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Affine regular pentagons in GS–quasigroups

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Abstract

The "geometric" concept of affine regular pentagon and affine regular star–shaped pentagon in general GS–quasigroup will be introduced. Some characteristics of the introduced concepts will be proved and the geometric interpretation in the GS–quasigroup $C(\frac{1}{2}(1+\sqrt{5}))$ will be given.

1. Introduction

A quasigroup $(Q,\cdot)$ is said to be *GS–quasigroup* if it satisfies (mutually equivalent) identities

$$a(ab\cdot c)\cdot c=b,\qquad a\cdot(a\cdot bc)c=b \tag{1}$$

and moreover the identity of *idempotency*

$$aa=a. \tag{2}$$

The considered GS–quasigroup $(Q,\cdot)$ satisfies the identities of *mediality, elasticity, left* and *right distributivity* i.e. we have the identities

$$ab\cdot cd=ac\cdot bd, \tag{3}$$

$$a\cdot ba=ab\cdot a, \tag{4}$$

$$a\cdot bc=ab\cdot ac,\qquad ab\cdot c=ac\cdot bc. \tag{5}$$

Further, the identities

$$a(ab\cdot b)=b,\qquad (b\cdot ba)a=b, \tag{6}$$

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$$a(ab \cdot c) = b \cdot bc, \qquad (c \cdot ba)a = cb \cdot b, \tag{7}$$

$$a(a \cdot bc) = b(b \cdot ac), \qquad (cb \cdot a)a = (ca \cdot b)b \tag{8}$$

and equivalencies

$$ab = c \Leftrightarrow a = c \cdot cb, \qquad ab = c \Leftrightarrow b = ac \cdot c \tag{9}$$

also hold. GS–quasigroups are studied in [1].

Example 1. Let C be the set of points of the Euclidean plane. For any two different points a, b we define $ab = c$ if the point b or a divides the pair a, c or the pair b, c, respectively, in the ratio of the golden section.

In [1] it is proved that $(Q, \cdot)$ is a GS–quasigroup in both cases. We shall denote these two quasigroups by $C(\frac{1}{2}(1+\sqrt{5}))$ and $C(\frac{1}{2}(1-\sqrt{5}))$ because we have $c = \frac{1}{2}(1+\sqrt{5})$ or $c = \frac{1}{2}(1-\sqrt{5})$ if $a = 0$ and $b = 1$. These quasigroups can give a motivation for the definition of "geometric" notions and proving of "geometric" properties of a general GS–quasigroup. In the quasigroup $C(\frac{1}{2}(1+\sqrt{5}))$ we shall illustrate (by figures) the properties of general GS–quasigroup.

The considered two quasigroups are equivalent because it can be shown that if the operations $\cdot$ and $\bullet$ on the set Q are connected with the identity $a \bullet b = b \cdot a$, then $(Q, \bullet)$ is a GS–quasigroup if and only if $(Q, \cdot)$ is a GS–quasigroup.

From now on, let $(Q, \cdot)$ be any GS–quasigroup. The elements of the set Q are said to be *points*. The points a, b, c, d are said to be the vertices of a *parallelogram* and we write $Par(a, b, c, d)$ if the identity $d = a \cdot b(ca \cdot a)$ holds. The points a, b, c, d successively are said to be the vertices of the *golden section trapezoid* and it is denoted by $\mathrm{GST}(a, b, c, d)$ if the identity $a \cdot ab = d \cdot dc$ holds.

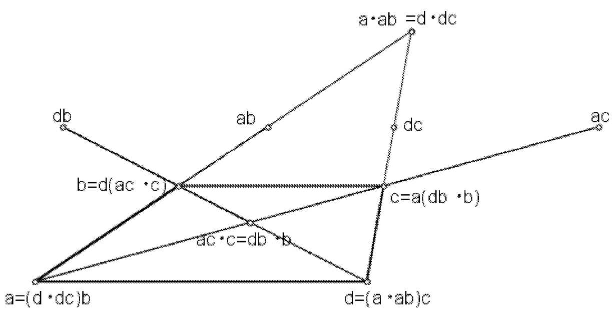

Figure 1.

In [2] the different properties of the quaternary relation GST on the set Q are proved. We shall mention just a few of them which will be used afterwards.

Theorem 1. *$GST(a,b,c,d)$ implies $GST(d,c,b,a)$.* □

If the relation $\mathrm{GST}(a,b,c,d)$ holds we shall say that the points c,a,d,b form a *GS–trapezoid of the second* kind and we shall write $\overline{GST}(c,a,d,b)$.

Remark 1. In [2] it is proved that a GS–trapezoid in one of the two quasigroups mentioned in Example 1 will be a GS–trapezoid of the second kind in the other quasigroup and vice versa. It means that it is a matter of convention which of the two quadrangles (a,b,c,d) or (c,a,d,b) will be called GS–trapezoid and which one a GS–trapezoid of the second kind, since we cannot differ them in the general GS–quasigroup.

Theorem 2. *The statement $GST(a,b,c,d)$ is equivalent to the equality $ac \cdot c = db \cdot b$ (Figure 1).* □

Theorem 3. *The statement $GST(a,b,c,d)$ is equivalent to any of the four equalities $a = (d \cdot dc)b$, $b = d(ac \cdot c)$, $c = a(db \cdot b)$, $d = (a \cdot ab)c$ (Figure 1).* □

Corollary 1. *GS–trapezoid is uniquely determined by any 3 of its vertices.*

Theorem 4.

(*i*) *Any two of the three statements $GST(a,b,c,d)$, $GST(b,c,d,e)$, $GST(c,d,e,a)$ imply the remaining statement (Figure 2).*

(*ii*) *Any two of the three statements $GST(a,b,c,d)$, $GST(b,c,d,e)$, $GST(d,e,a,b)$ imply the remaining statement (Figure 2).* □

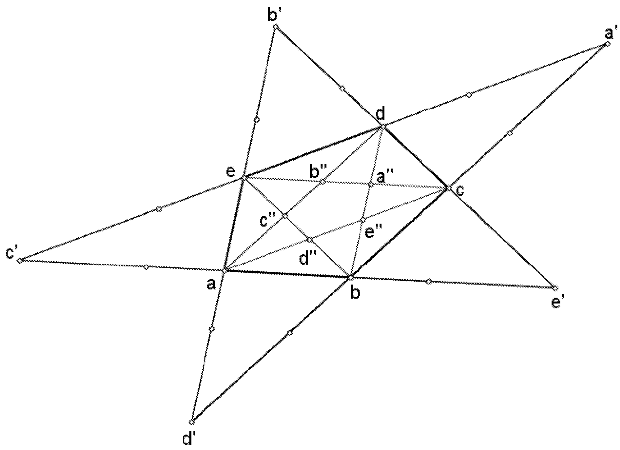

Figure 2.

If we apply Theorem 4 it immediately follows that any two of the five statements imply the remaining statement

$$GST(a,b,c,d), GST(b,c,d,e), GST(c,d,e,a), GST(d,e,a,b), GST(e,a,b,c).$$

Definition 1. The points a,b,c,d,e successively are said to be the vertices of the *affine regular pentagon* and it is denoted by $ARP(a,b,c,d,e)$ if any two (and then all five) of the above five statements are valid (Figure 2).

Based on Theorem 1 and Corollary 1 following three statements immediately follow.

Theorem 5. *If* (f,g,h,i,j) *is any cyclic permutation of* (a,b,c,d,e) *or of* (e,d,c,b,a)*, then* $ARP(a,b,c,d,e)$ *implies* $ARP(f,g,h,i,j)$*.* □

Theorem 6. *Affine regular pentagon is uniquely determined by any three of its vertices.* □

Theorem 7. *If the statement* $GST(a,b,c,d)$ *holds then there is one and only one point* e *such that the statement* $ARP(a,b,c,d,e)$ *holds.* □

Definition 2. If the relation $ARP(a,b,c,d,e)$ holds we shall say that the points a,c,e,b,d successively are the vertices of *affine regular star–shaped pentagon* and write $\overline{ARP}(a,c,e,b,d)$.

It is obvious, because of Theorem 5, that the equivalency of the statements $\overline{ARP}(a,b,c,d,e)$ and $ARP(a, c,e,b,d)$ is valid, it means that the relations ARP and $\overline{ARP}$ are mutually symmetric. From the Theorem about duality for GS–trapezoids (cf. [2]) now an analogous theorem follows.

Theorem 8 (about duality for affine regular pentagons).
From every theorem about affine regular pentagons we get an analogous theorem about affine regular star–shaped pentagons (and vice versa) if the roles of both factors are interchanged in all products which appear in the theorem. □

Corollary 2. *From every theorem about affine regular pentagons again we get a theorem about affine regular pentagons, if every statement of the form* $ARP(a,b,c,d,e)$ *is interchanged by the corresponding statement* $ARP(a,c,e,b,d)$*, and the roles of both factors are interchanged in all products.* □

In the interchanges mentioned in Theorem 8 and Corollary 2 it is not necessary to make an interchange in possible statements about the relation Par.

It follows from Remark 1 that in the general GS–quasigroup, whose model is the Euclidean plane, mentioned in Example 1 on one of the two ways, we cannot make out the difference between the affine regular pentagon and the affine regular star–shaped pentagon because what is an affine regular pentagon in one model that is an affine regular star–shaped pentagon in the other model (and vice versa); so it is just the matter of convention which of two pentagons we shall call affine regular and which affine regular star–shaped pentagon.

Theorem 6 can be expressed by the following theorem more precisely.

Theorem 9. *For any points* a, b, c *we have* $ARP((c \cdot cb)a,\ a,\ b,\ c,\ (a \cdot ab)c)$ *and* $ARP(a,\ c(ba \cdot a),\ b,\ a(bc \cdot c),\ c)$.

Proof. The second statement follows from the first applying the Corollary 2 and Theorem 5, and the first statement follows from the fact that $GST(a, b, c, d)$ is equivalent to $d = (a \cdot ab)c$, and $GST(e, a,\ b,\ c)$ to $e = (c \cdot cb)a$ (Theorem 3). □

From now on, let the statement $\mathrm{ARP}(a, b, c, d, e)$ be valid.

From $\mathrm{GST}(b, c, d, e)$ according to the definition of the relation GST and because of Theorem 2 follow the equations

$$b \cdot bc = e \cdot ed, \qquad bd \cdot d = ec \cdot c.$$

Set

$$a' = b \cdot bc = e \cdot ed, \qquad a'' = bd \cdot d = ec \cdot c,$$

and similarly the points b', b'', c', c'', d', d'', e', e'' (Figure 2) can be defined.

Let us prove several statements about these points.

Theorem 10. *The statements* $ARP(a', b', c', d', e')$ *and* $ARP(a'', b'', c'', d'', e'')$ *hold.*

Proof. According to Theorem 9 from [2] we have the statements

$$\mathrm{GST}(e \cdot ed,\ a \cdot ae,\ d \cdot de,\ e \cdot ea), \quad \mathrm{GST}(ec \cdot c,\ ce \cdot e,\ be \cdot e,\ eb \cdot b).$$

However,

$$e \cdot ed = a',\quad a \cdot ae = b',\quad d \cdot de = c',\quad e \cdot ea = d',$$
$$ec \cdot c = a'',\quad ce \cdot e = b'',\quad be \cdot e = c'',\quad eb \cdot b = d'',$$

so we get

$$\mathrm{GST}(a',b',c',d'), \qquad \mathrm{GST}(a'',b'',c'',d''),$$

and the remaining statements follow by the cyclical exchange of letters. □

Theorem 11. *The statements $Par(a,c,a',d)$, $Par(a,b',a',e')$, $Par(a,b,a'',e)$, $Par(a,c'',a'',d'')$ etc. hold.*

Proof. First, the equalities

$$\begin{aligned} a'c &= (b\cdot bc)c \overset{(6)}{=} b,\\ b'a &= (a\cdot ae)a \overset{(4)}{=} a(ae\cdot a) \overset{(7)}{=} e\cdot ea = d',\\ ba'' &= b(bd\cdot d) \overset{(6)}{=} d,\\ ac'' &= a(da\cdot a) \overset{(4)}{=} (a\cdot da)a \overset{(7)}{=} ad\cdot d = b'' \end{aligned}$$

are valid, and then, according to Lemma 2 from [2] we get these implications

$$\begin{aligned} \mathrm{GST}(d,c,b,a),\quad a'c = b &\Rightarrow \mathrm{Par}(a',d,a,c),\\ \mathrm{GST}\ (b',a',e',d'),\quad b'a = d' &\Rightarrow \mathrm{Par}(b',a',e',a),\\ \mathrm{GST}(b,a,e,d),\quad ba'' = d &\Rightarrow \mathrm{Par}(b,a,e,a''),\\ \mathrm{GST}(d'',c'',b'',a''),\quad ac'' = b'' &\Rightarrow \mathrm{Par}(a,d'',a'',c''), \end{aligned}$$

where, in the assumptions, the results of Theorem 10 are used. □

In the Theorem 11 we have come to some statements about the relation Par from the statements about the relation ARP. It can be also done in the opposite way, because the following theorem holds.

Theorem 12. *From $Par(a,b,c,d)$ follow $ARP(b,ab,c,ad,d)$ and $ARP(ba,d,c,b,da)$ (Figure 3).*

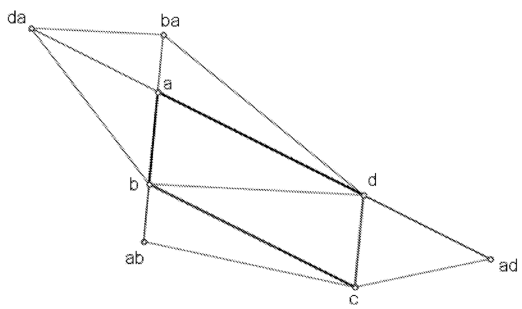

Figure 3.

Proof. According to Theorem 7 from [2] $GST(ad, d, b, ab)$ is valid, and from $\mathrm{Par}(a, d, c, b)$ by Lemma 2 from [2] it follows $GST(d, b, ab, c)$ so the first statement holds, and the second statement follows from the first one according to Corollary 2. □

Let us further prove

Theorem 13. *The statements* $ARP(a, b, c, d, e)$, $ARP(c, ac, f, ad, d)$ *hold where we have* $f = b \cdot bc = e \cdot ed$ (*Figure* 4).

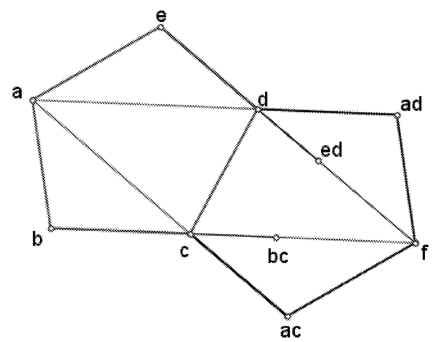

Figure 4.

Proof. According to Theorem 10 from [2], from the statement $\mathrm{GST}(a, b, c, d)$ it follows $\mathrm{GST}(c, d, ad, b \cdot bc)$, and from $\mathrm{GST}(a, e, d, c)$ it follows $\mathrm{GST}(d, c, ac, e{\cdot}ed)$. However, because of $\mathrm{GST}(b, c, d, e)$ we have the equality $b \cdot bc = e \cdot ed$. □

The following theorem about affine regular pentagons also holds.

Theorem 14. *Any two of the three statements* $ARP(a, b, c, d, e)$, $ARP(f, g, h, i, j)$, $ARP(af, bg, ch, di, ej)$ *imply the remaining statement* (*Figure* 5).

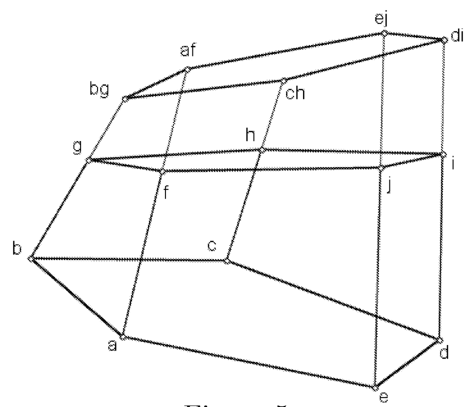

Figure 5.

Proof. It is sufficient to prove that any two of three statements $GST(a,b,c,d)$, $GST(f,g,h,i)$ and $GST(af,bg,ch,di)$ imply the remaining statement. However, according to (3) we have successively

$$[(af)\cdot(af)(bg)](ch)=[(af)\cdot(ab\cdot fg)](ch)=[(a\cdot ab)\cdot(f\cdot fg)](ch)$$
$$=(a\cdot ab)c\cdot(f\cdot fg)h$$

and then it is obvious that any two of the three equalities $(a\cdot ab)c=d$, $(f\cdot fg)h=i$ and $[ae\cdot(af)(bg)](ch)=di$ imply the remaining equality. □

Corollary 3. *$ARP(a,b,c,d,e)$ always implies $ARP(ab,bc,cd,de,ea)$, $ARP(ac,bd,ce,da,eb)$, $ARP(ad,be,ca,db,ec)$, $ARP(ae,ba,cb,dc,ed)$.* □

For any point p we have obviously ARP(p,p,p,p,p) and from Theorem 14 it follows further:

Corollary 4. *The statements $ARP(a,b,c,d,e)$, $ARP(ap,bp,cp,dp,ep)$, $ARP(pa,pb,pc,pd,pe)$ are mutually equivalent (for any point p).* □

Theorem 15. *From $c=(ob\cdot a)o$ it follows $a=(ob\cdot c)o$, and from $c=(ob\cdot a)o$ and $d=(oc\cdot b)o$ it follows $GST(a,b,c,d)$ (Figure 6).*

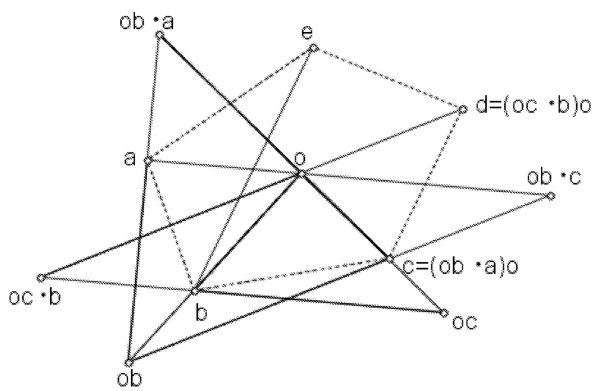

Figure 6.

Proof. We have successively

$$(ob\cdot c)o=[ob\cdot(ob\cdot a)o]o\overset{(1)}{=}a$$

$$(a\cdot ab)c=(a\cdot ab)\cdot(ob\cdot a)o\overset{(3)}{=}a(ob\cdot a)\cdot(ab\cdot o)\overset{(4)}{=}(a\cdot ob)a\cdot(ab\cdot o)$$
$$\overset{(3)}{=}(a\cdot ob)(ab)\cdot ao\overset{(5)}{=}a\cdot(ob\cdot b)o=(ob\cdot c)o\cdot(ob\cdot b)o$$
$$\overset{(5)}{=}(ob\cdot c)(ob\cdot b)\cdot o\overset{(5)}{=}(ob\cdot cb)o\overset{(5)}{=}(oc\cdot b)o=d.$$ □

Based on Theorem 15 this definition makes sense.

Definition 3. We say that the point o is the *centre of affine regular pentagon* with vertices a_o, a_1, a_2, a_3, a_4 if for each $i \in \{0,1,2,3,4\}$ is valid (modulo 5) the following equality

$$(oa_{i+1} \cdot a_i)o = a_{i+2} \qquad \text{respectively} \qquad (oa_{i-1} \cdot a_i)o = a_{i-2}.$$

On the Figure 6 the point o is the centre of affine regular pentagon with the vertices a, b, c, d, e.

Theorem 16. *Under the hypothesis of Theorem* 15 *equalities* $d = a\cdot(ob\cdot b)o$, $d = o(c \cdot ao)$, $b = o(c \cdot co) \cdot a$ *are valid.*

Proof. The first equality is proved in the proof of Theorem 15. Then we have successively

$$\begin{aligned} o(c \cdot ao) &\overset{(5)}{=} oc \cdot (o \cdot ao) \overset{(5)}{=} (oc \cdot o)(oc \cdot ao) = (oc \cdot o) \cdot [o \cdot (ob \cdot a)o](ao) \\ &\overset{(4)}{=} (oc \cdot o) \cdot [o(ob \cdot a) \cdot o](ao) \overset{(7)}{=} (oc \cdot o)[(b \cdot ba)o \cdot ao] \\ &\overset{(5)}{=} [oc \cdot (b \cdot ba)a]o \overset{(6)}{=} (oc \cdot b)o = d, \end{aligned}$$

and then from $(ob \cdot a)o = c$ because of (9) first follows $ob \cdot a = c \cdot co$, and then $ob = (c \cdot co) \cdot (c \cdot co)a$, and finally out of that according to (9) we get

$$\begin{aligned} b &= o[(c \cdot co) \cdot (c \cdot co)a] \cdot [(c \cdot co) \cdot (c \cdot co)a] \\ &\overset{(3)}{=} o(c \cdot co) \cdot [(c \cdot co) \cdot (c \cdot co)a][(c \cdot co)a] \\ &\overset{(5)}{=} o(c \cdot co) \cdot (c \cdot co)[(c \cdot co)a \cdot a] \overset{(6)}{=} o(c \cdot co) \cdot a., \end{aligned}$$

which completes the proof. □

If the point o is the centre of affine regular pentagon a_o, a_1, a_2, a_3, a_4 then the equalities from the Theorem 16 can be written in the form

$$\begin{aligned} a_i \cdot (oa_{i+1} \cdot a_{i+1})o &= a_{i+3}, \\ o(a_{i+2} \cdot a_i o) &= a_{i+3}, \\ o(a_{i+2} \cdot a_{i+2}o) \cdot a_i &= a_{i+1}, \end{aligned}$$

and similarly because of symmetry the equalities

$$\begin{aligned} a_i \cdot (oa_{i-1} \cdot a_{i-1})o &= a_{i-3}, \\ o(a_{i-2} \cdot a_i o) &= a_{i-3}, \\ o(a_{i-2} \cdot a_{i-2}o) \cdot a_i &= a_{i-1} \end{aligned}$$

are valid.

Under the hypothesis of the Theorem 15 and 16 and labels from Figure 2 the equalities

$$ab'' = d = a \cdot (ob \cdot b)o,$$
$$c'a = b = o(c \cdot co) \cdot a$$

are valid, and then immediately follows

$$b'' = (ob \cdot b)o,$$
$$c' = o(c \cdot co).$$

In general case, using analogous labels, we get the equalities

$$a_i' = o(a_i \cdot a_i o),$$
$$a_i'' = (oa_i \cdot a_i)o.$$

From previous considerations also follows

Theorem 17. *Affine regular pentagon is uniquely determined by its centre and with any two of its vertices.* □

References

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