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Some results on hyper BCK-algebras

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Abstract

In this paper by considering the notion of hyper BCK-algebra, we state and prove some theorems which determine the relationship among (weak) hyper BCK-ideals, positive implicative hyper BCK-ideals of types $1, 3, \ldots, 8$ and hypersubalgebras, under some suitable conditions. Moreover, we define the notions of commutative hyper BCK-ideals of types 1, 2, 3 and 4 and obtain some results.

1. Introduction

The study of BCK-algebras was initiated by Y. Imai and K. Iséki [5] in 1966 as a generalization of the concept of set-theoretic difference and propositional calculus. Since then a great deal of literature has been produced on the theory of BCK-algebras. In particular, emphasis seems to have been put on the ideal theory of BCK-algebras. The hyperstructure theory (called also multialgebras) was introduced in 1934 by F. Marty [8] at the 8th congress of Scandinavian Mathematicians. Around 40‘s, several authors worked on hypergroups, especially in France, United States, Italy, Greece and Iran. Hyperstructures have many applications to several sectors of both pure and applied sciences. In [7], Y.B. Jun, M.M. Zahedi, X. L. Xin and R.A. Borzooei applied the hyperstructures to BCK-algebras, and introduced the notion of a hyper BCK-algebra which is a generalization of BCK-algebra, and investigated some related properties. They also introduced the notions of hyper BCK-ideal and weak (strong) hyper BCK-ideal, and gave relations among this notions. Now we follow [3,6,7] and obtain some results, as mentioned in the abstract.

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2. Preliminaries

Definition 2.1. By a *hyper BCK-algebra* we mean a nonempty set H endowed with a hyperoperation $\circ$ and a constant 0 satisfies the following axioms:

(HK1) $(x \circ z) \circ (y \circ z) \ll x \circ y$,
(HK2) $(x \circ y) \circ z = (x \circ z) \circ y$,
(HK3) $x \circ H \ll \{x\}$,
(HK4) $x \ll y$ and $y \ll x$ imply $x = y$

for all $x, y, z \in H$, where $x \ll y$ is defined by $0 \in x \circ y$ and for every $A, B \subseteq H$, $A \ll B$ is defined by $\forall a \in A$, $\exists b \in B$ such that $a \ll b$. In such case, we call $\ll$ the *hyperorder* in H.

Theorem 2.2 [7]. *In any hyper BCK-algebra H, the following hold:*

(i) $0 \circ 0 = \{0\}$,
(ii) $0 \ll x$,
(iii) $x \ll x$,
(iv) $A \subseteq B$ *implies* $A \ll B$,
(v) $0 \circ x = \{0\}$,
(vi) $x \circ y \ll x$,
(vii) $x \circ 0 = \{x\}$,

for all $x, y, z \in H$ and for all nonempty subsets A and B of H.

Let I be a nonempty subset of a hyper BCK-algebra H. Then I is said to be a *hyper BCK-ideal* of H, if for all $x, y \in H$, $x \circ y \ll I$ and $y \in I$ imply $x \in I$, *weak hyper BCK-ideal* of H, if for all $x, y \in H$, $x \circ y \subseteq I$ and $y \in I$ imply $x \in I$, *strong hyper BCK-ideal* of H, if for all $x, y \in H$, $(x \circ y) \cap I \neq \emptyset$ and $y \in I$ imply $x \circ y \subseteq I$, *hyper BCK-subalgebra* of H, if I is a hyper BCK-algebra with respect to the hyperoperation $\circ$ on H.

Clear that, any strong hyper BCK-ideal of H is a hyper BCK-ideal and any hyper BCK-ideal of H is a weak hyper BCK-ideal. Moreover, let I be a nonempty subset of a hyper BCK-algebra H. Then I is a hypersubalgebra of H if and only if $x \circ y \subseteq I$ for all $x, y \in I$.

Definition 2.3. Let I be a nonempty subset of hyper BCK algebra H and $0 \in I$. Then I is said to be a *positive implicative hyper BCK-ideal* of

(i) *type* 1,
if $(x \circ y) \circ z \subseteq I$ and $y \circ z \subseteq I$ imply that $x \circ z \subseteq I$ for all $x, y, z \in H$,

(ii) *type* 2,
if $(x \circ y) \circ z \ll I$ and $y \circ z \subseteq I$ imply that $x \circ z \subseteq I$ for all $x, y, z \in H$,

(iii) *type* 3,
if $(x \circ y) \circ z \ll I$ and $y \circ z \ll I$ imply that $x \circ z \subseteq I$ for all $x, y, z \in H$,

(iv) *type* 4,
if $(x \circ y) \circ z \subseteq I$ and $y \circ z \ll I$ imply that $x \circ z \subseteq I$ for all $x, y, z \in H$,

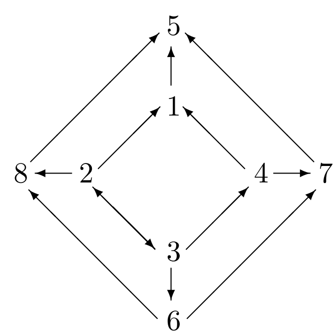
(v) *type 5*,
if $(x \circ y) \circ z \subseteq I$ and $y \circ z \subseteq I$ imply that $x \circ z \ll I$ for all $x, y, z \in H$,

(vi) *type 6*,
if $(x \circ y) \circ z \ll I$ and $y \circ z \ll I$ imply that $x \circ z \ll I$ for all $x, y, z \in H$,

(vii) *type 7*,
if $(x \circ y) \circ z \subseteq I$ and $y \circ z \ll I$ imply that $x \circ z \ll I$ for all $x, y, z \in H$,

$(viii)$ *type 8*,
if $(x \circ y) \circ z \ll I$ and $y \circ z \subseteq I$ imply that $x \circ z \ll I$ for all $x, y, z \in H$.

In the following diagram, we can see the relationship among all of types of positive implicative hyper BCK-ideals.



Let H be a hyper BCK-algebra and for each $a, b \in H$, $|a \circ b|$ be cardinality of $a \circ b$. An element $a \in H$ is said to be *left* (resp. *right*) *scalar* if $|a \circ x| = 1$ (resp. $|x \circ a| = 1$) for all $x \in H$. If $a \in H$ is both left and right scalar, we say that a is a *scalar* element.

We say that subset I of H satisfies the *closed* condition, if $x \ll y$ and $y \in I$ imply $x \in I$, for all $x, y \in H$.

Lemma 2.4. *If I is a hyper BCK-ideal and A is a nonempty subset of H, then I satisfies the closed condition and if $A \ll I$, then $A \subseteq I$.* $\square$

Theorem 2.5. *Let I be a nonempty subset of H satisfying the closed condition. If I is a positive implicative hyper BCK-ideal of type i, then I is a positive implicative hyper BCK-ideal of type j, for all $1 \leqslant i, j \leqslant 8$.*

Proof. By considering the Lemma 2.4 the proof is straightforward. □

Lemma 2.6 [3]. *Let $H = \{0,1,2\}$ be a hyper BCK-algebra of order 3. Then the following statements are hold.*

(a) *If H satisfies the simple condition (that is $1 \not\ll 2$ and $2 \not\ll 1$), then*
- (i) $1 \circ 1 \in \{\{0\},\{0,1\}\}$ *and* $1 \circ 2 = \{1\}$,
- (ii) $2 \circ 1 = \{2\}$ *and* $2 \circ 2 \in \{\{0\},\{0,2\}\}$.

(b) *If H satisfies the normal condition (that is $1 \ll 2$ or $2 \ll 1$), then*
- (iii) $1 \circ 1 \in \{\{0\},\{0,1\}\}$,
- (iv) $1 \circ 2 \in \{\{0\},\{0,1\}\}$,
- (v) $2 \circ 1 \in \{\{1\},\{2\},\{1,2\}\}$,
- (vi) $2 \circ 2 \in \{\{0\},\{0,1\},\{0,2\},\{0,1,2\}\}$.

Theorem 2.7 [3]. *Let H be a hyper BCK-algebra of order 3 which satisfies the normal condition. Then H has at most one proper hyper BCK-ideal.*

3. Positive implicative hyper BCK-ideals

In the sequel H denotes a hyper BCK-algebra.

Definition 3.1. A nonempty subset I of H is said to be *S-reflexive* if $(x \circ y) \bigcap I \neq \emptyset$ implies that $(x \circ y) \subseteq I$, for all $x, y \in H$.

Theorem 3.2. *Let I be a S-reflexive nonempty subset of H. If I is a positive implicative hyper BCK-ideal of type 1, then I is a strong hyper BCK -ideal of H and so is a positive implicative hyper BCK-ideal of type i for all $1 \leqslant i \leqslant 8$.*

Proof. Assume that I is a positive implicative hyper BCK-ideal of type 1, $(x \circ y) \bigcap I \neq \emptyset$ and $y \in I$ for $x, y \in H$. Since I is S-reflexive, then $x \circ y \subseteq I$. Hence by Theorem 2.2 (vii), $(x \circ y) \circ 0 = x \circ y \subseteq I$ and $y \circ 0 = \{y\} \subseteq I$. Since I is a positive implicative hyper BCK-ideal of type 1, then $\{x\} = x \circ 0 \subseteq I$ i.e $x \in I$. Thus I is a strong hyper BCK-ideal of H and so I is a hyper BCK-ideal of H. Hence by Lemma 2.4, I satisfy the closed condition and so by Theorem 2.5, I is a positive implicative hyper BCK-ideal of type i for all $1 \leqslant i \leqslant 8$. □

Example 3.3. Let H be a hyper BCK-algebra which is defined as follows:

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{0\}$
2	$\{2\}$	$\{1,2\}$	$\{0,1,2\}$

Then $I = \{0, 1\}$ is a positive implicative hyper BCK-ideal of type $1, 3, \ldots, 8$, but it is not a strong hyper BCK-ideal and it is not a S-reflexive. Because $2 \circ 1 = \{1, 2\} \nsubseteq I$, where $(2 \circ 1) \bigcap I \neq \emptyset$. Therefore, the S-reflexive condition is necessary in Theorem 3.2. □

Definition 3.4. (i) H is called a *positive implicative hyper* BCK*-algebra*, if for all $x, y, z \in H$, $(x \circ y) \circ z = (x \circ z) \circ (y \circ z)$.

(ii) H is called an *alternative quasi hyper* BCK*-algebra*, if for all $x, y \in H$, $(x \circ y) \circ y = x \circ (y \circ y)$.

Lemma 3.5. *Let A, B and I are nonempty subsets of H. If I is a weak hyper BCK-ideal of H, $A \circ B \subseteq I$ and $B \subseteq I$, then $A \subseteq I$.* □

Theorem 3.6. *If H is a positive implicative hyper BCK-algebra, then any weak hyper BCK-ideal of H is a positive implicative hyper BCK-ideal of types* 1 *and* 5.

Proof. Let I be a weak hyper BCK-ideal of H, $(x \circ y) \circ z \subseteq I$ and $y \circ z \subseteq I$, for $x, y, z \in H$. Since H is a positive implicative hyper BCK-algebra, then $(x \circ z) \circ (y \circ z) = (x \circ y) \circ z \subseteq I$. Hence by Lemma 3.5, we get that $x \circ z \subseteq I$. Therefore I is a positive implicative hyper BCK-ideal of type 1 and so by diagram in section 2, I is a positive implicative hyper BCK-ideal of type 5. □

Example 3.7. Let H be a hyper BCK-algebra which is defined as follows:

$\circ$	0	1	2	3
0	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{0\}$	$\{0\}$
2	$\{2\}$	$\{2\}$	$\{0\}$	$\{0\}$
3	$\{3\}$	$\{3\}$	$\{2\}$	$\{0, 2\}$

Then H is not a positive implicative hyper BCK-algebra. Since $(3 \circ 2) \circ 2 = 0 \neq 2 = (3 \circ 2) \circ (2 \circ 2)$. Moreover $I = \{0, 1\}$ is a weak hyper BCK-ideal of H but it is not a positive implicative hyper BCK-ideal of type 5. Since $(3 \circ 2) \circ 2 = \{0\} \subseteq I$ and $2 \circ 2 = \{0\} \subseteq I$, but $3 \circ 2 = \{2\} \not\ll I$ and so by diagram in section 2, I is not a positive implicative hyper BCK-ideal of type 1. Therefore, positive implicative condition is necessary in Theorem 3.6. □

Definition 3.8. A subset I of H is said to be *proper* if $\{0\} \subset I \subset H$.

Theorem 3.9. *Let $H = \{0, 1, 2\}$ be an alternative quasi hyper BCK-algebra. Then, there is at least one proper weak hyper BCK-ideal of H.*

Proof. We claim that $I=\{0,1\}$ is a weak hyper BCK-ideal of H. Let $x\circ y\subseteq I$ and $y\in I$ for $x,y\in H$. We must show that $x\in I$. Let $x\notin I$ (by contrary). Then $x=2$ and so $2\circ y\subseteq I$. Since $y\in I$ then $y=0$ or 1. If $y=0$ then by Theorem 2.2 (vii), $2\in\{2\}=2\circ 0\subseteq I$, which is a contradiction. Hence $y=1$. By Lemma 2.6, $2\circ 1=\{1\},\{2\}$ or $\{1,2\}$. If $2\circ 1=\{2\}$ or $\{1,2\}$, then $2\in 2\circ 1=x\circ y\subseteq I$, which is impossible. Hence $2\circ 1=\{1\}$. Moreover, by Lemma 2.6 (iii), $1\circ 1=\{0\}$ or $\{0,1\}$. If $1\circ 1=\{0\}$, then by Theorem 2.2 (vii)

$$(2\circ 1)\circ 1=1\circ 1=\{0\}\neq\{2\}=2\circ 0=2\circ(1\circ 1)$$

which is contradiction by alternative quasi. If $1\circ 1=\{0,1\}$, then $(2\circ 1)\circ 1=1\circ 1=\{0,1\}$. But $2\in\{2\}=2\circ 0\subseteq 2\circ(1\circ 1)$ and so $(2\circ 1)\circ 1\neq 2\circ(1\circ 1)$, which is a contradiction by alternative quasi hyper BCK-algebra. Therefore, $I=\{0,1\}$ is a weak hyper BCK-ideal of H. □

Theorem 3.10. *Let $H=\{0,1,2\}$ be a hyper BCK-algebra of order 3 and I be a proper subset of H. Then*

- (i) *I is a positive implicative hyper BCK-ideal of type 3 if and only if I is a hyper BCK-ideal,*
- (ii) *I is a positive implicative hyper BCK-ideal of type 1 if and only if I is a weak hyper BCK-ideal of H.*

Proof. (i) It is easy to check that, any positive implicative hyper BCK-ideal of type 3 is a hyper BCK-ideal of H.

Conversely, let I be a hyper BCK-ideal of H. We consider two following cases.

Case 1. H satisfies the normal condition. By Theorem 2.7, H has at most one proper hyper BCK-ideal which is $I=\{0,1\}$. Now, let $I=\{0,1\}$ be a hyper BCK-ideal of H. Then $2\circ 1\not\ll I$. Since $1\in I$, if $2\circ 1\ll I$, then $2\in I$, which is impossible. Hence $2\in 2\circ 1$ and so by Lemma 2.6 (v), $2\circ 1=\{2\}$ or $\{1,2\}$. Now, let $(x\circ y)\circ z\ll I$ and $y\circ z\ll I$, but $x\circ z\nsubseteq I$. Then $2\in x\circ z$. By Lemma 2.6 (iii) and (iv), $x\neq 1$. Moreover, $x\neq 0$. Since if $x=0$, then $2\in x\circ z=0\circ z=\{0\}$, which is impossible. Thus $x=2$. Since I is a hyper BCK-ideal of H, then

$$(x\circ y)\circ z\subseteq I \text{ and } y\circ z\subseteq I.$$

Now, we considering the following cases:

Case 1.1. If $z=0$, since $\{y\}=y\circ 0=y\circ z\subseteq I$, then $y=0$ or 1. If $y=0$, then $\{2\}=(2\circ 0)\circ 0=(x\circ y)\circ z\subseteq I$, which is a contradiction. If $y=1$, then $2\in 2\circ 1=(2\circ 1)\circ 0=(x\circ y)\circ z\subseteq I$, which is impossible.

Case 1.2. If $z = 1$, then $y \circ 1 = y \circ z \subseteq I$. Since I is a hyper BCK-ideal of H and $1 \in I$, then $y \in I$ and so $y = 0$ or 1. If $y = 0$, then by (HK2)

$$2 \in 2 \circ 1 = (2 \circ 1) \circ 0 = (2 \circ 0) \circ 1 = (x \circ y) \circ z \subseteq I,$$

which is a contradiction. If $y = 1$, then

$$2 \in 2 \circ 1 \subseteq (2 \circ 1) \circ 1 = (x \circ y) \circ z \subseteq I,$$

which is impossible.

Case 1.3. If $z = 2$, since $2 \in x \circ z$ and $x = z = 2$, then $2 \in 2 \circ 2$. Hence, by Lemma 2.6 (vi), $2 \circ 2 = \{0, 2\}$ or $\{0, 1, 2\}$. If $y = 0$, then

$$2 \in 2 \circ 2 = (2 \circ 0) \circ 2 = (x \circ y) \circ z \subseteq I,$$

which is a contradiction. If $y = 1$, then by (HK2)

$$2 \in 2 \circ 1 \subseteq (2 \circ 2) \circ 1 = (2 \circ 1) \circ 2 = (x \circ y) \circ z \subseteq I,$$

which is impossible. If $y = 2$, then

$$2 \in 2 \circ 2 \subseteq (2 \circ 2) \circ 2 = (x \circ y) \circ z \subseteq I,$$

which is impossible. Therefore, $x \circ z \subseteq I$ and so I is a positive implicative hyper BCK-ideal of type 3.

Case 2. H satisfies the simple condition. By Theorem 3.1 [3], there are only three hyper BCK-algebras of order 3 which satisfies the simple condition. Now, we can show that the $I_1 = \{0, 1\}$ and $I_2 = \{0, 2\}$ are hyper BCK-ideals and positive implicative hyper BCK-ideal of type 3 in the this three hyper BCK-algebras.

(ii) The proof is similar to the proof of case (i). □

Theorem 3.11. *Let $H = \{0, 1, 2\}$ be an alternative quasi hyper BCK-algebra. Then there is at least one proper positive implicative hyper BCK-ideal of type* $1, 3, \ldots, 8$.

Proof. By the proof of Theorem 3.9, $I = \{0, 1\}$ is a weak hyper BCK-ideal of H and so by Theorem 3.10 (ii), I is a positive implicative hyper BCK-ideal of type 1.

Now, we show that I is a hyper BCK-ideal of H. Let $x \circ y \ll I$ and $y \in I$ but $x \notin I$ (by contrary). Then $x = 2$. Since $y \in I$, then $y = 0$ or 1. If $y = 0$, then $\{2\} = 2 \circ 0 \ll I_1$ and so $2 \ll 1$. Hence $0 \in 2 \circ 1$, which is impossible by Lemma 2.6. If $y = 1$, then we consider the following two cases.

Case 1. Let H satisfies the simple condition. Then by Lemma 2.6 (ii), $\{2\} = 2 \circ 1 \ll I_1 = \{0, 1\}$ and so $2 \ll 1$, which is a contradiction.

Case 2. Let H satisfies the normal condition. Then by Lemma 2.6 (v), $2 \circ 1 = \{1\}, \{2\}$ or $\{1, 2\}$. If $2 \circ 1 = \{2\}$ or $\{1, 2\}$, then $2 \in 2 \circ 1 \ll I_1 = \{0, 1\}$ and so $2 \ll 1$. Hence $0 \in 2 \circ 1$ which is a impossible by Lemma 2.6. If $2 \circ 1 = \{1\}$, then $2 \circ 1 \subseteq I$. Since I is a weak hyper BCK-ideal of H, and $1 \in I$, then $2 \in I = \{0, 1\}$ which is impossible. Hence, I is a hyper BCK-ideal of H. Therefore, by Lemma 2.4 and Theorem 2.5 since I is a positive implicative hyper BCK-ideal of type 1, then I is a positive implicative hyper BCK-ideal of type i, for all $1 \leqslant i \leqslant 8$. □

Theorem 3.12. *Let H be a positive implicative and an alternative quasi hyper BCK-algebra. Then every hyper BCK-subalgebra of H is a positive implicative hyper BCK-ideal of type* 1.

Proof. Let I be a hyper BCK-subalgebra of H, $(x \circ y) \circ z \subseteq I$ and $y \circ z \subseteq I$, for $x, y, z \in H$. Since H is a positive implicative hyper BCK-algebra, then $(x \circ z) \circ (y \circ z) = (x \circ y) \circ z \subseteq I$. Then for all $t \in x \circ z$ and $s \in y \circ z$, $t \circ s \subseteq I$. Since by Theorem 2.2 (iii) and (vii), $0 \in s \circ s$ and for all $t \in x \circ z$, $t \in \{t\} = t \circ 0$, hence

$$t \in t \circ 0 \subseteq t \circ (s \circ s) = (t \circ s) \circ s \subseteq I \circ s \subseteq I,$$

since I is a hyper BCK-subalgebra and $s \in I$. Thus $x \circ z \subseteq I$. Therefore, I is a positive implicative hyper BCK-ideal of type 1. □

Example 3.13. Consider the following tables:

$\circ_1$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0,1,2\}$	$\{1\}$
2	$\{2\}$	$\{0,2\}$	$\{0,2\}$

$\circ_2$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0,2\}$	$\{2\}$
2	$\{2\}$	$\{0\}$	$\{0\}$

$\circ_3$	0	1	2	3
0	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{0\}$	$\{0\}$
2	$\{2\}$	$\{2\}$	$\{0,2\}$	$\{0\}$
3	$\{3\}$	$\{3\}$	$\{3\}$	$\{0,3\}$

$(H, \circ_1)$ is a positive implicative and alternative quasi hyper BCK-algebra and $I = \{0, 1\}$ is a positive implicative hyper BCK-ideal of type 1 but it is not a hyper BCK-subalgebra of H. Since $1 \in I$, but $1 \circ 1 \not\subseteq I$. Therefore, the converse of Theorem 3.12 is not correct in general.

$(H, \circ_2)$ is a hyper BCK-algebra but it is not a positive implicative hyper BCK-algebra. Since, $(1 \circ 1) \circ 1 \neq (1 \circ 1) \circ (1 \circ 1)$. Moreover, $I = \{0, 2\}$ is a hyper BCK-subalgebra of H, but it is not a positive implicative hyper BCK-ideal of type 1. Since $(1 \circ 2) \circ 0 \subseteq I$ and $2 \circ 0 \subseteq I$ but $1 \circ 0 \not\subseteq I$.

$(H, \circ_3)$ is a hyper BCK-algebra but it is not an alternative quasi hyper BCK-algebra. Since, $(2 \circ 3) \circ 3 \neq 2 \circ (3 \circ 3)$. Moreover, $I = \{0, 1, 3\}$ is a hyper BCK-subalgebra of H, but it is not a positive implicative hyper BCK-ideal of type 1. Since $(2 \circ 3) \circ 0 \subseteq I$ and $3 \circ 0 \subseteq I$ but $2 \circ 0 \not\subseteq I$. □

4. Commutative hyper BCK-ideals

Definition 4.1. Let I be a subset of H such that $0 \in I$. Then I is said to be a *commutative hyper BCK-ideal* of

(i) *type* 1, if $(x \circ y) \circ z \subseteq I$ and $z \in I$ imply $x \circ (y \circ (y \circ x)) \subseteq I$,

(ii) *type* 2, if $(x \circ y) \circ z \subseteq I$ and $z \in I$ imply $x \circ (y \circ (y \circ x)) \ll I$,

(iii) *type* 3, if $(x \circ y) \circ z \ll I$ and $z \in I$ imply $x \circ (y \circ (y \circ x)) \subseteq I$,

(iv) *type* 4, if $(x \circ y) \circ z \ll I$ and $z \in I$ imply $x \circ (y \circ (y \circ x)) \ll I$,

for all $x, y, z \in H$.

Theorem 4.2. *Let I be a nonempty subset of H. Then the following statements hold:*

(i) *if I is a commutative hyper BCK-ideal of type 3, then I is a commutative hyper BCK-ideal of type 1 and 4,*

(ii) *if I is a commutative hyper BCK-ideal of type 1 or 4, then I is a commutative hyper BCK-ideal of type 2.* □

Example 4.3. (i) Let H be the hyper BCK-algebra which is defined as follows:

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{1\}$
2	$\{2\}$	$\{0, 2\}$	$\{0, 2\}$

Thus, $I = \{0, 1\}$ is a commutative hyper BCK-ideal of type 1, 2 and 4 but it is not of type 3. Because, $(2 \circ 1) \circ 1 = \{0, 2\} \circ 1 = \{0, 2\} \ll I$ and $1 \in I$, but $2 \circ (1 \circ (1 \circ 2)) = 2 \circ (1 \circ 1) = 2 \circ 0 = \{2\} \not\subseteq I$.

(ii) Let $H = \{0, 1, 2, 3\}$. The following table shows a hyper BCK-algebra structure on H:

$\circ$	0	1	2	3
0	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{0\}$	$\{0\}$
2	$\{2\}$	$\{1\}$	$\{0,1\}$	$\{1\}$
3	$\{3\}$	$\{1\}$	$\{0\}$	$\{0,1\}$

Then $I = \{0, 2\}$ is a commutative hyper BCK-ideal of type 2 and 4, but it is not commutative hyper BCK-ideal of type 1. Since, $(2 \circ 1) \circ 2 = 1 \circ 2 = \{0\} \subseteq I$ and $2 \in I$ but $2 \circ (1 \circ (1 \circ 2)) = 2 \circ (1 \circ 0) = 2 \circ 1 = \{1\} \not\subseteq I$.

Moreover, $I = \{0, 3\}$ is a commutative hyper BCK-ideal of type 2, but it is not commutative hyper BCK-ideal of type 4. Since, $(2 \circ 0) \circ 3 = 2 \circ 3 = \{1\} \ll I$ and $3 \in I$ but $2 \circ (0 \circ (0 \circ 2)) = 2 \circ 0 = \{2\} \not\ll I$. □

Theorem 4.4. *Let I be a nonempty subset of H. Then:*

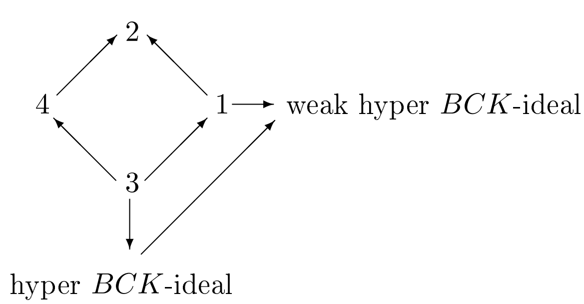
(i) *if I is a commutative hyper BCK-ideal of type 3, then I is a hyper BCK-ideal of H,*

(ii) *if I is a commutative hyper BCK-ideal of type 1, then I is a weak hyper BCK-ideal of H.*

Proof. (i) Let I be a commutative hyper BCK-ideal of type 3, $x \circ y \ll I$ and $y \in I$, for $x, y \in H$. Since $(x \circ 0) \circ y = x \circ y \ll I$ and $y \in I$, then by hypothesis we get that $\{x\} = x \circ 0 = x \circ (0 \circ (0 \circ x)) \subseteq I$. Therefore, I is a hyper BCK-ideal of H.

(ii) The proof is similar to the proof (i). □

We summarize the Theorems 4.2 and 4.4 in the following diagram:



Lemma 4.5. *Let A, B and I are nonempty subsets of a hyper BCK-algebra H. If I is a hyper BCK-ideal of H, then $A \circ B \ll I$ and $B \subseteq I$ imply $A \subseteq I$.* □

Theorem 4.6. *Let $H = \{0, 1, 2\}$ be a hyper BCK-algebra of order 3 and I be a nonempty subset of H. Then:*

- (i) *I is a commutative hyper BCK-ideal of type 3 if and only if I is a hyper BCK-ideal of H,*
- (ii) *I is a commutative hyper BCK-ideal of type 1 if and only if I is a weak hyper BCK-ideal of H,*
- (iii) *if I is a commutative hyper BCK-ideal of type 1, then I is a commutative hyper BCK-ideal of type 4.*

Proof. (i) $(\Longrightarrow)$ The proof follows from Theorem 4.4 (i).

$(\Longleftarrow)$ Let $I = \{0, 1\}$ be a hyper BCK-ideal of H, $(x \circ y) \circ z \ll I$ and $z \in I$ but $x \circ (y \circ (y \circ x)) \nsubseteq I$, for $x, y, z \in H$. Thus $2 \in x \circ (y \circ (y \circ x))$ and so $x \neq 0$. Because, if $x = 0$, then $2 \in 0 \circ (y \circ (y \circ 0)) = \{0\}$, which is impossible. Since $z \in I$ and I is a hyper BCK-ideal, then by Lemma 4.5, $x \circ y \subseteq I$. This implies that $2 \notin x \circ y$. If $y \in I$ (i.e. $y = 0$ or 1), then $x \in I$ and since $x \neq 0$, then $x = 1$. Now, if $y = 0$, then by hypothesis, $2 \in 1 \circ (0 \circ (0 \circ 1)) = 1 \circ 0 = \{1\}$, which is a contradiction. If $y = 1$, since $1 \circ 1 = x \circ y \subseteq I$, thus $1 \circ 1 = \{0\}$ or $\{0, 1\}$ and so $2 \in 1 \circ (1 \circ (1 \circ 1)) \subseteq \{0, 1\}$, which is impossible.

Now, let $y = 2$. Hence, $x \circ 2 = x \circ y \subseteq I$ and $2 \notin x \circ 2$. We consider two cases:

Case 1. Let H satisfies the simple condition. By Lemma 2.6 (a), $x = 1$ and $1 \circ 2 = \{1\}$ or $x = 2$ and $2 \circ 2 = \{0\}$. If $x = 1$, since by Lemma 2.6 (a), $2 \circ 1 = \{2\}$ and $2 \circ 2 = \{0\}$ or $\{0, 2\}$, thus

$$2 \in x \circ (y \circ (y \circ x)) = 1 \circ (2 \circ (2 \circ 1)) = 1 \circ (2 \circ 2) = \{1\}$$

which is a contradiction. If $x = 2$, then

$$2 \in 2 \circ (2 \circ (2 \circ 2)) = 2 \circ (2 \circ 0) = 2 \circ 2 = \{0\},$$

which is impossible.

Case 2. H satisfies the normal condition. If $x = 1$, then by Lemma 2.6 (iii) and (iv), for all $t \in H$, $2 \notin 1 \circ t$ and so

$$2 \notin \bigcup_{t \in y \circ (y \circ 1)} 1 \circ t = 1 \circ (y \circ (y \circ 1)) = x \circ (y \circ (y \circ x)$$

which contradicts the contrary hypothesis. If $x = 2$, since $2 \circ 2 = x \circ 2 \subseteq I$, then $2 \circ 2 = \{0\}$ or $\{0, 1\}$ and so $2 \in 2 \circ (2 \circ (2 \circ 2)) = \{0\}$ or $\{0, 1\}$, which is a contradiction.

Now, let $I = \{0, 2\}$ be a hyper BCK-ideal of H, $(x \circ y) \circ z \ll I$ and $z \in I$ but $x \circ (y \circ (y \circ x)) \not\subseteq I$, for $x, y, z \in I$. So, $1 \in x \circ (y \circ (y \circ x))$ and so $x \neq 0$. Since $z \in I$ and I is a hyper BCK-ideal of H, then by lemma 4.5, $x \circ y \subseteq I$. Thus $1 \notin x \circ y$. If $y \in I$, then $x \in I$ and since $x \neq 0$, thus $x = 2$. Now, if $y = 0$, then by hypothesis, $1 \in 2 \circ (0 \circ (0 \circ 2)) = 2 \circ 0 = \{2\}$ which is a contradiction. If $y = 2$, since $2 \circ 2 = x \circ y \subseteq I$, then $2 \circ 2 = \{0\}$ or $\{0, 2\}$. Hence, $1 \in 2 \circ (2 \circ (2 \circ 2)) = \{0\}$ or $\{0, 2\}$ which is impossible.

Now let $y = 1$. Since $x \circ 1 = x \circ y \subseteq I$, then $x \circ 1 = \{0\}$ or $\{2\}$ or $\{0, 2\}$. We consider the following cases:

Case 1. H satisfies the simple condition. By Lemma 2.6 (a) we have $x = 1$ and $1 \circ 1 = \{0\}$ or $x = 2$ and $2 \circ 1 = \{2\}$. If $x = 1$, then

$$1 \in 1 \circ (1 \circ (1 \circ 1)) = 1 \circ (1 \circ 0) = 1 \circ 1 = \{0\},$$

which is a contradiction. If $x = 2$, since by Lemma 2.6 (a), $1 \circ 1 = \{0\}$ or $\{0, 1\}$ and $1 \circ 2 = \{1\}$, thus

$$1 \in 2 \circ (1 \circ (1 \circ 2)) = 2 \circ (1 \circ 1) = \{2\}$$

which is impossible.

Case 2. H satisfies the normal condition. By Lemma 2.6 (b), we have $x = 1$ and $1 \circ 1 = \{0\}$ or $x = 2$ and $2 \circ 1 = \{2\}$. If $x = 1$, similar to the preceding case we get a contradiction. If $x = 2$, since by Lemma 2.6 (iv), $1 \circ 2 = \{0\}$ or $\{0, 1\}$, then $1 \in 2 \circ (1 \circ (1 \circ 2)) = \{2\}$, which is impossible.

(ii) The proof is similar to the proof (i).

(iii) Let I be a commutative hyper BCK-ideal of type 1, $(x \circ y) \circ z \ll I$ and $z \in I$ but $x \circ (y \circ (y \circ x)) \not\ll I$, for $x, y, z \in H$. If $I = \{0, 1\}$, thus $2 \in x \circ (y \circ (y \circ x))$ and $2 \not\ll 1$. Since $(x \circ y) \circ z \ll I$, then $2 \notin (x \circ y) \circ z$ and so $(x \circ y) \circ z = \{0\}$ or $\{1\}$ or $\{0, 1\}$. Hence, $(x \circ y) \circ z \subseteq I$. Since $z \in I$ and I is a commutative hyper BCK-ideal of type 1, then $x \circ (y \circ (y \circ x)) \subseteq I$ and so $x \circ (y \circ (y \circ x)) \ll I$, which is a contradiction.

The proof of the case $I = \{0, 2\}$ is similar. □

Example 4.7. Let $H = \{0, 1, 2, 3\}$. The following table shows a hyper BCK-algebra structure on H:

$\circ$	0	1	2	3
0	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{0\}$	$\{0\}$
2	$\{2\}$	$\{2\}$	$\{0\}$	$\{0\}$
3	$\{3\}$	$\{3\}$	$\{3\}$	$\{0, 3\}$

Then $I = \{0,1\}$ is a weak hyper BCK-ideal and a hyper BCK-ideal of H, but it is not commutative hyper BCK-ideal of type 1 and 3. Since, $(2 \circ 3) \circ 1 = 0 \circ 1 = \{0\} \subseteq I$ and $1 \in I$ but

$$2 \circ (3 \circ (3 \circ 2)) = 2 \circ (3 \circ 3) = 2 \circ (3 \circ 3) = 2 \circ \{0,3\} = \{0,2\} \nsubseteq I$$

Hence, $I = \{0,1\}$ is not commutative hyper BCK-ideal of type 1 and so is not commutative hyper BCK-ideal of type 3. □

Corollary 4.8. *Let $H = \{0,1,2\}$ be a hyper BCK-algebra of order 3 and I be a nonempty subset of H. Then:*

- (i) *I is a positive implicative hyper BCK-ideal of type 3 if and only if is a commutative hyper BCK-ideal of type 3,*
- (ii) *I is a positive implicative hyper BCK-ideal of type 1 if and only if is a commutative hyper BCK-ideal of type 1.*

Proof. The proof is a consequence of Theorems 3.10 and 4.6. □

Theorem 4.9. *In any hyper BCK-algebra of order 3, there is at least one commutative hyper BCK-ideal of type 2 and 4.*

Proof. Let $H = \{0,1,2\}$ be hyper BCK-algebra of order 3. We show that $I = \{0,2\}$ is a commutative hyper BCK-ideal of type 2 and 4 of H. But, by Theorem 4.2 (ii), it is enough to show that $I = \{0,2\}$ is a commutative hyper BCK-ideal of type 4. Let $(x \circ y) \circ z \ll I$ and $z \in I$ but $x \circ (y \circ (y \circ x)) \not\ll I$, for $x, y, z \in H$. Thus $1 \in x \circ (y \circ (y \circ x))$ and $1 \not\ll 2$. Moreover, by Theorem 2.2 (v), $x \neq 0$. Since $z \in I$, thus $z = 0$ or $z = 2$.

Now we consider two following cases:

Case 1. Let $z = 0$. Then $x \circ y = (x \circ y) \circ 0 = (x \circ y) \circ z \ll I$. Since $1 \not\ll 2$, then $1 \notin x \circ y$. Hence $x \circ y = \{0\}$ or $\{2\}$ or $\{0,2\}$.

Case 1.1. Let $x \circ y = \{0\}$. Then by Lemma 2.6, $x = y = 1$ or $x = y = 2$ or $x = 1, y = 2$. If $x = y = 1$ or $x = y = 2$, then by hypothesis $1 \in x \circ (y \circ (y \circ x)) = \{0\}$, which is impossible. If $x = 1$ and $y = 2$, then $1 \circ 2 = \{0\}$ and so $1 \ll 2$, which is impossible.

Case 1.2. Let $x \circ y = \{2\}$. Then by Lemma 2.6, $x = 2$ and $y = 1$. Since $2 \circ 1 = \{2\}$, then $2 \not\ll 1$ and so H satisfies the simple condition. But in this case, $1 \in x \circ (y \circ (y \circ x)) = 2 \circ (1 \circ (1 \circ 2)) = 2 \circ (1 \circ 1) \subseteq 2 \circ \{0,1\} = \{2\}$, which is impossible.

Case 1.3. Let $x \circ y = \{0,2\}$. Then by Lemma 2.6, $x = 2$ and $y = 2$. Hence $2 \circ 2 = \{0,2\}$ and so

$$1 \in x \circ (y \circ (y \circ x)) = (2 \circ (2 \circ 2)) = 2 \circ (2 \circ \{0,2\}) = 2 \circ \{0,2\} = \{0,2\},$$

which is impossible.

Case 2. Let $z = 2$. Hence $(x \circ y) \circ 2 \ll I$. Since $1 \not\ll 2$, then by Lemma 2.6, $1 \circ 2 = \{1\}$ and $1 \notin (x \circ y) \circ 2$.

Case 2.1. Let $y = 0$. Then

$$1 \in x \circ (y \circ (y \circ x)) = x \circ (0 \circ (0 \circ x)) = x \circ 0 = \{x\}$$

and $1 \notin (x \circ y) \circ 2 = (x \circ 0) \circ 2 = x \circ 2$. Thus, $x = 1$, and so $1 \notin 1 \circ 2 = \{1\}$, which is impossible.

Case 2.2. Let $y = 1$. Then

$$1 \in x \circ (y \circ (y \circ x)) = x \circ (1 \circ (1 \circ x)) \quad \text{and} \quad 1 \notin (\mathrm{x} \circ \mathrm{y}) \circ 2 = (\mathrm{x} \circ 1) \circ 2$$

If $x = 1$, then $1 \in 1 \circ (1 \circ (1 \circ 1))$ and so $1 \circ 1 \neq \{0\}$. Hence, by Theorem 2.6, $1 \circ 1 = \{0, 1\}$. But, in this case, $1 \notin (x \circ 1) \circ 2 = \{0, 1\} \circ 2 = \{0, 1\}$, which is impossible.

If $x = 2$, then

$$1 \in 2 \circ (1 \circ (1 \circ 2)) = 2 \circ (1 \circ 1) \quad , \quad 1 \notin (2 \circ 1) \circ 2$$

By Theorem 2.6, $2 \circ 1 = \{1\}$ or $\{2\}$ or $\{1, 2\}$. If $2 \circ 1 = \{1\}$, then $1 \notin (2 \circ 1) \circ 2 = 1 \circ 2 = \{1\}$, which is impossible. If $2 \circ 1 = \{2\}$, then $1 \in 2 \circ (1 \circ 1) \subseteq 2 \circ \{0, 1\} = \{2\}$, which is impossible. If $2 \circ 1 = \{1, 2\}$, then $1 \notin (2 \circ 1) \circ 2 \subseteq \{1, 2\} \circ 2 \subseteq \{0, 1, 2\}$, which is impossible.

Case 2.3. Let $y = 2$. Then

$$1 \in x \circ (y \circ (y \circ x)) = x \circ (2 \circ (2 \circ x)) \quad , \quad 1 \notin (x \circ y) \circ 2 = (x \circ 2) \circ 2$$

If $x = 1$, then $1 \notin (1 \circ 2) \circ 2 = \{1\} \circ 2 = \{1\}$, which is impossible. If $x = 2$, then $1 \in 2 \circ (2 \circ (2 \circ 2))$ and $1 \notin (2 \circ 2) \circ 2$. If $1 \in 2 \circ 2$, then $\{1\} = 1 \circ 2 \subseteq (2 \circ 2) \circ 2$, which is impossible. Since $0 \in 2 \circ 2$, hence $2 \circ 2 = \{0\}$ or $\{0, 2\}$. If $2 \circ 2 = \{0\}$, then $1 \in 2 \circ (2 \circ (2 \circ 2)) = \{0\}$, which is impossible. If $2 \circ 2 = \{0, 2\}$, then $1 \in 2 \circ (2 \circ (2 \circ 2)) = \{0, 2\}$, which is impossible.

Therefore, $I = \{0, 2\}$ is a commutative hyper BCK-ideal of type 4. □

Corollary 4.10. *Let $H = \{0, 1, 2\}$ be a hyper BCK-algebra of order 3 and I be a nonempty subset of H. Then I is a commutative hyper BCK-ideal of type 2 if and only if I is a commutative hyper BCK-ideal of type 4.*

Proof. ($\Longleftarrow$) The proof follows by Theorem 4.2 (*ii*).

($\Longrightarrow$) Let I be a commutative hyper BCK-ideal of type 2 of $H = \{0, 1, 2\}$. If $I = \{0, 2\}$, then by the proof of Theorem 4.9, I is a commutative hyper BCK-ideal of type 4. If $I = \{0, 1\}$, then by Theorems 3.1, 3.3, 3.4 and 3.5 of [3], there are only 3 non-isomorphic hyper BCK-algebra of order 3 such that $I = \{0, 1\}$ is not a hyper BCK-ideal of them, which are as follows:

$\circ_1$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{0\}$
2	$\{2\}$	$\{1\}$	$\{0\}$

$\circ_2$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{0\}$
2	$\{2\}$	$\{1\}$	$\{0, 1\}$

$\circ_3$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0, 1\}$	$\{0, 1\}$
2	$\{2\}$	$\{1\}$	$\{0, 1\}$

Moreover, in the above hyper BCK-algebras, $I = \{0, 1\}$ is not a commutative hyper BCK-ideal of type 2. Since, in all of them, $(2 \circ 0) \circ 1 = 2 \circ 1 = \{1\} \subseteq \{0, 1\}$ and $1 \in \{0, 1\}$ but $2 \circ (0 \circ (0 \circ 2)) = 2 \circ 0 = \{2\} \not\ll \{0, 1\}$.

Now, since except of the above 3 hyper BCK-algebras, $I = \{0, 1\}$ is a hyper BCK-ideal of H, then by Theorem 4.6(i), $I = \{0, 1\}$ is a commutative hyper BCK-ideal of type 3 and so by Theorem 4.2(i), it is a commutative hyper BCK-ideal of type 4. □

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