *of* G *acts on* Ω *k-homogeneously, then* $G = G_{(\Delta)}H$, *where* Δ *is a k-subset of* Ω *and* $G_{(\Delta)}$ *denotes its global stabilizer.*

Now it is easy to determine the indices of subgroups of $\mathbb{A}_7$ and $\mathbb{S}_7$. If $H \leqslant \mathbb{A}_7$, then $[\mathbb{A}_7 : H]$ may be one of the following numbers: 1, 7, 15, 21, 35, 42, 70, 105, 120, 126, 140, 210, 252, 280, 315, 360, 420, 504, 630, 840, 1260 or 2520. And if $H \leqslant \mathbb{S}_7$, then $[\mathbb{S}_7 : H]$ is one of the following numbers: 1, 2, 7, 14, 21, 30, 35, 42, 70, 84, 105, 120, 126, 140, 210, 240, 252, 280, 315, 360, 420, 504, 560, 630, 720, 840, 1008, 1260, 1280, 2520 or 5040. Therefore if $\mathbb{A}_7$ (or $\mathbb{S}_7$) acts transitively on a set of size n, then $n = [\mathbb{A}_7 : H]$ (or $n = [\mathbb{S}_7 : H]$) is one of the above numbers. The action is faithful if and only if $n \neq 1$ in the case of $\mathbb{A}_7$ and $n \neq 1, 2$ in the case of $\mathbb{S}_7$. It is well-known that if $\mathbb{S}_7$ has a k-homogeneous (k-transitive) action, $k > 1$, on a set Ω, then $|\Omega| = 7$ and $2 \leqslant k \leqslant 7$, but for $\mathbb{A}_7$ we have the same result in addition with the 2-transitive action of $\mathbb{A}_7$ on 15 points, see [3]. Since we need factorizations of the alternating groups involving $\mathbb{S}_7$ or $\mathbb{A}_7$, hence using [14] we will prove the following results.

Lemma 3. *Let* $\mathbb{A}_m$ *denote the alternating group of degree* m. *If* $\mathbb{A}_m = AB$ *is a non-trivial factorization of* $\mathbb{A}_m$, A *a non-abelian simple subgroup of* $\mathbb{A}_m$ *and* $B \cong \mathbb{A}_7$, *then one of the following cases occurs:*

(a) $\mathbb{A}_m = \mathbb{A}_{m-1}\mathbb{A}_7$, *where* $m =$ 15, 21, 35, 42, 70, 105, 120, 126, 140, 210, 252, 280, 315, 360, 420, 504, 630, 840, 1260 *or* 2520.

(b) $\mathbb{A}_{15} = \mathbb{A}_{13}\mathbb{A}_7$,

(c) $\mathbb{A}_8 = L_2(7)\mathbb{A}_7$, $\mathbb{A}_9 = L_2(8)\mathbb{A}_7$, $\mathbb{A}_{11} = M_{11}\mathbb{A}_7$, $\mathbb{A}_{12} = M_{12}\mathbb{A}_7$.

Proof. It is obvious that m is at least 8. By [14] either $m = 6$, 8, 10 or one of A or B is k-homogeneous on m letters, $1 \leqslant k \leqslant 5$. Factorization of $\mathbb{A}_m$ if $m = 6$, 8 or 10 does not involve $\mathbb{A}_7$. Therefore we will consider the following cases.

CASE (i). $\mathbb{A}_{m-k} \trianglelefteq A \trianglelefteq \mathbb{S}_{m-k} \times \mathbb{S}_k$ for some k with $1 \leqslant k \leqslant 5$, and B k-homogeneous on m letters.

Since A is assumed to be simple we obtain $\mathbb{A}_{m-k} = 1$ or A. If $\mathbb{A}_{m-k} = 1$, then $m - k = 1$ or 2, hence $k = m - 1$ or $m - 2$. But then from $1 \leqslant k \leqslant 5$ we will obtain $2 \leqslant m \leqslant 6$ or $3 \leqslant m \leqslant 7$, a contradiction because $m \geqslant 8$.

Therefore $A = \mathbb{A}_{m-k}$ and $B \cong \mathbb{A}_7$ is k-homogeneous on m letters, $1 \leqslant k \leqslant 5$. If $k = 1$, then the size of the set Ω on which $\mathbb{A}_7$ can act transitively is as stated in the Lemma and all the factorizations in case (a) occur. If $k \geqslant 2$, then $m = 7$ or 15. If $m = 15$, then $\mathbb{A}_7$ has a transitive action on 15 letters and hence $\mathbb{A}_{15} = \mathbb{A}_{14}\mathbb{A}_7$ and $\mathbb{A}_{15} = \mathbb{A}_{13}\mathbb{A}_7$ which is case (b).

Case (ii). $\mathbb{A}_{m-k} \trianglelefteq B \leqslant \mathbb{S}_{m-k} \times \mathbb{S}_k$, $1 \leqslant k \leqslant 5$, and A is k-homogeneous on m letters.

Since $B \cong \mathbb{S}_7$, we obtain $m-k = 1$ or 7. From $1 \leqslant k \leqslant 5$ we get $2 \leqslant m \leqslant 6$ or $8 \leqslant m \leqslant 12$. Therefore only $m = 8$, 9, 10, 11 or 12 are possible which correspond to $k = 1$, 2, 3, 4, 5 respectively. But now from [3] and [12] for possible (m, k) we obtain:

$(m, k) = (8, 1)$, $\mathbb{A}_8 = L_2(7)\mathbb{A}_7$,
$(m, k) = (9, 2)$, $\mathbb{A}_9 = L_2(8)\mathbb{A}_7$,
$(m, k) = (11, 4)$, $\mathbb{A}_{11} = M_{11}\mathbb{A}_7$,
$(m, k) = (12, 5)$, $\mathbb{A}_{12} = M_{12}\mathbb{A}_7$,
and these are all the possibilities in (c) of the Lemma. □

Lemma 4. *Let $\mathbb{A}_m = AB$ be a non-trivial factorization of $\mathbb{A}_m$, where A and B are subgroups of $\mathbb{A}_m$ with A a non-abelian simple group and $B \cong \mathbb{S}_7$. Then*

- (a) $\mathbb{A}_m = \mathbb{A}_{m-1}\mathbb{S}_7$ *where* $m = 14$, 21, 30, 35, 42, 70, 84, 105, 120, 126, 140, 210, 240, 252, 280, 315, 360, 420, 504, 560, 630, 720, 840, 1008, 1260, 2520 *or* 5040.
- (b) $\mathbb{A}_9 = L_2(8)\mathbb{S}_7$, $\mathbb{A}_{11} = M_{11}\mathbb{S}_7$, $\mathbb{A}_{12} = M_{12}\mathbb{S}_7$.

Proof. In this case we have $m \geqslant 9$. Using [14] again we obtain the groups listed in (a) in case $B \cong \mathbb{S}_7$ is a k-homogeneous group on m letters. If the simple group A is k-homogeneous on $m-$letters again using [3] and [12] together with Lemma 2 we will obtain the groups listed in (b) and the Lemma is proved. □

Remark 1. The factorizations $A_m = AB$ in cases (a), (b) and (c) of Lemma 3 all occur because actually A_m has subgroups isomorphic to A and B. The same is true for case (b) of Lemma 4. But for case (a) of Lemma 4 the equality $A_m = A_{m-1}S_7$ happens only if A_m has a subgroup isomorphic to S_7.

3. Main result

To find the structure of the factorizable simple groups $G = AB$ with A simple and $B \cong \mathbb{A}_7$ we need to know about the primitive groups of certain degrees which are equal to the indices of subgroups of $\mathbb{A}_7$. Simple primitive groups of degree at most 1000 are given in [8] and the index of most of the subgroups of $\mathbb{A}_7$ are less than 1000 except two indices which are 1260 and 2520. Therefore first we deal with these indices.

Lemma 5. *Let G be a non-abelian simple group which is not an alternating group. If G is a primitive group of degree 1260 or 2520, then G does not have a factorization $G = AB$ with A simple and $B \cong \mathbb{A}_7$.*

Proof. By the classification Theorem for the finite simple groups, G is isomorphic either to a sporadic simple group or a simple group of Lie type. By [10] there is no factorization as mentioned in the Lemma for a sporadic group. Therefore we will assume that G is a simple group of Lie type. If the rank of G is 1 or 2, then by [11] no desired factorization occurs. Hence we will assume that the Lie rank of G is at least 3. We will use results on the minimum index of a subgroup of a simple group of Lie type.

Case (a). $G = L_n(q)$, $n \geqslant 4$. In this case the minimum index of a proper subgroup of G is $\frac{(q^n-1)}{(q-1)}$. If $\frac{(q^n-1)}{(q-1)} \leqslant 2520$, then calculations reveal the following possibilities for G: $L_4(2)$, $L_4(3)$, $L_4(4)$, $L_4(5)$, $L_4(7)$, $L_4(8)$, $L_4(9)$, $L_4(11)$, $L_4(13)$, $L_5(2)$, $L_5(3)$, $L_5(4)$, $L_5(5)$, $L_5(7)$, $L_6(2)$, $L_6(3)$, $L_6(4)$, $L_7(2)$, $L_7(3)$, $L_8(2)$, $L_9(2)$, $L_{10}(2)$ or $L_{11}(2)$.

By [15], Proposition 4.8, the groups $L_4(q)$ with $q \not\equiv 1(8)$ are ruled out because they cannot have $\mathbb{A}_7$ in their factorization. Therefore among the possibilities of the form $L_4(q)$ only $L_4(9)$ needs examination. Assume $L_4(9) = A\mathbb{A}_7$ where A is a simple non-abelian group. Therefore $|A| = 2^7.3^{10}.5.13.41\ |A \cap \mathbb{A}_7|$. Since $A \cap \mathbb{A}_7$ is a proper subgroup of $\mathbb{A}_7$, hence $|A \cap \mathbb{A}_7|$ is one of the numbers: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 18, 20, 21, 24, 36, 60, 72, 120, 168 or 360. But by inspecting the simple groups A whose orders do not exceed $|L_4(9)|$ (at the end of [4]) with $13, 41 \mid |A|$, we find only one possibility for A, namely $A = O_8^-(3)$ of order $2^{10}.3^{12}.5.7.13.41$. But then we must have $|A \cap \mathbb{A}_7| = 2^3.3^2.7 = 504$ which is not the case. Therefore all the possibilities $L_4(q)$ are ruled out.

For the groups $L_5(q)$, again by [15], Proposition 4.7, if $q \equiv 3\ (4)$ there is no such factorization as mentioned in the Lemma. Hence the groups $L_5(3)$ and $L_5(7)$ are ruled out. For the groups $L_5(2)$, $L_5(4)$ and $L_5(5)$

similar arguments as used above rule out any factorization of these groups involving $\mathbb{A}_7$ and a simple subgroup. Factorization of the remaining groups in this case involving $\mathbb{A}_7$ are ruled out similarly and we omit the details.

Case (b). $G = U_n(q)$, $n \geqslant 6$. In this case a proper subgroup has index at least $\frac{(q^n-(-1)^n)(q^{n-1}-(-1)^{n-1})}{(q^2-1)}$ and if this number is less than or equal to 2520 we obtain only $G = U_6(2)$. But by [4] the group $U_6(2)$ has no maximal subgroup of index 1260 or 2520.

Case (c). $G = S_{2m}(q)$, $m \geqslant 3$. In this case if $q > 2$, then the index of a proper subgroup of G is at least $\frac{(q^{2m}-1)}{(q-1)}$ and if $q = 2$ then this number is $2^m(2^m-1)$. For these numbers to be less than or equal to 2520 we will obtain the following groups: $S_6(2)$, $S_6(3)$, $S_6(4)$, $S_8(2)$, $S_{10}(2)$ or $S_{12}(2)$. Now using [4] we see that the groups $S_6(2)$, $S_6(3)$ and $S_8(2)$ do not have maximal subgroups of index 1260 or 2520. For the groups $S_6(4)$, $S_{10}(2)$ and $S_{12}(2)$ similar arguments as used in case (a) rule out the possibility of factorizing these groups as product of a simple group and a group isomorphic to $\mathbb{A}_7$.

Case (d). $G = O^{\epsilon}_{2m}(q)$, $m \geqslant 4$, $\epsilon = \pm$. In this case the index of a proper subgroup is at least $\frac{(q^m-1)(q^{m-1}+1)}{(q-1)}$ when $\epsilon = +$, and is at least $\frac{(q^m+1)(q^{m-1}-1)}{(q-1)}$ when $\epsilon = -$ except in the case $(q,\epsilon) = (2,+)$ where this index is at least $2^{m-1}(2^m-1)$. For $G = O_{2m+1}(q)$, $m \geqslant 3$, q odd, $q > 3$, the index of a proper subgroup is at least $\frac{(q^{2m}-1)}{(q-1)}$ and if $q = 3$, this index is at least $\frac{(q^{2m}-q^m)}{(q-1)}$. Again calculations show that if an index is less than or equal to 2520, then $G = O_7(3)$, $O^{\pm}_8(2)$, $O^{\pm}_8(3)$, $O^{\pm}_{10}(2)$ or $O^{\pm}_{12}(2)$. Now again using [4] we ruled out any factorization of these groups involving $\mathbb{A}_7$.

Case (e). Finally we may assume that G is an exceptional simple group of Lie type. In this case by [14] factorizations of G are known and none of them involves $\mathbb{A}_7$. The Lemma is proved now. □

Theorem 1. *Let $G = AB$ be a non-trivial factorization of a simple group G with A a simple non-abelian group and $B \cong \mathbb{A}_7$. Then one of the following occurs:*

(a) $G = \mathbb{A}_m = \mathbb{A}_{m-1}\mathbb{A}_7$, *where* $m = 15, 21, 35, 42, 70, 105, 120, 126, 140, 210, 252, 280, 315, 360, 420, 504, 630, 840, 1260$ *or* 2520.

(b) $G = \mathbb{A}_{15} = \mathbb{A}_{13}\mathbb{A}_7$

(c) $G = \mathbb{A}_8, \mathbb{A}_9, \mathbb{A}_{11}$ *or* $\mathbb{A}_{12}$ *with appropriate factorizations:* $\mathbb{A}_8 = L_2(7)\mathbb{A}_7$, $\mathbb{A}_9 = L_2(8)\mathbb{A}_7$, $\mathbb{A}_{11} = M_{11}\mathbb{A}_7$, $\mathbb{A}_{12} = M_{12}\mathbb{A}_7$

(d) $G = O^{+}_8(2) = S_6(2)\mathbb{A}_7$.

Proof. Suppose $G = AB$ is a factorization of a simple group G with A a simple non-abelian group and $B \cong \mathbb{A}_7$. We remind that by a factorization we mean a non-trivial factorization. If M is a maximal subgroup of G containing A, then $G = AB$, hence $[G : M] \mid [B : B \cap M]$. Since $d = [B : B \cap M]$ is equal to the index of a subgroup of $\mathbb{A}_7$, therefore G is a primitive permutation group of degree d. We have $d = 1$, 7, 15, 21, 35, 42, 70, 105, 120, 126, 140, 210, 252, 280, 315, 360, 420, 504, 630, 840, 1260 or 2520. Obviously $d \neq 1, 7$. By Lemma 5 if $d = 1260$ or 2520, then G is isomorphic to an alternating group of these degrees. If G is an alternating group, then by Lemma 3 we obtain the cases (a), (b) and (c) in the Theorem. We will prove if G is not an alternating group, then $G \cong O_8^+(2)$.

Since the remaining degrees d are less than 1000, hence we may use [8]. By Table I in [7] which is obtained from [8] we need only consider primitive simple groups G of degree 21, 105, 120, 126, 280, 315 and 840. Now using [10] and [11] the only cases that we should consider are $S_6(2)$, $S_8(2)$ or $O_8^+(2)$.

If $S_6(2) = A\mathbb{A}_7$, then $|A| = 2^6.3^2\, |A \cap \mathbb{A}_7|$. But $|A|$ must be divisible by at least three distinct primes. Therefore if $A \cap \mathbb{A}_7$ is a proper subgroup of $\mathbb{A}_7$ we must have $|A \cap \mathbb{A}_7| = 5$, 10, 20, 60, 360, 7, 21, 168. Hence $|A| = 2^6.3^2.5$, $2^7.3^2.5$, $2^8.3^2.5$, $2^9.3^3.5$, $2^9.3^4.5$, $2^6.3^2.7$, $2^6.3^3.7$, $2^9.3^3.7$. But by [4] there is no simple group of the above orders.

If $S_8(2) = A\mathbb{A}_7$, then $|A| = 2^{13}.3^3.5.17\, |A \cap \mathbb{A}_7|$. By [4] there is no simple group A such that $2^{13}.3^3.5.17 \mid |A| \mid |S_8(2)|$.

If $O_8^+(2) = A\mathbb{A}_7$, then $|A| = 2^9.3^3.5\, |A \cap \mathbb{A}_7|$. Now $2^9.3^3.5 \mid |A|$ and $|A| \mid 2^{12}.3^5.5^2.7 = |O_8^+(2)|$. By [4] the only possibility is $A \cong S_6(2)$. Again by [4] and using Lemma 1 we obtain $O_8^+(2) = S_6(2)\mathbb{A}_9$. The intersection of the two factors is a group $H = L_2(8) : 3 = P\Gamma L_2(8)$ and since it acts 2-transitively on 9 points we have $\mathbb{A}_9 = P\Gamma L_2(8).\mathbb{A}_7$, hence $O_8^+(2) = S_6(2)\mathbb{A}_7$ and the Theorem is proved. □

In conclusion we will prove the following Corollary.

Corollary. *Suppose that $G = AB$ with A a simple group and B isomorphic to $\mathbb{A}_7$. Then, either $G = A \supseteq B$, $G \cong A \times B$, or G is as in the Theorem* 1.

Proof. By induction, if G is not simple, G is not isomorphic to $A \times B$, and G is a minimal normal subgroup of G, then $\frac{G}{N}$ is simple. By lemma 1 of [17], $|N|$ divides the order of $\mathbb{A}_7$, $|N| = 8$ (which is impossible as $C(N) = N$ and hence, $\mathbb{A}_7$ is isomorphic to a subgroup of $Aut(N)$) or $|N| = p$ where p is a prime dividing $|\mathbb{A}_7|$ for which the Sylow subgroup is non-abelian. It follows that $p = 2$ and $N = Z(G)$. Thus, G is a covering group of the simple group $\frac{G}{N} = (\frac{AN}{N})(\frac{BN}{N})$ which is as in the Theorem 1. But this is impossible as theorem 10 of [17] shows that $\frac{G}{N}$ cannot be isomorphic to an alternating group and a simple order argument shows $\frac{G}{N}$ cannot be isomorphic to $O_8^+(2)$. The result follows. □

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