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Fuzzy isomorphism and quotient of fuzzy subpolygroups

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Abstract

The aim of this note is the study of fuzzy isomorphism and quotient of fuzzy subpolygroups. In this regards first we introduce the notion of fuzzy isomorphism of fuzzy subpolygroups and then we study the quotient of fuzzy subpolygroups. Finally we obtain some related basic results.

1. Introduction

Hyperstructure theory was born in 1934 when Marty defined hypergroups, began to analyse their properties and applied them to groups, rational algebraic functions. Now they are widely studied from theoretical point of view and for their applications to many subjects of pure and applied properties and applied mathematics. In 1981 Ioulidis introduced the notion of *polygroup* as a hypergroup containing a scalar identity ([14]). Polygroups are studied in [5, 6] were connections with color schemes, relational algebras, finite permutation groups and Pasch geometry.

Following the introduction of fuzzy set by L. A. Zadeh in 1965 ([20]), the fuzzy set theory developed by Zadeh himself and others in mathematics and many applied areas. Rosenfeld in 1971 defined and studied the concept of a fuzzy subgroups [19]. Zahedi and others introduced and study the notion of fuzzy hyper-algebraic structures (for example see [1, 2, 3, 8, 11, 13, 21]). In this note by considering the notion of polygroups, first we introduce the

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notions of isomorphism, quotient and composition of fuzzy subpolygroups. Finally we study the relation of isomorphism and level subpolygroups.

2. Preliminaries

Let H be a nonempty set by $P_*(H)$ we mean the family of all nonempty subsets of H. A map $\cdot : H \times H \longrightarrow P_*(H)$ is called a *hyperoperation* or *join operation*. A *hypergroup* is a structure $(H, \cdot)$ that satisfies two axioms:

(Associativity) $a(bc) = (ab)c$ for all $a, b, c \in H$,

(Reproduction) $aH = H = Ha$ for all $a \in H$.

Let H be a hypergroup and K a nonempty subset of H. Then K is a *subhypergroup* of H if itself is a hypergroup under hyperoperation restricted to K. Hence it is clear that a subset K of H is a subhypergroup if and only if $aK = Ka = K$, under the hyperoperation on H (See [7]).

A hypergroup is called a *polygroup* if

(1) $\exists e \in H$ such that $e \circ x = x = x \circ e \forall x \in H$,

(2) $\forall x \in H$ there exists an unique element, say $x' \in H$ such that $e \in x \circ x' \cap x' \circ x$ (we denote x' by x^{-1}),

(3) $\forall x, y, z \in H,\ z \in xoy \implies x \circ y \implies x \in z \circ y^{-1} \implies y \in x^{-1} \circ z$.

A *canonical hypergroup* is a commutative polygroup. A nonempty subset A of a polygroup $(H, \cdot)$ is called a *subpolygroup* if $(A, \cdot)$ is itself a polygroup. In this case we write $A <_P H$. A subpolygroup A is called *normal* in H if

$$xNx^{-1} \subseteq N, \quad \forall x \in H.$$

In this case we write $N \lhd_P H$.

Lemma 2.1 [21]. *Let $A <_P H$. Then*

(1) $\forall a \in A\ Aa = aA = A$,

(2) $AA = A$,

(3) $(a^{-1})^{-1} = a$.

Lemma 2.2 [6]. *Let $N \lhd_P H$. Then*

(1) $Na = aN\ \forall a \in H$,

(2) $(Na)(Nb) = Nab$.

Let $A \leq_P H$, $x \in H$. Then Ax is called a *right coset* of A and we denote the set of all right costs of A in H by H/A, that is $H/A = \{Ax \mid x \in H\}$.

Define on H/A two hyperoperations:

$$Ax \circ Ay = \{Az \mid z \in Ax \cdot Ay\}, \qquad Ax \otimes Ay = \{Az \mid z \in xy\}.$$

Lemma 2.3. *Let H be a polygroup and A a normal subpolygroup of H. Then $(H/A, \otimes)$ and $(H/A, \circ)$ are polygroups, which are coincide together.*

Proof. Indeed, for $x, y \in H$, we have $xN \odot yN = \{zN \mid z \in xy\} = xyN = \bigcup_{z\in xy} zN = xyN = xN \otimes yN$. $\square$

Definition 2.4 [16]. Let H_1 and H_2 be two polygroups. A function $f : H_1 \longrightarrow H_2$ is called

(1) a *homomorphism* if $f(xy) \subseteq f(x)f(y)$,
(2) a *good homomorphism* if $f(xy) = f(x)f(y)$,
(3) a *homomorphism* of *type* 2, if $f^{-1}(f(x)f(y)) = f^{-1}f(xy)$,
(4) a *homomorphism* of *type* 3, if $f^{-1}(f(x)f(y)) = f^{-1}f(x)f^{-1}f(y)$,
(5) a *homomorphism* of *type* 4, if
$$f^{-1}(f(x)f(y)) = f^{-1}f(xy) = f^{-1}f(x)f^{-1}f(y),$$
(6) a *good isomorphism* if it is an isomorphism and good homomor– phism.

Proposition 2.5 [16]. *Every homomorphism (one-to-one homomorphism) of any of type* 1 *through* 4 *is a homomorphism (isomorphism).*

Definition 2.6. Let $(G, \cdot)$ be a group, $FS(G)$ the set of all fuzzy subset of G. Then $\mu \in FS(G)$ is a *fuzzy subgroup* of G if $\forall a, b \in G$ the following conditions are satisfied:

(i) $\mu(z) \geqslant min(\mu(x), \mu(y))$,
(ii) $\mu(x^{-1}) \geqslant \mu(x)$.

We denote the fuzzy subgroup μ by $\mu <_F G$.

Definition 2.7 [21]. Let $(H, \cdot)$ be a polygroup and $\mu \in FS(H)$. Then μ is a *fuzzy subpolygroup* of H if

(i) $\mu(z) \geqslant min(\mu(x), \mu(y))$, $\forall x, y \in H$ *and* $\forall z \in xy$,
(ii) $\mu(x^{-1}) \geqslant \mu(x)$.

In this case we write $\mu <_{FP} H$.

Definition 2.8. A fuzzy subpolygroup μ of H is called *fuzzy normal* if for every $x, y \in H$, $z \in xy$, $z' \in yx$ we have $\mu(z) = \mu(z')$. We denote this fact by $\mu \lhd_{FP} H$.

Lemma 2.9 [21]. *Let $\mu <_{FP} H$. Then*

(i) *$\mu(e) \geqslant \mu(x)$ for all $x \in H$,*
(ii) *$\mu(x^{-1}) = \mu(x)$ for all $x \in H$.*

Theorem 2.10 [21]. *Let μ be a fuzzy subset of H. Then $\mu <_{FP} H$ (resp. $\mu \lhd_{FP} H$) if and only if $\mu(e) \geqslant \mu(x)$ for all $x \in H$ and $\mu_t <_{FP} H$ (resp. $\mu_t \lhd_{FP} H$) for all $t \in [0, \mu(e)]$.*

Let $\mu \lhd_{FP} H$. Then we define fuzzy subset $x^{\hat{\mu}}$ by

$$x^{\hat{\mu}}(g) = \sup_{z \in x^{-1}g} \mu(z),$$

which is called a *fuzzy left coset* of μ. Similarly a fuzzy right coset, $\hat{\mu}_x$ of μ is defined.

Suppose that μ is a fuzzy subset of X. Then for $t \in [0,1]$ the *level subset* μ_t is defined by $\mu_t = \{x \in X \mid \mu(x) \geqslant t\}$. The *support* of μ, is defined by

$$Supp(\mu) = \{x \in H \mid \mu(x) > 0\}.$$

If G is a group and μ is a fuzzy subset of G, then we define μ^a as follows:

$$\mu^a = \{x \in G \mid \mu x = \mu a\}.$$

Also we define $a\mu^e$ and $\mu^a \mu^b$ by

$$a\mu^e = \{ax \mid x \in \mu^e\}, \quad \mu^a \mu^b = \{xy \mid x \in \mu^a, y \in \mu^b\}.$$

Theorem 2.11 [1]. *Let G be a group and μ be a fuzzy subset of G. Define $o_\mu : G \times G \longrightarrow P_*(G)$ by $ao_\mu b = \mu^a \mu^b$. Then o_μ is a hyperoperation on G. Moreover, if μ is a fuzzy normal subgroup of G, then (G, o_μ) is a polygroup.*

Extension Principal: Any function $f : X \longrightarrow Y$ induces two functions

$$f : FS(X) \longrightarrow FS(Y) \quad \text{and} \quad f^{-1} : FS(Y) \longrightarrow FS(X),$$

which are defined by

$$f(\mu)(y) = \sup\{\mu(x) \mid y = f(x)\}$$

for all $\mu \in FS(X)$, and

$$f^{-1}(\nu)(x) = \nu(f(x))$$

for all $\nu \in FS(Y)$.

3. Main results

In the sequel by H we mean a polygroup.

Theorem 3.1.

(*i*) *If $\mu <_{FP} H$, then $Supp(\mu) <_P H$.*

(*ii*) *If $\mu \lhd_{FP} H$, then $Supp(\mu) \lhd_P H$.*

Proof. It is easy to verify that $Supp(\mu) = \bigcup_{t\in Im(\mu)\setminus\{0\}} \mu_t$. Then, by Theorem 2.11 and the fact that the sets of level subsets of μ constitute a totally ordered set, $Supp(\mu)$ is a subpolygroup of H. □

Remark 3.2. The converse of Theorem 3.1 is not true. For example the set $H = \{e, a, b\}$ with the hyperoperation

$.$	e	a	b
e	e	a	b
a	a	e	b
b	b	b	$\{e, a\}$

is a polygroup. Define a fuzzy subset μ on H by $\mu(e) = 1$, $\mu(a) = 1/4$, $\mu(b) = 1/3$. Then μ is a fuzzy subpolygroup, but $\mu_{1/3} = \{e, b\}$ is not a subhypergroup of H, since $b \in \mu_{1/3}$, but $b \cdot b = \{e, a\} \not\subseteq \mu_{1/3}$. Thus, by Theorem 2.15, $\mu_{1/3}$ is not a subpolygroup of H. So, $\mu_{1/3}$ is not normal in H, but $supp(\mu) = H$ is a normal subpolygroup of H.

Theorem 3.3. *Let H be a fuzzy polygroup and $\mu <_{FP} H$. Then the set $\mathcal{I}_\mu = \{x^{\hat{\mu}} \mid x \in H\}$ with the hyperoperation $x^{\hat{\mu}} \cdot y^{\hat{\mu}} = \{z^{\hat{\mu}} \mid z \in xy\}$ is a polygroup.*

Proof. The associativity immediately follows from the associativity. Obviously $e^{\hat{\mu}}$ is the identity element. The inverse of $x^{\hat{\mu}}$ is $(x^{-1})^{\hat{\mu}}$. Now, if $x^{\hat{\mu}}, y^{\hat{\mu}}, z^{\hat{\mu}} \in \mathcal{I}_\mu$, then from $z^{\hat{\mu}} \in x^{\hat{\mu}} \cdot y^{\hat{\mu}}$ it is concluded that $z \in xy$. Thus $x \in zy^{-1}$ and hence $x^{\hat{\mu}} \in (x^{-1})^{\hat{\mu}} \cdot y^{\hat{\mu}}$. Therefore $\mathcal{I}_\mu$ is a polygroup. □

Definition 3.4. Let $\mu <_{FP} H$. Then μ is called *Abelian* if μ_t is Abelian (or a canonical hypergroup) for every $t \in [0, \mu(e)]$.

Theorem 3.5. *Let $\mu <_{FP} H$. Then μ is Abelian if and only if $Supp(\mu)$ is Abelian.*

Proof. Suppose that $Supp(\mu)$ is Abelian. Then for every $t \in (0, \mu(e)]$ we have $\mu_t \subseteq Supp(\mu)$. Thus μ_t is Abelian for every $t \in [0, \mu(e)]$. Therefore μ is Abelian.

Conversely, suppose that for every $t \in (0, \mu(e)]$, μ_t is Abelian. Let $a, b \in Supp(\mu)$. Thus there are μ_{t_1} and μ_{t_2} such that $a \in \mu_{t_1}$ and $b \in \mu_{t_2}$, $t_1, t_2 \in (0, \mu(e)]$. Suppose that $t_1 \leqslant t_2$, then $\mu_{t_2} \leqslant \mu_{t_1}$, and hence $a, b \in \mu_{t_1}$. Thus $ab = ba$. This complete the proof. □

Definition 3.6. Let H_1 and H_2 be polygroups. If $\mu <_{FP} H_1$ and $\nu <_{FP} H_2$, then a good isomorphism $f : Supp(\mu) \longrightarrow Supp(\nu)$ is called a *fuzzy good isomorphism* from μ to ν if there exists a positive real number k such that

$$\mu(x) = k\nu(f(x)), \quad \forall x \in Supp(\mu) \setminus \{e\}.$$

In this case we write $\mu \simeq \nu$ and say that μ and ν are isomorphic. It is clear that $\simeq$ is an equivalence on the set of all fuzzy subpolygroups of H.

Remark 3.7. Note that if two fuzzy polygroups are isomorphic it dose not imply that the underling polygroups are being isomorphic. For instance consider $S_3 = \{e, a, a^2, b, ab, a^2b\}$ and $Z_6 = \{\overline{0}, \overline{1}, \overline{2}, \overline{3}, \overline{4}, \overline{5}\}$. Define the fuzzy subsets μ and ν on S_3 and Z_6 respectively as follows:

$$\mu(e) = 1, \mu(a) = 1/2 = \mu(a^2), \mu(b) = 0 = \mu(ab) = \mu(b^2),$$
$$\nu(\overline{0}) = 1, \nu(\overline{2}) = 1/3 = \nu(\overline{4}), \nu(\overline{1}) = \nu(\overline{5}) = \nu(\overline{3}) = 0.$$

Then $(S_3, \circ_\mu)$ and $(Z_6, \circ_\nu)$ are polygroups by Theorem 2.14.

Now we define the mapping $f : Supp(\mu) \longrightarrow Supp(\nu)$ by $f(e) = \overline{0}$, $f(a) = \overline{2}$, $f(a^2) = \overline{4}$. It is easy to verify that $\mu \simeq \nu$, $\mu <_{FP} S_3$ and $\mu <_{FP} Z_6$. Thus $\mu \simeq \nu$, but $(S_3, o_\mu) \not\simeq (Z_6, o_\nu)$.

Theorem 3.8. Let $\mu <_{FP} H_1$ and $\nu <_{FP} H_2$. If $\mu \simeq \nu$, then μ is Abelian if and only if ν is Abelian.

Proof. Let μ be Abelian. We show that also ν is Abelian. By Theorem 3.6 it is enough we show that $Supp(\nu)$ is Abelian. Let $x, y \in Supp(\nu)$. Then there are $a, b \in Supp(\mu)$ such that $x = f(a)$ and $y = f(b)$. On the other hand by hypothesis there exists a positive number k such that

$$\mu(a) = k\nu(f(a)), \quad \mu(b) = k\nu(f(b)).$$

Since $k > 0$, then $\nu(f(a)) > 0$, so $\mu(a) > 0$, $\mu(b) > 0$ and, in the consequence $a, b \in Supp(\mu)$. Thus $ab = ba$ and $f(ab) = f(ba)$. Then $f(a)f(b) = f(b)f(a)$. Thus $xy = yx$. Therefore ν is Abelian.

Conversely, suppose that ν is Abelian. Let $a, b \in Supp(\mu)$. Then $f(a), f(b) \in Supp(\nu)$, henceforth $f(a)f(b) = f(b)f(a)$, that is $ab = ba$. Therefore ν is fuzzy Abelian. □

Theorem 3.9. *Let $\mu <_{FP} H_1$ and $\nu <_{FP} H_2$. If $\mu \simeq \nu$, then for every $t \in (0, \mu(e)]$ there exists an element $s \in (0, \nu(e)]$ such that $\mu_t \simeq \nu_s$.*

Proof. Let $f : Supp(\mu) \longrightarrow Supp(\nu)$ be a fuzzy isomorphism such that $\mu(x) = k\nu(f(x))$ for all $x \in Supp(\mu) \setminus \{e\}$ and for some positive real number k. Let $s = t/k$. Consider $g : \mu_t \longrightarrow \nu_s$, as the restriction of f to μ_t. Let $x \in \mu_t$, then $\mu(x) \geqslant t$, and hence $k\nu(f(x)) \geqslant t$. Thus $f(x) \in \nu_s$ and so g is well-defined. Clearly g is injective and $g(ab) = g(a)g(b)$, $\forall a, b \in \mu_t$. Now suppose that $y \in \nu_s$. Then $\nu(y) \geqslant s$. On the other hand there exists an element $x \in Supp(\mu)$ such that $y = f(x)$, thus $k\nu(f(x)) \geqslant t$, and hence $x \in \nu_t$. Therefore g is surjective and hence $\mu_t \simeq \nu_s$. □

Theorem 3.10. *Let* $\mu <_{FP} H_1,\ \nu <_{FP} H_2,\ \mu \simeq \nu$ *and* $\mu \lhd_{FP} supp(\mu)$. *Then* $\nu \lhd_{FP} supp(\nu)$.

Proof. We must prove that for all $x, y \in Supp(\nu)$ we have:

$$\nu(z) = \nu(z') \ \ \forall z \in xy,\ z' \in yx.$$

For $x, y \in Supp(\nu)$ there are $a, b \in Supp(\mu)$ such that $f(a) = x,\ f(b) = y$. Then $xy = f(ab)$ and $yx = f(ba)$. Now let $z \in xy = f(ab)$ and $z' \in yx = f(ba)$, thus there are $t, t' \in Supp(\nu)$ such that $z = f(t),\ z' = f(t')$, hence $t' \in ba$ and, by hypothesis, $\mu(t) = \mu(t')$. But we have $\mu(t) = k\nu(z)$ and $\mu(t') = k\nu(z')$. Thus $\nu(z) = \nu(z')$. Therefore μ is fuzzy normal. □

Definition 3.11. Let $\mu <_{FP} H_1$, $\nu <_{FP} H_2$ and $Supp(\mu) \subseteq Supp(\nu)$. We define the quotient of μ/ν as follows:

$$\mu/\nu : H/Supp(\nu) \longrightarrow [0, 1],$$
$$(\mu/\nu)(xSupp(\nu)) = Sup\{\mu(a) \mid aSupp(\nu) = xSupp(\nu)\}.$$

Remark 3.12. Note that in general $\mu_1/\nu = \mu_2/\nu$ dose not implies that $\mu_1 = \mu_2$. For example, consider the polygroup $H = \{e, a, b\}$ from Remark 3.2 and define the fuzzy subsets μ_1 and μ_2 on H as follows:

$$\mu_1(e) = 1,\ \mu_1(a) = 1/2,\ \mu_1(b) = 1/4,$$

and

$$\mu_2(e) = 1,\ \mu_2(a) = 1/3,\ \mu_2(b) = 1/4,$$

and

$$\nu(e) = 1,\ \nu(a) = 1/4,\ \nu(b) = 0.$$

Clearly $\mu_1, \mu_2 <_{FP} H,\ \nu \unlhd H$ and $\mu_1/\nu = \mu_2/\nu$, but $\mu_1 \neq \mu_2$.

Theorem 3.13. *If* $\mu <_{FP} H_1$, *then* $\mu/\mu_e \simeq \mu$ *and* $\mu/\mu \simeq \mu_e$, *where* $\mu_e(t) = \mu(e)$, *if* $t = e$ *and* 0, *otherwise.*

Proof. Define $f : Supp(\mu/\mu_e) \longrightarrow Supp(\mu)$ putting $f(xSupp(\mu_e)) = x$. Since $Supp(\mu_e) = \{e\}$ and $\mu/\mu_e(xSupp(\mu_e)) = \mu(x)$, then we conclude that f is a fuzzy isomorphism. Now define $g : Supp(\mu_e) \longrightarrow Supp(\mu/\mu)$, by $g(e) = Supp(\mu)$. Clearly $\mu(e) = (\mu/\mu)(g(e))$. Thus $\mu/\mu \simeq \mu_e$. □

Proposition 3.14. *Let H be a polygroup and N its normal subpolygroup. Then the map* $\phi_H : H \longrightarrow (H/N, \circ)$ *defined by* $\phi_H(x) = xN$ *is an onto homomorphism of type* 3.

Proof. Clearly ϕ_H is onto. In view of Definition 2.5 we must show that

$$\phi_H^{-1}(\phi_H(x)) \circ \phi_H(y)) = \phi_H^{-1}(\phi_H(x)) \circ \phi_H^{-1}(\phi_H(y)) \ \ \forall x, y \in H.$$

Let $t \in \phi_H^{-1}(\phi_H(x) \circ \phi_H(y))$, then $\phi_H(t) \in \phi_H(x) \circ \phi_H(y)$, yields $t \in xy$ by Lemma 2.4, and hence $t \in \phi_H^{-1}(\phi_H(x)) \circ \phi_H^{-1}(\phi_H(y))$.

Conversely, suppose that $z \in \phi_H^{-1}(\phi_H(x)) \circ \phi_H^{-1}(\phi_H(y)$. Then there exist $u \in \phi_H^{-1}(\phi_H(x))$ and $v \in \phi_H^{-1}(\phi_H(y))$ such that $z \in uv$. Thus $\phi_H(z) \subseteq \phi_H(uv) \subseteq \phi_H(u) \circ \phi_H(v) = \phi_H(x) \circ \phi_H(y)$. Therefore $z \in \phi_H^{-1}(\phi_H(x)) \circ \phi_H^{-1}(\phi_H(y))$. So, ϕ_H is a homomorphism of type 3. □

The map ϕ_H is called a *canonical epimorphism* and for simplicity will be denoted by ϕ.

Let μ be a fuzzy subpolygroup of H and N its normal subpolygroup. Then we can define on H/N the fuzzy set $\overline{\mu}$ putting

$$\overline{\mu}(z) = \sup_{xN=zN} \mu(x).$$

In fact, by the principal extension, we have $\overline{\mu} = \phi(\mu)$. So, from just proved results we conclude

Corollary 3.15. *Let $\mu <_{FP} H_1$, $\nu <_{FP} H_2$ and $Supp(\mu) \subseteq Supp(\nu)$. Then $\phi(\mu) = \mu/\nu$, where $\phi : H \longrightarrow H/Supp(\nu)$ is the canonical epimorphism.* □

The composition of fuzzy subpolygroups μ and ν of H is defined by

$$\mu\nu(x) = \sup_{x \in uv} min(\mu(u), \nu(v)).$$

Lemma 3.16. *If $\mu \leqslant_F PH$, then $\mu^2 = \mu$, and hence $\mu^n = \mu$.*

Proof. For every $x \in H$ we have $\mu^2(x) = \sup_{x \in uv} min(\mu(u), \mu(v)) \leqslant \mu(x)$, since μ is a fuzzy polygroup. On the other hand, $\mu^2(x) \geqslant min(\mu(x), \mu(e)) = \mu(x)$. Thus $\mu^2 = \mu$ and, by induction, $\mu^n = \mu$. □

Theorem 3.17. *Let $\mu \in FS(H)$. Then μ is a fuzzy subpolygroup of H if and only if $\mu^2 = \mu$ and $\mu(x) = \mu(x^{-1})$ for all $x \in H$.*

Proof. If μ is a fuzzy subpolygroup, then by Lemma 3.16 and Definition 2.8 we have $\mu(x) = \mu(x^{-1})$ for all $x \in H$.

Conversely, let $x \in uv$. Then by the hypothesis we have

$$\mu(x) = \mu^2(x) = \sup_{x \in uv} min(\mu(u), \mu(v)) \geqslant min(\mu(u), \mu(v)).$$

Thus μ is a fuzzy subpolygroup of H. □

Corollary 3.18. *If μ and ν are fuzzy subpolygroups of H and $\nu_t \trianglelefteq \mu_t$, for all $t \in Im(\mu)$, then $Supp(\mu) \trianglelefteq Supp(\nu)$.* □

Proposition 3.19. *Let μ and ν are fuzzy subpolygroups of H such that $\mu\nu = \nu\mu$. Then $\mu\nu$ is a fuzzy subpolygroup of H.*

Proof. First we show that $\mu\nu(x) = \mu\nu(x^{-1})$. Indeed,

$$\begin{aligned}\mu\nu(x) &= \sup_{x\in x_1x_2}(\mu(x_1)\mu(x_2)) = \sup_{x^{-1}\in x_2^{-1}x_1^{-1}} min(\mu(x_1),\nu(x_2))\\ &= \sup_{x^{-1}\in x_2^{-1}x_1^{-1}} min(\nu(x_1),\mu(x_2)) = \nu\mu(x^{-1}) = \mu\nu(x^{-1}).\end{aligned}$$

On the other hand $\mu\nu = \mu^2\nu^2 = \mu[(\mu\nu)\nu] = \mu[(\nu\mu)\nu] = (\mu\nu)(\mu\nu) = (\mu\nu)^2$. Then, by Theorem 3.17, $\mu\nu$ is a fuzzy subpolygroup. □

Proposition 3.20. *If $\mu \leqslant_{FP} H$ and $K \trianglelefteq H$. Define $\nu(x) = \mu(x)$, if $x \in K$ and $\nu(x) = 0$ otherwise. Then $\nu_t \trianglelefteq \mu_t$ for all $t \in (0, 1]$.*

Proof. We must show that $x\nu_t x^{-1} \subseteq \nu_t$, $\forall x \in \mu_t$ and $\forall t \in (0, 1]$.

Let $z \in xax^{-1} \subseteq x\nu_t x^{-1}$. If $a \notin K$, then $\nu(a) = 0 \geqslant t > 0$, which is a contradiction. Thus $a \in K$ and hence $\mu(a) = \nu(a) \geqslant t$.

If $a \in K$, then $\mu(a) = \nu(a)$ and $\mu(z) \geqslant min(\mu(x), \mu(a)) \geqslant t$. Hence $\nu(z) \geqslant t$, i.e. $z \in \nu_t$. Therefore $\nu_t \trianglelefteq \mu_t$. □

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