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Zeroids and idempoids in AG-groupoids

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Abstract

Clifford and Miller (Amer. J. Math. 70, 1948) and Dawson (Acta Sci. Math. 27, 1966) have studied semigroups having left or right zeroids in a semigroup. In this paper, we have investigated AG-groupoids, and AG-groupoids with weak associative law, having zeroids or idempoids. Some interesting characteristics of these structures have been explored.

An *Abel-Grassman's groupoid* [8], abbreviated as *AG-groupoid*, is a groupoid G whose elements satisfy the *left invertive law*: $(ab)c = (cb)a$. It is also called a *left almost semigroup* [4, 5, 6, 7]. In [3], the same structure is called a *left invertive groupoid*. In this note we call it an AG-groupoid. It is a useful non-associative algebraic structure, midway between a groupoid and a commutative semigroup, with wide applications in the theory of flocks.

AG-groupoid is *medial* [5], that is, $(ab)(cd) = (ac)(bd)$ for all a, b, c, d in G. It has been shown in [5] that if an AG-groupoid contains a left identity then it is unique. It has been proved also that an AG-groupoid with right identity is a commutative monoid, that is, a semigroup with identity element. An element $a_\circ$ of an AG-groupoid G is called a *left zero* if $a_\circ a = a_\circ$ for all $a \in G$.

It has been shown in [5] that if $ab = cd$ then $ba = dc$ for all a, b, c, d in an AG-groupoid with left identity. If for all a, b, c in an AG-groupoid G, $ab = ac$ implies that $b = c$, then G is called *left cancellative*. Similarly, if $ba = ca$ implies that $b = c$, then G is called *right cancellative*. It is known [5] that every left cancellative AG-groupoid is right cancellative but the converse is not true. However, every right cancellative AG-groupoid with left identity is left cancellative.

Clifford and Miller [1] have defined an element z_l as a *left zeroid* in a semigroup G if for each element x in G, there exists a in G such that $ax = z_l$.

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A *right zeroid* is similarly defined. An element is a *zeroid* in G if it is both left and right zeroid.

Dawson [2] has studied semigroups having left or right zeroid elements and investigated some of their properties. In this paper we introduce the concept of left idempoids in AG-groupoid and investigate some of their properties.

Next we prove the following result.

Theorem 1. *An AG-groupoid G is a semigroup if and only if $a(bc) = (cb)a$ for all $a, b, c \in G$.*

Proof. Let $a(bc) = (cb)a$. Since G is an AG-groupoid, $(ab)c = (cb)a$. As the right hand sides of the two equations are equal, we conclude that $(ab)c = a(bc)$. Thus G is a semigroup.

Conversely, suppose that an AG-groupoid G is a semigroup. This means that $(ab)c = (cb)a$ and $(ab)c = a(bc)$. Since the left hand sides of these equations are equal, we get $a(bc) = (cb)a$ for all $a, b, c \in G$. □

An element z_r of an AG-groupoid G is called a *right idempoid* if, for each $x \in G$, there exists $a \in G$ such that $(xa)a = z_r$.

Note that G contains a right idempoid because for any $x, y \in G$ there exists $a \in G$ such that $ax, ay \in G$. So $(ax)(ay) = (aa)(xy) = (aa)z = (za)a$, where $z = xy$ is an arbitrary element in G, implies that G contains a right idempoid.

Proposition 1. *An AG-groupoid G is a semigroup if and only if $z_r = a(ax)$ is a right idempoid for some fixed a and any $x \in G$.*

Proof. The proof follows directly from Theorem 1. □

Theorem 2. *An AG-groupoid G with $G^2 = G$ is a commutative semigroup if and only if $(ab)c = a(cb)$ for all $a, b, c \in G$.*

Proof. Suppose $(ab)c = a(cb)$. Since G is an AG-groupoid, $(cb)a = (ab)c$. Combining the two equations we obtain $(cb)a = a(cb)$ implying that G is commutative. Thus $(ab)c = (cb)a = a(cb) = a(bc)$ shows that G is a commutative semigroup.

The converse follows immediately. □

Corollary 1. *An AG-groupoid is a commutative semigroup if and only if $z_r = xa^2$ is a right idempoid for fixed $a \in G$ and any $z \in G$.*

Proof. The proof follows immediately from Theorem 2. □

Proposition 2. *The square of every left zeroid in an AG-groupoid G with an idempotent is a right idempoid.*

Proof. Let x be an idempotent and z_l a left zeroid in G. Since z_l is a left zeroid, there exists a in G such that $ax = z_l$. Therefore

$$z_l z_l = (ax)(ax) = (aa)(xx) = (aa)x = (xa)a = z_r,$$

which completes the proof. □

Corollary 2. *In an AG-groupoid G there exists a left zeroid element.*

Proof. If we define a mapping $l_a : G \to G$ by $(x)l_a = ax$ by for all x in G, then obviously these mappings are related to left zeroids in a natural way. □

In the following we shall examine the necessary and sufficient conditions for l_a to be an epimorphism, endomorphism, automorphism, monomorphism and anti-homomorphism.

Theorem 3. *If in a left cancellative AG-groupoid G we define for a fixed a and some x, a mapping $l_a : x \mapsto ax$, from G onto G, then the following statements are equivalent:*
(i) *l_a is an epimorphism,*
(ii) *a is an idempotent in G,*
(iii) *l_a is an automorphism.*

Proof. Suppose (i) holds. Then there exists x in G such that for some fixed a, $ax = y$, in G. This implies that for some x in G and a fixed a in G, there exists an element y in G such that $y = (x)l_a$. Now $(a)l_a y = (a)l_a(x)l_a = (aa)(ax)$ and $(a)l_a(x)l_a = (ax)l_a = a(ax) = ay$ imply that $(a)l_a = a$, that is, a is an idempotent in G. Hence (i) implies (ii).

Also $(x)l_a(y)l_a = (ax)(ay) = (aa)(xy) = a(xy)$ because a is idempotent. This implies that $(x)l_a(y)l_a = (xy)l_a$, which further implies that l_a is an endomorphism. In order to show that l_a in an automorphism it is sufficient to show that l_a is one-to-one. But this is obvious since $(x)l_a = (y)l_a$ and $ax = ay$ implies that $x = y$ by virtue of left cancellation. Thus (ii) implies (iii).

Since l_a is an automorphism, (iii) implies (i). □

Theorem 4. *In an AG-groupoid G the following statements are equivalent:*
(i) *G has a right zero,*
(ii) *$l_a : x \longmapsto ax$ an automorphism and G has an idempotent element,*
(iii) *G has a zero.*

Proof. If x is a right zero of G, then $ax = x$ for some$a \in G$. But $x = ax = (x)l_a$ for every x in G. This implies that l_a is the identity mapping, which is an automorphism and, in particular, $a = (a)l_a$. It follows that $a = aa$, that is, a is an idempotent. Thus (i) implies (ii).

Further, for any x and some a in G, we have $a(xx) = (xx)l_a = xx$ and $(xx)a = (ax)x = (x)l_a x = xx$. This implies that $a(xx) = (xx)a = xx$, showing that xx is a zero in G. Hence (ii) implies (iii).

(iii) obviously implies (i). □

Theorem 5. *If* $(G)l_a = \{(x)l_a : x \in G\}$, *where* a *is a fixed idempotent of an* AG*-groupoid* G, *then* $(G)l_a$ *is an* AG*-groupoid with an idempotent* a.

Proof. Let $(x)l_a$, $(y)l_a$ belong to $(G)l_a$. Then

$$(x)l_a(y)l_a = (ax)(ay) = (aa)(xy) = a(xy) = (xy)l_a.$$

This implies that $(x)l_a(y)l_a \in (G)l_a$. Now

$$(x)l_a(y)l_a(z)l_a = ((ax)(ay))(az) = ((az)(ay))(ax) = ((z)l_a(y)(x)l_a.$$

Hence $(G)l_a$ is an AG-groupoid. □

Theorem 6. *If* $(G)l_a = \{(x)l_a : x \in G\}$, *where* a *is a fixed element of a right cancellative* AG*-groupoid* G, *then* l_a *is an endomorphism if and only if* a *is an idempotent of* G.

Proof. Let l_a be an endomorphism. Then $(xx') = (x)l_a(x')l_a$. Hence

$$a(xx') = (ax)(ax') = (aa)(xx')$$

imply that $a = aa$.

Conversely, if $a = aa$ then

$$(x)l_a(x')l_a = (ax)(ax') = (aa)(xx') = a(xx') = (xx')l_a,$$

which completes our proof. □

Theorem 7. *If* G *is an* AG*-groupoid with an idempotent* a *and* l_a *is an anti-homomorphism, then* a *commutes with every element of* G.

Proof. Let x be an arbitrary element of G. Then there exists $x' \in G$ such that $(x')l_a = x$. Consider xa for any x and some idempotent a in G. Then

$$xa = x(aa) = x(a)l_a = (x')l_a(a)l_a = (ax')l_a = a(ax') = a(x')l_a = ax.$$

This implies that a commutes with every x in G. □

Theorem 8. *In a right cancellative AG-groupoid G with an idempotent a, if $l_a : x \mapsto ax$ is an anti-homomorphism, then the following statements are equivalent:*

(i) *l_a is an anti-epimorphism,*

(ii) *G is a commutative monoid,*

(iii) *l_a is an anti-automorphism.*

Proof. Suppose (i) holds. Then for a fixed $a \in G$, there exist x and y in G such that, $y = ax = (x)l_a$. Now

$$ya = y(aa) = (x)l_a(a)l_a = (ax)l_a = a(ax) = a(x)l_a = ay$$

because l_a is an anti-epimorphism.

Further $ay = (aa)y = (ya)a$, which implies that $ya = (ya)a$. So $y = ya = ay$. Hence a is the identity of G. But an AG-groupoid with right identity is a commutative monoid by a result in [5]. Hence (i) implies(ii).

Now, since a is the identity in G, then for any x in G, we have $ax = x$ implying that $(x)l_a = x$ and so l_a is the identity mapping. This implies that l_a is an anti-automorphism. It follows that (ii) implies (iii).

Also, (iii) implies (i), follows immediately since an anti-automorphism must necessarily be an anti-epimorphism. □

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