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On decomposable hyper BCK-algebras

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Abstract

In this manuscript, we introduce the concept of decomposable hyper BCK-algebras and we give a condition for a hyper BCK-algebra to be a decomposable hyper BCK-algebra. Moreover, we state and prove some theorems about (weak, implicative) strong hyper BCK-ideal of a decomposable hyper BCK-algebra. Finally, we give a characterization of some decomposable hyper BCK-algebras.

1. Introduction

The study of BCK-algebras was initiated by Y. Imai and K. Iséki [5] in 1966 as a generalization of the concept of set-theoretic difference and propositional calculus. Since then a great deal of literature has been produced on the theory of BCK-algebras. The hyperstructure theory(called also multialgebras) was introduced in 1934 by F. Marty [9] at the 8th congress of Scandinavian Mathematiciens. In [8], Y.B. Jun et al. applied the hyperstructures to BCK-algebras, and introduced the notion of a hyper BCK-algebra which is a generalization of BCK-algebra, and investigated some related properties. Now we follow [7] and [8] and introduce the concept of decomposable hyper BCK-algebra and give a condition for a hyper BCK-algebra to be a decomposable hyper BCK-algebra. Moreover, we state and prove some theorems about (weak, implicative) strong hyper BCK-ideal a of decomposable hyper BCK-algebra.

2. Preliminaries

Definition 2.1. [8] By a *hyper BCK-algebra* we mean a non-empty set H endowed with a hyperoperation "$\circ$" and a constant 0 satisfying the following

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axioms:

(HK1) $(x \circ z) \circ (y \circ z) \ll x \circ y$,
(HK2) $(x \circ y) \circ z = (x \circ z) \circ y$,
(HK3) $x \circ H \ll \{x\}$,
(HK4) $x \ll y$ and $y \ll x$ imply $x = y$

for all $x, y, z \in H$, where $x \ll y$ is defined by $0 \in x \circ y$ and for every $A, B \subseteq H$, $A \ll B$ is defined by $\forall a \in A$, $\exists b \in B$ such that $a \ll b$. In such case, we call "$\ll$" the *hyperorder* in H.

Theorem 2.2. [8] *In any hyper BCK-algebra H, the following hold:*

(i) $0 \circ 0 = \{0\}$,
(ii) $0 \ll x$,
(iii) $x \ll x$,
(iv) $A \ll A$,
(v) $A \subseteq B$ *implies* $A \ll B$,
(vi) $0 \circ x = \{0\}$,
(vii) $x \circ y \ll x$,
(viii) $x \circ 0 = \{x\}$,
(ix) $y \ll z$ *implies* $x \circ z \ll x \circ y$

for all $x, y, z \in H$ and for all non-empty subsets A and B of H.

Definition 2.3. Let I be a subset of a hyper BCK-algebra H and $0 \in I$. Then I is said to be a *week hyper BCK-ideal* of H if $x \circ y \subseteq I$ and $y \in I$ imply $x \in I$ for all $x, y \in H$, *hyper BCK-ideal* of H if $x \circ y \ll I$ and $y \in I$ imply $x \in I$ for all $x, y \in H$, *strong hyper BCK-ideal* if $(x \circ y) \cap I \neq \emptyset$ and $y \in I$ imply $x \in I$ for all $x, y \in H$, *reflexive hyper BCK-ideal* of H if I is a hyper BCK-ideal of H and $x \circ x \subseteq I$ for all $x \in H$.

Theorem 2.4. [6, 7, 8] *Let H be a hyper BCK-algebra. Then,*

(i) *any strong hyper BCK-ideal of H is a hyper BCK-ideal of H,*
(ii) *if I is a hyper BCK-ideal of H and A be a nonempty subset of H, then $A \ll I$ implies $A \subseteq I$,*
(iii) *H is a BCK-algebra if and only if $H = \{x \in H : x \circ x = \{0\}\}$.*

Definition 2.5. [3] Let H be a hyper BCK-algebra, Θ be an equivalence relation on H and $A, B \subseteq H$. Then,

(i) we write $A\Theta B$, if there exist $a \in A$ and $b \in B$ such that $a\Theta b$,
(ii) we write $A\bar{\Theta}B$, if for all $a \in A$ there exists $b \in B$ such that $a\Theta b$ and for all $b \in B$ there exists $a \in A$ such that $a\Theta b$,
(iii) Θ is called a *congruence relation* on H, if $x\Theta y$ and $x'\Theta y'$, then

$x \circ x^{'} \bar{\Theta} y \circ y^{'}$, for all $x, y \in H$,

(iv) Θ is called a *regular relation* on H if $x \circ y \Theta \{0\}$ and $y \circ x \Theta \{0\}$, then $x \Theta y$ for all $x, y \in H$.

Theorem 2.6. [3] *Let Θ and $\Theta^{'}$ are two regular congruence relations on H such that $[0]_{\Theta} = [0]_{\Theta^{'}}$. Then $\Theta = \Theta^{'}$.*

Theorem 2.7. [3] *Let Θ be a regular congruence relation on H and $H/\Theta = \{I_x : x \in H\}$, where $I_x = [x]_{\Theta}$, for all $x \in H$. Then $\frac{H}{\Theta}$ with hyperoperation $I_x \circ I_y = \{I_z : z \in x \circ y\}$ and hyper order $I_x < I_y \Longleftrightarrow I \in I_x \circ I_y$ is a hyper BCK-algebra which is called quotient hyper BCK-algebra.*

Theorem 2.8. [3] (Isomorphism Theorem) *Let Θ be a regular congruence relation on hyper BCK-algebra H. If $f : H \longrightarrow H^{'}$ is a homomorphism of hyper BCK-algebras such that $Kerf = [0]_{\Theta}$, then $H/\Theta \cong f(H)$.*

3. Decomposable hyper BCK-algebras

Definition 3.1. A hyper BCK-algebra H is called *decomposable* if there exists a nontrivial family $\{A_i\}_{i \in \Lambda}$ of hyper BCK-ideals of H such that

(i) $H \neq A_i \neq \{0\}$ for all $i \in \Lambda$,

(ii) $H = \bigcup_{i \in \Lambda} A_i$,

(iii) $A_i \cap A_j = \{0\}$ for all $i \neq j \in \Lambda$.

In this case, we say that $H = \bigcup_{i \in \Lambda} A_i$ is a decomposition of H and we write $H = \bigoplus_{i \in \Lambda} A_i$.

Example 3.2. (i) Let H be a hyper BCK-algebra with the following Cayley table:

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0, 1\}$	$\{1\}$
2	$\{2\}$	$\{2\}$	$\{0, 2\}$

It is easy to check that $A_1 = \{0, 1\}$ and $A_2 = \{0, 2\}$ are hyper BCK-ideals of H such that $H = A_1 \cup A_2$ and $A_1 \cap A_2 = \{0\}$. Therefore, H is decomposable.

(ii) Let $H = N \cup \{0\}$. Consider the hyperoperation

$$x \circ y = \begin{cases} \{0\} & \text{if } x = 0 \text{ or } x = y, \\ \{x\} & \text{otherwise.} \end{cases}$$

It is easily verified that $(H, \circ, 0)$ is a hyper BCK-algebra and $A_n = \{0, n\}$ is a hyper BCK-ideal of H, for all $n \in N$. Now, since $H = \bigcup_{n\in N} A_n$ and $A_n \cap A_m = \{0\}$, for each $n \neq m \in N$. Therefore, H is decomposable.

(iii) Let $N = \{0, 1, 2, 3, ...\}$ and hyper operation "$\circ$" on N is defined as follow:

$$x \circ y = \begin{cases} \{0, x\} & \text{if } x \leqslant y, \\ \{x\} & \text{if } x > y \end{cases}$$

for all $x, y \in H$. Then $(N, \circ, 0)$ is a hyper BCK-algebra but it is not a decomposable hyper BCK-algebra. Since every hyper BCK-ideal of H is equal to H or $\{0, 1, 2, ..., n-1\}$, for some $n \in N$.

Note. From now on, we let H be a hyper BCK-algebra.

Theorem 3.3. *Let H be decomposable with decomposition $H = \bigoplus_{i\in\Lambda} A_i$. Then A_i is a strong hyper BCK-ideal of H for all $i \in \Lambda$.*

Proof. Let $H = \bigoplus_{i\in\Lambda} A_i$ be a decomposition of H and let $(x \circ y) \cap A_i \neq \emptyset$ and $y \in A_i$ for $x \in H$ and $i \in \Lambda$. Then there exists $t \in x \circ y$ such that $t \in A_i$. From $x \in H = \bigcup_{i\in\Lambda} A_i$ we conclude that there exists $j \in \Lambda$ such that $x \in A_j$. Since $x \circ y \ll x \in A_j$, then $x \circ y \ll A_j$ and so by Theorem 2.4, $x \circ y \subseteq A_j$. Therefore, $t \in A_i \cap A_j$. Now, we consider the following two cases. If $j = i$, then $A_j = A_i$ and so $x \in A_i$. If $j \neq i$, then $t \in A_i \cap A_j = \{0\}$ that $t = 0$ and so $0 \in x \circ y$. This implies that $x \ll y$. It follow from $y \in A_i$ and Theorem 2.4 (ii) $x \in A_i$. Therefore, A_i is a strong hyper BCK-ideal of H. □

Theorem 3.4. *Let H be decomposable with decomposition $H = \bigoplus_{i\in\Lambda} A_i$. Then $A_i \cup A_j$ is a strong hyper BCK-ideal of H for all $i, j \in \Lambda$.*

Proof. Let $i, j \in \Lambda$ and $x, y \in H$ be such that $(x \circ y) \cap (A_i \cup A_j) \neq \emptyset$ and $y \in A_i \cup A_j$. Without loss of generality, assume that $y \in A_i$. Since $(x \circ y) \cap (A_i \cup A_j) \neq \emptyset$, then there exists $t \in H$ such that $t \in (x \circ y) \cap (A_i \cup A_j)$ and so $t \in A_i$ or $t \in A_j$. If $t \in A_i$, since A_i is a strong hyper BCK-ideal of H and $y \in A_i$, then $x \in A_i \subseteq A_i \cup A_j$. If $t \in A_j$, then by $x \in H = \bigcup_{i\in\Lambda} A_i$ there exists $k \in \Lambda$ such that $x \in A_k$. It follow from $x \circ y \leqslant x \in A_k$ and Theorem 2.4 (i,ii) that $x \circ y \ll A_k$ and so $x \circ y \subseteq A_k$. Hence we have $t \in A_j \cap A_k$. If $j = k$ then $A_j = A_k$ and so $x \in A_j \subseteq A_i \cup A_j$. If $j \neq k$, then $t \in A_j \cap A_k = \{0\}$ and so $t = 0$. Then $0 \in x \circ y$ and so $x \ll y$. Now, since $y \in A_i$ and A_i is a hyper BCK-ideal of H then $x \in A_i \subseteq A_i \cup A_j$. Therefore, $A_i \cup A_j$ is a strong hyper BCK-ideal of H. □

Theorem 3.5. *Let H be decomposable with decomposition $H = \bigoplus_{i\in\Lambda} A_i$. Then $\bigcup_{i\in\Omega} A_i$ is a strong hyper BCK-ideal of H for all $\emptyset \neq \Omega \subseteq \Lambda$.*

Proof. We proceed by induction on $|\Omega|$. For $\Omega \subseteq \Lambda$ with $|\Omega| = 1$ the result holds by Theorem 3.3. Suppose that for $2 \leqslant m \in N$ and all $\Omega \subseteq \Lambda$ with $|\Omega| \leqslant m$ the result hold and let $\Omega \subseteq \Lambda$ be such that $|\Omega| = m + 1$. Let i, j be arbitrary elements of Ω. Taking $A_{ij} = A_i \cup A_j$ and by using Theorems 3.4 and 2.4(i), we conclude that A_0 is a hyper BCK-ideal of H. Taking $\Omega' = (\Omega - \{i, j\}) \cup \{ij\}$ and by using the hypothesis of induction, we conclude that $\bigcup_{i\in\Omega'} A_i$ is a strong hyper BCK-ideal of H. Now, since $\bigcup_{i\in\Omega} A_i = \bigcup_{i\in\Omega'} A_i$ then $\bigcup_{i\in\Omega} A_i$ is a strong hyper BCK-ideal of H. Therefore for all $\emptyset \neq \Omega \subseteq \Lambda$, $\bigcup_{i\in\Omega} A_i$ is a strong hyper BCK-ideal of H. □

Corollary 3.6. *Let H be decomposable. Then there exist nontrivial strong hyper BCK-ideals A, B of H such that $H = A \cup B$ and $A \cap B = \{0\}$, that is $H = A \bigoplus B$.*

Proof. The proof come immediately from Theorem 3.5. □

Theorem 3.7. *Let H be a hyper BCK-algebra. Then H is decomposable if and only if there exists a nontrivial strong hyper BCK-ideal A of H such that $0 \notin (A' \circ B) \circ B$, where $A' = A - \{0\}$ and $B = H - A'$.*

Proof. ($\Longrightarrow$) Let H be decomposable. Then by Corollary 3.6 there exist nontrivial strong hyper BCK-ideals A and B of H such that $H = A \bigoplus B$. Let $0 \in (A' \circ B) \circ B$, by contrary. Since, $(A' \circ B) \circ B = \bigcup_{b\in B,\, t\in A'\circ B} t \circ b$, then there exist $t \in A' \circ B$ and $b \in B$ such that $0 \in t \circ b$. Now, since $b \in B$ and B is a strong hyper BCK-ideal of H, then $t \in B$. But, $t \in A' \circ B$ implies that there exist $a \in A'$ and $b_1 \in B$ such that $t \in a \circ b_1$ and so $a \circ b_1 \cap B \neq \emptyset$ and this implies that $a \in B$. Hence, $0 \neq a \in A \cap B = \{0\}$, which is impossible. Therefore, $0 \notin (A' \circ B) \circ B$.

($\Longleftarrow$) It is enough to prove that B is a hyper BCK-ideal of H. Let for $a, b \in H$, $a \circ b \ll B$ and $b \in B$ but $a \notin B$. Hence, $a \in A'$. Since $a \circ b \ll B$, then there exist $t \in a \circ b$ and $b_1 \in B$ such that $t \ll B_1$ and so $0 \in t \circ B_1$. Hence

$$0 \in t \circ b_1 \subseteq (a \circ b) \circ b' \subseteq (A' \circ B) \circ B$$

which is impossible. □

Theorem 3.8. *Let H be decomposable with decomposition $H = A \bigoplus B$. Then A and B are implicative hyper BCK-ideals of H if and only if for all $x, y \in H$ $x \circ (y \circ x) = \{0\}$ imply $x = 0$.*

Proof. Let A and B be implicative hyper BCK-ideals of H and $x \circ (x \circ y) = \{0\}$ for $x, y \in H$. Then $x \circ (y \circ x) \ll A$ and $x \circ (y \circ x) \ll B$ and so by Theorem 2.4 (iii), $x \in A \cap B = \{0\}$.

Conversely, let for $x, y \in H$, $x \circ (y \circ x) \ll A$ but $x \notin A$, by contrary. Hence, $0 \neq x \in B$. By Theorem 2.2 (vii), $x \circ (y \circ x) \ll x \in B$ and so by Theorem 2.4 (ii), $x \circ (y \circ x) \subseteq B$. On the other hand, since $x \circ (y \circ x) \ll A$ then by Theorem 2.4 (ii), $x \circ (y \circ x) \subseteq A$. Hence $x \circ (y \circ x) \subseteq A \cap B = \{0\}$ and so $x \circ (y \circ x) = \{0\}$. Now, by hypothesis $x = 0$, which is a contradiction. Therefore, $x \in A$ and so by Theorem 2.4 (iii) A is a implicative hyper BCK-ideal of H. The proof of case B is similar. □

Proposition 3.9. *Let H be decomposable with decomposition $H = A \bigoplus B$. If A and B are reflexive, then H is a BCK-algebra.*

Proof. Let A and B be reflexive. Then we have $x \circ x \subseteq A$ and $x \circ x \subseteq B$ for all $x \in H$. Hence $x \circ x \subseteq A \cap B = \{0\}$ and so $x \circ x = 0$. It follows from Theorem 2.4 (iv) that H is a BCK-algebra. □

Definition 3.10. Let $\emptyset \neq A \subset H$. Then subset I of H is called a *weak hyper BCK-ideal of H related to A* if

(r1) $0 \in I$,

(r2) $x \circ y \subseteq I$ and $y \in I$ imply $x \in I$ for all $x \in A$.

Note that, for all nonempty subset A of H if I is a weak hyper BCK-ideal of H, then I is a weak hyper BCK-ideal of H related to A. But the converse is not true in general.

Example 3.11. Consider a hyper BCK-algebra H with the following Cayley table:

$\circ$	0	1	2	3
0	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{0\}$	$\{0\}$
2	$\{2\}$	$\{1\}$	$\{0\}$	$\{0\}$
3	$\{3\}$	$\{3\}$	$\{3\}$	$\{0, 3\}$

Then $I = \{0, 2\}$ is a weak hyper BCK-ideal of H related to $A = \{0, 2, 3\}$. But, I is not a weak hyper BCK-ideal of H. Since $1 \circ 2 \subseteq I$ and $2 \in I$ but $1 \notin I$. □

Theorem 3.12. *Let H be decomposable with decomposition $H = A \bigoplus B$ and $I \subseteq A$. If I is a weak hyper BCK-ideal of H related to A, then I is a weak hyper BCK-ideal of H.*

Proof. Let I be a weak hyper BCK-ideal of H related to A and $x \circ y \subseteq I$ and $y \in I$, for $x, y \in H$. If $x \in A$, then by hypothesis $x \in I$. Now, let $x \in B$. Then by Theorem 2.2 (vii), $x \circ y \ll B$, which implies that $x \circ y \subseteq B$ by Theorem 2.4 (i,ii). Hence $x \circ y \subseteq A \cap B = \{0\}$, which implies that $x \circ y = \{0\}$ and so $x \ll y$. Since $y \in I \subseteq A$, we have $x \ll A$ and so by Theorem 2.4, we get $x \in A$. Thus $x \in A \cap B = \{0\}$. This implies that $x = 0$ and so $x \in I$. Therefore, I is a weak hyper BCK-ideal of H. □

Definition 3.13. Let $\emptyset \neq A \subset H$. Then subset I of H is called a *hyper BCK-ideal of H related to* A if

(r1) $0 \in I$,

(r3) $x \circ y \ll I$ and $y \in I$ imply $x \in I$ for all $x \in A$.

Note that, for all nonempty subset A of H if I is a hyper BCK-ideal of H, then I is a hyper BCK-ideal of H related to A. But the converse is not true in general.

Example 3.14. Let $J = \{0, 1\}$ and $B = \{0, 1, 3\}$ in Example 3.12. It is easy to show that J is a hyper BCK-ideal of H related to B, but J is not hyper BCK-ideal of H. Since $2 \circ 1 \ll J$ and $1 \in J$ but $2 \notin J$. □

Theorem 3.15. *Let H be decomposable with decomposition $H = A \bigoplus B$ and $I \subseteq A$. If I is a hyper BCK-ideal of H related to A, then I is a hyper BCK-ideal of H.*

Proof. The proof is similar to the proof of Theorem 3.12 by some modification. □

4. Quotient structure

Theorem 4.1. *Let H be decomposable with decomposition $H = A \bigoplus B$. Then there exists a regular congruence relation Θ on H and a hyper BCK-algebra X of order 2 such that $H/\Theta \cong X$.*

Proof. Let relation Θ on H is defined as follows:

$$x\Theta y \iff x, y \in A \text{ or } x, y \in B - \{0\}.$$

Since $H = A \bigoplus B$ is a decomposition of H, then it is easily verified that Θ is an equivalence relation on H. Now, let $x, y \in H$ such that $x\Theta y$. Then $x, y \in A$ or $x, y \in B - \{0\}$. Without loss of generality we can suppose that $x, y \in A$. It follow from Theorem 2.2 (vii) and Theorem 2.4 we get that

$x \circ a \subseteq A$ ($y \circ a \subseteq A$), which implies that $x \circ a \bar{\Theta} y \circ a$ for all $a \in H$. On the other hand by using Theorem 2.2 (vii) and Theorem 2.4 (i,ii), we get $a \circ x \subseteq A$ ($a \circ y \subseteq A$) if $a \in A$, and $a \circ x \subseteq B$ ($a \circ y \subseteq B$) if $a \in B$, for all $a \in H$ and so $a \circ x \overline{\Theta} a \circ y$. Hence Θ is a congruence relation on H. Now, let $x, y \in H$ such that $x \circ y \Theta \{0\}$ and $y \circ x \Theta \{0\}$. Then there exist $s \in x \circ y$ and $t \in y \circ x$ such that $s \Theta 0$ and $t \Theta 0$, which imply that $s, t \in A$. Hence, we have $(x \circ y) \cap A \neq \emptyset$ and $(y \circ x) \cap A \neq \emptyset$. Now, if $x \in A$ since $(y \circ x) \cap A \neq \emptyset$ and A is a strong hyper BCK-ideal of H then $y \in A$ and so $x \Theta y$.

Similarly, if $y \in A$, then we get that $x \in A$ and so $x \Theta y$. Now, remind only the case $x, y \in B - \{0\}$. But in this case by definition of Θ, we get that $x \Theta y$. Hence, Θ is a regular relation on H. Therefore, Θ is a regular congruence relation on H and so by Theorem 2.7, H/Θ is a hyper BCK-algebra. Now, it is easy to prove that $H/\Theta = \{[0]_\Theta = A, [b]_\Theta = B\}$, where $b \in B - \{0\}$. Hence $|H/\Theta| = 2$. Now, since we have only to hyper BCK-algebra $X = \{0, a\}$ of order 2 which are as follows:

$\circ_1$	0	a
0	$\{0\}$	$\{0\}$
a	$\{a\}$	$\{0\}$

$\circ_2$	0	a
0	$\{0\}$	$\{0\}$
a	$\{a\}$	$\{0, a\}$

Now, if $b \circ b = \{0\}$ then $[b]_\Theta \circ [b]_\Theta = \{[0]_\Theta\}$ and so $H/\Theta \cong (X, \circ_1)$ and if $b \circ b \neq \{0\}$ then $[b]_\Theta \circ [b]_\Theta = \{[0]_\Theta, [b]_\Theta\}$ and so $H/\Theta \cong (X, \circ_2)$. □

Theorem 4.2. *Let H be decomposable with decomposition $H = A \bigoplus B$ and let $b \circ x = b \circ y$ for all $b \in B$ and $x, y \in A$. Then there exists a regular congruence relation Γ on H such that $H/\Gamma \cong B$.*

Proof. Define the relation Γ on H as follows:

$$x \Gamma y \iff x, y \in A \quad \text{or} \quad x = y \notin A.$$

It is easy to prove that Γ is an equivalence relation on H. Let $x, y \in H$ be such that $x \Gamma y$. Then $x, y \in A$ or $x = y \notin A$.

Case 1. Let $x, y \in A$. Then by Theorem 2.2 (vii), $x \circ a \ll x$ ($y \circ a \ll y$) and so by Theorem 2.4, we get that $x \circ a \subseteq A$ ($y \circ a \subseteq A$), which implies that $x \circ a \bar{\Gamma} y \circ a$ for all $a \in H$. Now, we prove that $a \circ x \bar{\Gamma} a \circ y$, for all $a \in H$. If $a \in A$, the by the similar way in the above proof, we can show that $a \circ x \bar{\Gamma} a \circ y$. If $a \notin A$, then $a \in B$ and so by the hypothesis we have $a \circ x = a \circ y$, which implies that $a \circ x \bar{\Gamma} a \circ y$.

Case 2. Let $x = y \notin A$. Then $x \circ a = y \circ a$ and $a \circ x = a \circ y$ for all $a \in H$, which implies that $x \circ a \bar{\Gamma} y \circ a$ and $a \circ x \bar{\Gamma} a \circ y$ for all $a \in H$.

Therefore, Γ is a congruence relation on H. Now, let $x, y \in H$ such that $x \circ y\Gamma\{0\}$ and $y \circ x\Gamma\{0\}$. Then, there exist $s \in x \circ y$ and $t \in y \circ x$ such that $s\Gamma 0$ and $t\Gamma 0$ and so $s, t \in A$ and this implies that $(x \circ y) \cap A \neq \emptyset$ and $(y \circ x) \cap A \neq \emptyset$. Now, if $x \in A(y \in A)$, then since A is a strong hyper BCK-ideal of H, then $y \in A(x \in A)$, which implies that $x\Gamma y$. If $x, y \notin A$, then $x, y \in B - \{0\}$. Hence, by Theorem 2.2 (vii), $x \circ y \ll x$ $(y \circ x \ll y)$ and so by Theorem 2.4, $x \circ y \subseteq B$ $(y \circ x \subseteq B)$. So, $t, s \in A \cap B = \{0\}$ and this implies that $s = t = 0)$. Thus $x \ll y$ and $y \ll x$ and so $x = y$, which implies that $x\Gamma y$. Therefore, Γ is a regular congruence relation on H. Now, we define the function $f : H \longrightarrow H$ by

$$f(x) = \begin{cases} 0 & \text{if } x \in A, \\ x & \text{if } x \in B. \end{cases}$$

It follows from $A \cap B = \{0\}$, that f is well-defined. Now, let $x, y \in H$. We consider the following four cases:

Case 1. $x, y \in A$.

In this case, by Theorem 2.2 (vii), $x \circ y \ll x$ and so by Theorem 2.4, we get $x \circ y \subseteq A$. Hence,

$$f(x \circ y) = f(\bigcup_{t \in x \circ y} t) = \bigcup_{t \in x \circ y \subseteq A} \{f(t)\} = \{0\} = 0 \circ 0 = f(x) \circ f(y)$$

Case 2. $x, y \in B$.

Similar to the proof of Case 1, we get that $x \circ y \subseteq B$. Hence,

$$f(x \circ y) = f(\bigcup_{t \in x \circ y} t) = \bigcup_{t \in x \circ y \subseteq B} \{f(t)\} = \bigcup_{t \in x \circ y} \{t\} = x \circ y = f(x) \circ f(y)$$

Case 3. $x \in A$ and $y \in B - \{0\}$.

Similar to the proof of Case 1, we get that $x \circ y \subseteq A$ and so $f(x \circ y) = \{0\}$. On the other hand, since $f(x) = 0$, we have $f(x) \circ f(y) = 0 \circ y = \{0\}$. Hence

$$f(x \circ y) = f(x) \circ f(y)$$

Case 4 $x \in B - \{0\}$ and $y \in A$.

By hypothesis, we have $x \circ y = x \circ 0 = \{x\}$ and so

$$f(x \circ y) = \{f(x)\} = f(x) \circ 0 = f(x) \circ f(y)$$

Therefore, $f(x \circ y) = f(x) \circ f(y)$ for all $x, y \in H$ and so f is a homomorphism. It is easy to check that $\text{Ker} f = A = [0]_\Gamma$ and $f(H) = B$. Hence by Theorem 2.8, we have $H/\Gamma \cong B$. □

Corollary 4.3. *Let H be decomposable with decomposition $H = A \bigoplus B$ and let $b \circ x = b \circ y$ for all $b \in B$ and $x, y \in A$. Then $|B| = 2$.*

Proof. Let regular congruence relations Θ and Γ on H are as Theorems 4.1 and 4.2, respectively. Since $[0]_\Theta = A = [0]_\Gamma$, then by Theorem 2.6 that $\Theta = \Gamma$ and so $H/\Theta = H/\Gamma$. Now, by Theorem 4.1, $H/\Theta \cong X$, where X is a hyper BCK-algebra of order 2 and by Theorem 4.2, $H/\Gamma \cong B$. Hence,

$$X \cong H/\Theta = H/\Gamma \cong B$$

and so $|B| = |X| = 2$. □

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