m-Systems and n-systems in ordered semigroups

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Abstract

The aim of this short note is to introduce the concepts of m-systems and n-systems in ordered semigroups. These concepts are related to the concepts of weakly prime and weakly semiprime ideals, play an important role in studying the structure of ordered semigroups, so it seems to be interesting to study them.

There were several attempts to define the m-systems and n-systems in ordered semigroups. These concepts being related to the concepts of weakly prime and weakly semiprime ideals, play an important role in studying the structure of ordered semigroups. The aim of this note is to introduce the concepts of m-systems and n-systems in ordered semigroups. We begin our consideration by proving the relation between m-systems and weakly prime ideals, n-systems and weakly semiprime ideals. We prove that if S is an ordered semigroup, I a weakly prime (resp. weakly semiprime) proper ideal of S, then the complement $S\backslash I$ of I to S is an m-system (resp. an n-system) of S. "Conversely", if I is an ideal and $S\backslash I$ an m-system (resp. an n-system) of S, then I is weakly prime (resp. weakly semiprime). Thus a proper ideal I of an ordered semigroup S is weakly prime (resp. weakly semiprime) if and only if the complement $S\backslash I$ of I to S is an m-system (resp. an n-system). Moreover, each n-system of an ordered semigroup S containing an element a of S, contains an m-system that contains the same element a.

If $(S, \cdot)$ is a semigroup or an ordered semigroup, a subset T of S is called *weakly prime* if for all ideals A, B of S such that $AB \subseteq T$, we have $A \subseteq T$ or $B \subseteq T$. The subset T of S is called *weakly semiprime* if for every ideal A of S such that $A^2 \subseteq T$, we have $A \subseteq T$ [2]. For $H \subseteq S$, we define

$$(H] := \{t \in S \mid t \leqslant h \text{ for some } h \in H\}.$$

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A non-empty subset A of an ordered semigroup S is called an *ideal* of S if 1) $AS \subseteq A$, $SA \subseteq A$, 2) $a \in A$, $S \ni b \leqslant a$ implies $b \in A$ [2]. An ideal I of S is called *proper* if $I \neq S$. A non-empty subset B of an ordered semigroup S is called a *bi-ideal* of S if 1) $BSB \subseteq B$ and 2) if $a \in B$ and $S \ni c \leqslant a$, then $c \in B$ [3].

If $(S, \cdot)$ is a semigroup, a non-empty subset A of S is called an *m-system* of S if for each $a, b \in A$ there exists $x \in S$ such that $axb \in A$ [4].

The set A is called an *n-system* of S if for each $a \in A$ there exists $x \in S$ such that $axa \in A$.

In ordered semigroups the m-system and the n-system are defined as follows:

Definition 1. Let S be an ordered semigroup and $\emptyset \neq A \subseteq S$. The set A is called an *m-system* of S if for each $a, b \in A$ there exist $c \in A$ and $x \in S$ such that $c \leqslant axb$.

Equivalent Definition: For each $a, b \in A$ there exists $c \in A$ such that $c \in (aSb]$.

Definition 2. Let S be an ordered semigroup and $\emptyset \neq A \subseteq S$. The set A is called an *n-system* of S if for each $a \in A$ there exist $c \in A$ and $x \in S$ such that $c \leqslant axa$.

Equivalent Definition: For each $a \in A$ there exists $c \in A$ such that $c \in (aSa]$.

Remark 1. Each m-system is an n-system. Each bi-ideal is an m-system.

Remark 2. If $(S, \cdot)$ is a semigroup, we endow S with the order relation defined by $\leqslant := \{(a, b) \mid a = b\}$. Then $(S, \cdot, \leqslant)$ is an ordered semigroup. Moreover, the set A is an m-system (resp. an n-system) of $(S, \cdot)$ if and only if A is an m-system (resp. an n-system) of $(S, \cdot, \leqslant)$.

Lemma. (cf. [1]). *Let S be an ordered semigroup and I an ideal of S. Then I is weakly prime if and only if for each $a, b \in S$ such that $aSb \subseteq I$, we have $a \in I$ or $b \in I$.* □

We remark that since I is an ideal of S, we have

$$(aSb] \subseteq I \text{ if and only if } aSb \subseteq I.$$

Proposition 1. *Let S be an ordered semigroup and I an ideal of S. Then:*

1) *if I is weakly prime and $S \backslash I \neq \emptyset$, then $S \backslash I$ is an m-system,*
2) *if $S \backslash I$ is an m-system, then I is weakly prime.*

Proof. 1) Clearly, $\emptyset \neq S \backslash I \subseteq S$. Let $a, b \in S \backslash I$. Then, there exists $c \in S \backslash I$ such that $c \in (aSb]$. In fact, let $c \notin (aSb]$ for every $c \in S \backslash I$. We prove that $aSb \subseteq I$. Then, since I is weakly prime, by the Lemma, we have $a \in I$ or $b \in I$, which is impossible.

Let $aSb \nsubseteq I$. Then, there exists $y \in S$ such that $ayb \notin I$, so $ayb \in S \backslash I$. For the element $ayb \in S \backslash I$, we have $ayb \in aSb \subseteq (aSb]$.

2) Let $a, b \in S$, $aSb \subseteq I$. Then $a \in I$ or $b \in I$. Indeed, let $a, b \in S \backslash I$. Since $S \backslash I$ is an m-system, there exist $c \in S \backslash I$ and $x \in S$ such that $c \leqslant axb \in aSb \subseteq I$. Since I is an ideal of S, we have $c \in I$. Impossible. □

Corollary 1. *An ideal I of an ordered semigroup S is weakly prime if and only if either $S \backslash I = \emptyset$ or the set $S \backslash I$ is an m-system. A proper ideal I of an ordered semigroup S is weakly prime if and only $S \backslash I$ is an m-system.* □

In a similar way, we prove the following:

Proposition 2. *Let S be an ordered semigroup and I an ideal of S. Then:*

1) *if I is weakly semiprime and $S \backslash I \neq \emptyset$, then $S \backslash I$ is an n-system,*
2) *if $S \backslash I$ is an n-system, then I is weakly semiprime.* □

Corollary 2. *An ideal I of an ordered semigroup S is weakly semiprime if and only if either $S \backslash I = \emptyset$ or the set $S \backslash I$ is an n-system. A proper ideal I of an ordered semigroup S is weakly semiprime if and only if $S \backslash I$ is an n-system.* □

According to Remark 2, by Propositions 1 and 2, we get the Corollaries 3 and 4 below which are referred to semigroups without order.

Corollary 3. *Let S be a semigroup and I an ideal of S. Then:*

1) *if I is weakly prime and $S \backslash I \neq \emptyset$, then $S \backslash I$ is an m-system,*
2) *if $S \backslash I$ is an m-system, then I is weakly prime.* □

Corollary 4. *Let S be a semigroup and I an ideal of S. Then:*

1) *if I is weakly semiprime and $S \backslash I \neq \emptyset$, then $S \backslash I$ is an n-system,*
2) *if $S \backslash I$ is an n-system, then I is weakly semiprime.* □

In the rest of this note we prove that, in ordered semigroups, each n-system containing an element a, contains an m-system which contains the same element. As a consequence, the same result is true for semigroups without order, as well (Corollary 5). This interesting result has been first noticed by R. D. Giri and A. K. Wazalwar in [1] extending the corresponding known result of rings.

Proposition 3. *Let S be an ordered semigroup. If N is an n-system of S and $a \in N$, then there exists an m-system M of S such that $a \in M \subseteq N$.*

Proof. Since N is an n-system and $a \in N$, there exists $c_1 \in N$ such that $c_1 \in (aSa]$, then $(aSa] \cap N \neq \emptyset$. Take $a_1 \in (aSa] \cap N$. Since N is an n-system and $a_1 \in N$, there exists $c_2 \in N$ such that $c_2 \in (a_1Sa_1]$, then $(a_1Sa_1] \cap N \neq \emptyset$. Take $a_2 \in (a_1Sa_1] \cap N$. We continue this way. Take $a_i \in (a_{i-1}Sa_{i-1}] \cap N$. Since N is an n-system and $a_i \in N$, there exists $c_{i+1} \in N$ such that $c_{i+1} \in (a_iSa_i]$, then $(a_iSa_i] \cap N \neq \emptyset$.
We put $a_0 = a$, $M = \{a_0, a_1, a_2, \ldots,_i, a_{i+1}, \ldots.\}$. We have $a_0 \in M$ and $a_n \in N$ for all $n = 0, 1, 2, \ldots, i, \ldots$, so $a \in M \subseteq N$.
The set M is an m-system. Indeed: $\emptyset \neq M \subseteq S$ $(a \in M)$.Let $a_i, a_j \in M$.
If $i = j$ then, for the element $a_{i+1} \in S$, we have $a_{i+1} \in (a_iSa_i] = (a_iSa_j]$.
If $i < j$ then, for the element $a_{j+1} \in S$, we have

$$a_{j+1} \in (a_jSa_j] \subseteq ((a_{j-1}Sa_{j-1}]Sa_j] \subseteq (a_{j-1}Sa_j] \subseteq \ldots \subseteq (a_iSa_j].$$

If $j < i$ then, for the element $a_{i+1} \in S$, we have

$$a_{i+1} \in (a_iSa_i] \subseteq (a_iS(a_{i-1}Sa_{i-1})] \subseteq (a_iSa_{i-1}] \subseteq \ldots \subseteq (a_iSa_j],$$

which completes the proof. □

Corollary 5. *Let S be a semigroup. If N is an n-system of S and $a \in N$, then there exists an m-system M of S such that $a \in M \subseteq N$.* □

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