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Hyper I–algebras and polygroups

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Abstract

In this note first we give the notion of hyper I-algebra, which is a generalization of BCI-algebra and also it is a generalization of hyper K-algebra. Then we obtain some fundamental results about this notion. Finally we give some relationships between the notion of hyper I-algebra and the notions of hypergroup and polygroup. In particular we study these connections categorically. In other words by considering the categories of hyper I-algebrs, hypergroups and commutative polygroups, we give some full and faithful functors.

1. Introduction

The hyperalgebraic structure theory was introduced by F.Marty [8] in 1934. Imai and Iseki [7] in 1966 introduced the notion of a BCK-algebra. Recently [2], [9] Borzooei, Jun and Zahedi et.al. applied the hypersrtucture to BCK-algebras and introduced the concepts of hyper K-algebra which is a generalization of BCK-algebra. In [5] 1988 Dudek obtained some connections between BCI-algebras and (quasi)groups. Bonansinga and Corsini [1] in 1982 introduced the notion of quasi-canonical hypergroup, called polygroup by Comer [3]. Now in this note we consider all of the above referred papers and introduce the notion of hyper I-algebra and then we obtain some results as mentioned in the abstract.

2. Preliminaries

By a *hyperstructure* $(H, \circ)$ we mean a nonempty set H with a *hyperoperation* $\circ$, i.e. a function $\circ$ from $H \times H$ to $\mathcal{P}(H) \setminus \{\emptyset\}$.

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Definition 2.1. A hyperstructure $(H, \circ)$ is called *hypergroup* if:

(i) $(x \circ y) \circ z = x \circ (y \circ z)$ for all $x, y, z \in H$,

(ii) $a \circ H = H \circ a = H$ for all $a \in H$,

(i.e. for all $a, b \in H$ there exist $c, d \in H$ such that $b \in c \circ a$ and $b \in a \circ d$).

Definition 2.2. A hyperstructure $(H, \circ)$ is called *quasi-canonical hypergroup* or *polygroup* if it satisfies the following conditions:

(i) $(x \circ y) \circ z = x \circ (y \circ z)$ for all $x, y, z \in H$ (*associative law*),

(ii) there exists $e \in H$ such that $e \circ x = \{x\} = x \circ e$ for all $x \in H$ (*identity element*),

(iii) for all $x \in H$ there exists a unique element $x' \in H$ such that $e \in (x \circ x') \bigcap (x' \circ x)$, we denote x' by x^{-1} (*inverse element*),

(iv) for all $x, y, z \in H$ we have: $z \in x \circ y \Longrightarrow x \in z \circ y^{-1} \Longrightarrow y \in x^{-1} \circ z$ (*reversibility property*).

If $(H, \circ)$ is a polygroup and $x \circ y = y \circ x$ holds for all $x, y \in H$, then we say that H is a *commutative polygroup*.

If $A \subseteq H$, then by A^{-1} we mean the set $\{a^{-1} : a \in A\}$.

Lemma 2.3. *Let $(H, \circ)$ be a polygroup. Then for all $x, y \in H$, we have:*

(i) $(x^{-1})^{-1} = x$,

(ii) $e = e^{-1}$,

(iii) *e is unique,*

(iv) $(x \circ y)^{-1} = y^{-1} \circ x^{-1}$.

Proof. See [4]. □

Lemma 2.4. *Let $(H, \circ)$ be a polygroup. Then $(A \circ B) \circ C = A \circ (B \circ C)$ for all nonempty subsets A, B and C of H.* □

3. Hyper I-algebra

Definition 3.1. A hyperstructure $(H, \circ)$ is called a *hyper I-algebra* if it contains a constant 0 and satisfies the following axioms:

(HK1) $(x \circ z) \circ (y \circ z) < x \circ y$,

(HK2) $(x \circ y) \circ z = (x \circ z) \circ y$,

(HK3) $x < x$,

(HK4) $x < y,\ y < x \Longrightarrow x = y$,

(HI5) $x < 0 \Longrightarrow x = 0$,

for all $x, y, z \in H$, where $x < y$ is defined by $0 \in x \circ y$ and for every $A, B \subseteq H$, $A < B$ is defined by $\exists a \in A$, $\exists b \in B$ such that $a < b$.

A simple example of a hyper I-algebra is a BCI-algebra $(H,*,0)$ with the hyperopration $\circ$ defined by $x\circ y=\{x*y\}$. Also it is not difficult to see that a hyper I-algebra is a generalization of hyper K-algebras considered in [2] and [9]. The following example shows that there are hyper I-algebras which are not a hyper K-algebras.

Example 3.2. Let $H=\{0,1,2\}$. Then the following tables show the hyper I-algebra structures on H.

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{2\}$
1	$\{1\}$	$\{0\}$	$\{2\}$
2	$\{2\}$	$\{2\}$	$\{0,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,1\}$	$\{2\}$
1	$\{1\}$	$\{0\}$	$\{2\}$
2	$\{2\}$	$\{0\}$	$\{0,1,2\}$

Note that none of the above hyper I-algebras is not a hyper K-algebra, because $0\not< 2$. □

Theorem 3.3. *Let $(H,\circ,0)$ be a hyper I-algebra. Then for all $x,y,z\in H$ and for all non-empty subsets A, B and C of H the following hold:*

(*i*) $x\circ y<z \Longleftrightarrow x\circ z<y$,

(*ii*) $(x\circ z)\circ(x\circ y)<y\circ z$,

(*iii*) $x\circ(x\circ y)<y$,

(*iv*) $(A\circ B)\circ C=(A\circ C)\circ B$,

(*v*) $A\subseteq B \Longrightarrow A<B$,

(*vi*) $A<A$,

(*vii*) $(A\circ C)\circ(A\circ B)<B\circ C$,

(*viii*) $(A\circ C)\circ(B\circ C)<A\circ B$,

(*ix*) $A\circ B<C \Leftrightarrow A\circ C<B$.

Proof. The proof is similar to the proof of Proposition 2.5 of [2]. □

Example 3.4. Let $H=\{0,1,2\}$. Then the following table shows a hyper I-algebra structure on H such that $x\circ y\not< x$, because $1\circ 2=2\not< 1$.

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{2\}$
1	$\{1\}$	$\{0,1\}$	$\{2\}$
2	$\{2\}$	$\{2\}$	$\{0\}$

Lemma 3.5. *Let H be a hyper I-algebra. Then for all x in H we have:*

(*i*) $x\circ 0<x$,

(*ii*) $x\in x\circ 0$.

Proof. (*i*) We have $0\in 0\circ 0\subseteq(x\circ x)\circ 0=(x\circ 0)\circ x$. So there exists $t\in x\circ 0$ such that $0\in t\circ x$. Thus $t<x$, and hence $x\circ 0<x$.

(ii) By (i) $x \circ 0 < x$. So there exists $t \in x \circ 0$ such that $t < x$. Since $t \in x \circ 0$, then $x \circ 0 < t$ and hence $x \circ t < 0$, by Theorem 3.3(i). Thus there exists $h \in x \circ t$ such that $h < 0$, so by (HI5) we have $h = 0$. Therefore $0 \in x \circ t$ and hence $x < t$. Since $t < x$, then by (HK4) we get that $t = x$. Therefore $x \in x \circ 0$. □

Definition 3.6. Let $(H, \circ, 0)$ be a hyper I-algebra. We define

$$H^+ = \{x \in H \mid 0 \in 0 \circ x\}.$$

Note that $H^+ \neq \emptyset$ because $0 \in 0 \circ 0$.

Proposition 3.7. *Let $(H, \circ, 0)$ be a hyper I-algebra. Then $(H^+, \circ, 0)$ is a hyper K-algebra if and only if $x \circ y \subseteq H^+$, for all x,y in H^+.*

Proof. Straightforward. □

Example 3.8. (i) Let $H = \{0, 1, 2\}$. Then the following tables show two different hyper I-algebra structures on H:

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{2\}$
1	$\{1\}$	$\{0,1\}$	$\{2\}$
2	$\{2\}$	$\{2\}$	$\{0,1\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{2\}$
1	$\{1\}$	$\{0,1\}$	$\{0,2\}$
2	$\{2\}$	$\{2\}$	$\{0,1,2\}$

We can seen that $H^+ = \{0, 1\}$ and it is a hyper K-algebra.

(ii) The following table shows a hyper I-algebra structure on $H = \{0, 1, 2\}$, where $H^+ = \{0, 1\}$ and it is not a hyper K-algebra, since $1 \in H^+$ but $1 \circ 1 \not\subseteq H^+$.

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{2\}$
1	$\{1\}$	$\{0,2\}$	$\{0,2\}$
2	$\{2\}$	$\{2\}$	$\{0,2\}$

Theorem 3.9. *Let $(H, \circ, e)$ be a commutative polygroup. Then $(H, \diamond, e)$ is a hyper I-algebra, where the hyperopration $\diamond$ is defined by $x \diamond y = x \circ y^{-1}$. Furthermore we have:*

(i) $H^+ = \{e\}$,

(ii) $e \diamond (e \diamond x) = x$ *for all x in H.*

Proof. $(HK1)$ Let $A = (x \diamond y) \diamond (z \diamond y)$. Then by considering Lemma 2.3 we have $A = (x \diamond y) \diamond (z \diamond y) = \bigcup_{\substack{a \in x \diamond y \\ b \in z \diamond y}} a \diamond b = \bigcup_{\substack{a \in x \circ y^{-1} \\ b \in z \circ y^{-1}}} a \circ b^{-1} = \bigcup_{\substack{a \in x \circ y^{-1} \\ b^{-1} \in y \circ z^{-1}}} a \circ b^{-1}$.

Thus, by Lemma 2.4, we get that

$$A = (x \circ y^{-1}) \circ (y \circ z^{-1}) = x \circ (y^{-1} \circ (y \circ z^{-1})) = x \circ ((y^{-1} \circ y) \circ z^{-1}).$$

By Lemma 2.3 we have

$$A \diamond (x \diamond z) = \bigcup_{\substack{a \in A \\ b \in x \diamond z}} a \diamond b = \bigcup_{\substack{a \in A \\ b \in x \circ z^{-1}}} a \circ b^{-1} = A \circ (z \circ x^{-1}).$$

Since $e \in y^{-1} \circ y$, hence $e \circ z^{-1} \subseteq (y^{-1} \circ y) \circ z^{-1}$, so

$$x \circ (e \circ z^{-1}) \subseteq x \circ ((y^{-1} \circ y) \circ z^{-1}) = A.$$

Thus we get that

$$(x \circ z^{-1}) \circ (z \circ x^{-1}) = (x \circ (e \circ z^{-1})) \circ (z \circ x^{-1}) \subseteq A \circ (z \circ x^{-1}) = A \diamond (x \diamond z).$$

Now, by Definition 2.2 and Lemma 2.4 we have

$$x \circ ((z^{-1} \circ z) \circ x^{-1}) \\ = x \circ (z^{-1} \circ (z \circ x^{-1})) = (x \circ z^{-1}) \circ (z \circ x^{-1}) \subseteq A \diamond (x \diamond z).$$

Since $e \in z^{-1} \circ z$ and $e \in x \circ x^{-1}$, then we have $e \in A \diamond (x \diamond z)$, so $A < x \diamond z$. Therefore $(x \diamond y) \diamond (z \diamond y) < x \diamond z$.

(HK2) By Definition 2.2 and hypothesis we get that $(x \diamond y) \diamond z = (x \circ y^{-1}) \diamond z = (x \circ y^{-1}) \circ z^{-1} = x \circ (y^{-1} \circ z^{-1}) = x \circ (z^{-1} \circ y^{-1}) = (x \circ z^{-1}) \circ y^{-1} = (x \diamond z) \diamond y$. Therefore (HK2) holds.

(HK3) Since $e \in x \circ x^{-1} = x \diamond x$ we conclude that $x < x$ and hence (HK3) holds.

(HK4) To show that $(HK4)$ holds, we prove that $x < y$ implies that $x = y$. Let $x < y$. Then $e \in x \diamond y = x \circ y^{-1}$. By Definition 2.2 (vi) we have $y \in e^{-1} \circ x = e \circ x = \{x\}$, thus $y = x$.

(HI5) Let $x < e$. Then by the proof of $(HK4)$ we get that $e = x$, and hence (HI5) holds.

Therefore $(H, \diamond, e)$ is a hyper I-algebra.

The proofs of the statements (i) and (ii) are routine. □

Category of commutative polygroups: $\mathcal{CPG}$

Consider the class of all polygroups. For any two polygroups $(H_1, \circ_1, e_1)$ and $(H_2, \circ_2, e_2)$ we define a morphism $f : H_1 \longrightarrow H_2$ as a strong homomorphism between H_1 and H_2 (i.e. $f(x \circ_1 y) = f(x) \circ_2 f(y)\ \forall x, y \in H$), which satisfies $f(e_1) = e_2$. Then it can easily checked that the class of all polygroups and the above morphisms construct a category which is denoted by $\mathcal{CPG}$.

Remark 3.10. It is well known that if $f \in \mathcal{CPG}(H_1, H_2)$, then $f(x^{-1}) = (f(x))^{-1}$ for all $x \in H_1$.

Category of hyper I-algebras: $\mathcal{IALG}$

Consider the class of all hyper I-algebras. For any two I-algebras $(H_1, \circ_1, 0_1)$ and $(H_2, \circ_2, 0_2)$ we define a morphism $f : H_1 \longrightarrow H_2$ as a strong homomorphism between H_1 and H_2, which satisfies the condition $f(0_1) = 0_2$. Then it can easily checked that the class of all hyper I-algebras and the above morphisms construct a category which is denoted by $\mathcal{IALG}$.

Theorem 3.11. *$F : \mathcal{CPG} \longrightarrow \mathcal{IALG}$ is a faithful functor, where $F(H, \circ, e) = (H, \diamond, e)$ and $F(f) = f$ for all $H \in \mathcal{CPG}$ and $f \in \mathcal{CPG}(H_1, H_2)$.*

Proof. Let $(H, \circ, e)$ be a polygroup. Then by Theorem 3.9 $(H, \diamond, e)$ is a hyper I-algebra, hence $F(H)$ is an object in $\mathcal{IALG}$. Now let $f \in \mathcal{CPG}(H_1, H_2)$ we prove that $Ff \in \mathcal{IALG}(F(H_1), F(H_2))$. By Theorem 3.9 we have

$$\begin{aligned} Ff(x \diamond_1 y) &= f(x \diamond_1 y) = f(x \circ_1 y^{-1}) = f(x) \circ_2 f(y^{-1}) \\ &= f(x) \circ_2 (f(y))^{-1} = f(x) \diamond_2 f(y) = (Ff)(x) \diamond_2 (Ff)(y). \end{aligned}$$

Now it is easy to see that F satisfies to the other conditions of a functor. Since F maps $\mathcal{CPG}(H_1, H_2)$ injectively to $\mathcal{IALG}(FH_1, FH_2)$, hence F is faithful. □

Problem: *Is the functor F (defined in Theorem 3.11) full embedding ?*

Definition 3.12. A hyperstructure $(H, \circ)$ is called a *semipolygroup* if it satisfies the following axioms:

(*i*) $(x \circ y) \circ z = x \circ (y \circ z)$ for all $x, y, z \in H$,

(*ii*) there exists $e \in H$ such that $e \circ x = \{x\} = x \circ e$ for all $x \in H$,

(*iii*) for all $x \in H$ there exists a unique element $x' \in H$ such that $e \in (x \circ x') \bigcap (x' \circ x)$, we denote x' by x^{-1}.

Example 3.13. Let $H = \{0, 1, 2\}$ and the hyperopration $\circ$ on H is given by the following table:

$\circ$	0	1	2
0	$\{0\}$	$\{1\}$	$\{2\}$
1	$\{1\}$	$\{2\}$	$\{0,1\}$
2	$\{2\}$	$\{0,1\}$	$\{1,2\}$

Then H is a semipolygroup, but it is not a polygroup because the reversibility does not hold. Indeed, $1 \in 1\circ 2 = \{0, 1\}$ but $1 \notin 1\circ 2^{-1} = 1\circ 1 = \{2\}$. □

Lemma 3.14. *Any group can be cosidered as a semipolygroup.* □

Lemma 3.15. *Let $(H, \circ, 0)$ be a hyper I-algebra. If $H^+ \neq \{0\}$, then $0 \circ (0 \circ x) \neq x$ for all nonzero elements $x \in H^+$.*

Proof. Let $x \neq 0$ be in H^+. Then $0 \in (0 \circ x)$. Thus $0 \in (0 \circ 0) \subseteq 0 \circ (0 \circ x)$, hence $0 \in 0 \circ (0 \circ x)$. Since $x \neq 0$, so $0 \circ (0 \circ x) \neq x$. □

Note that the following example shows that if $H^+ = \{0\}$, then it may be that the equality $0 \circ (0 \circ x) = x$ holds or does not hold.

Example 3.16. (i) Let $H = \{0, 1, 2\}$. Then the following table shows a hyper I-algebra structure on H such that $H^+ = \{0\}$, while $0 \circ (0 \circ 2) = 0 \circ 1 = 1 \neq 2$.

$\circ$	0	1	2
0	$\{0\}$	$\{1\}$	$\{1\}$
1	$\{1\}$	$\{0,1\}$	$\{0,1\}$
2	$\{2\}$	$\{1\}$	$\{0,1,2\}$

(ii) The following table shows a hyper I-algebra structure on $H = \{0, 1, 2\}$. Then $H^+ = \{0\}$ and $0 \circ (0 \circ x) = x$ for all $x \in H$.

$\circ$	0	1	2
0	$\{0\}$	$\{2\}$	$\{1\}$
1	$\{1\}$	$\{0,1\}$	$\{2\}$
2	$\{2\}$	$\{1,2\}$	$\{0,1\}$

Theorem 3.17. *Let $(H, \circ, 0)$ be a hyper I-algebra. If $H^+ = \{0\}$ and $0\circ(0\circ x) = x$ for all $x \in H$, then $(H, \odot, 0)$ is a commutative semipolygroup, where the hyperopration $\odot$ is defined by $x \odot y = x \circ (0 \circ y)$.*

Proof. By Theorem 3.3(iv) we get that $x\odot y = x\circ(0\circ y) = (0\circ(0\circ x))\circ(0\circ y) = (0 \circ (0 \circ y)) \circ (0 \circ x) = y \circ (0 \circ x) = y \odot x$, namely $(H, \odot)$ is commutative.

Now we show that $(H, \odot)$ is associative. We have

$$\begin{aligned}
(x \odot y) \odot z &= (x \circ (0 \circ y)) \circ (0 \circ z) \\
&= (x \circ (0 \circ z)) \circ (0 \circ y) && \text{by Theorem 3.3 (iv)} \\
&= ((0 \circ (0 \circ x)) \circ (0 \circ z)) \circ (0 \circ y) && \text{by hypothesis} \\
&= ((0 \circ (0 \circ z)) \circ (0 \circ x)) \circ (0 \circ y) && \text{by Theorem 3.3 (iv)} \\
&= (z \circ (0 \circ y)) \circ (0 \circ x) && \text{by Theorem 3.3 (iv)} \\
&= (z \odot y) \odot x \\
&= x \odot (z \odot y) && \text{by commutativity} \\
&= x \odot (y \odot z) && \text{by commutativity}
\end{aligned}$$

Thus $(H, \odot)$ is associative.

Now, we prove that $0 \circ x$ has only one element for all $x \in H$. On the contrary, let $x_1, x_2 \in 0 \circ x$ and $x_1 \neq x_2$. Then by hypothesis we have $0 \circ x_1 \subseteq 0 \circ (0 \circ x) = x$, hence $0 \circ x_1 = x$ and similarly $0 \circ x_2 = x$. Thus $0 \circ (0 \circ x_1) = x_1$ and $0 \circ x_1 = x$ imply that $0 \circ x = x_1$. Since $x_2 \in 0 \circ x$, hence $x_1 = x_2$ which is a contradiction.

Since $0 \circ x$ has only one element for all $x \in H$, hence $0 \in 0 \circ 0$, thus we conclude that $0 \circ 0 = 0$. By Theorem 3.3 (iv) and hypothesis we get that $x \circ 0 = (0 \circ (0 \circ x)) \circ 0 = (0 \circ 0) \circ (0 \circ x) = 0 \circ (0 \circ x) = x$. Hence $x \circ 0 = x$. Therefore $0 \odot x = x \odot 0 = x \circ (0 \circ 0) = x \circ 0 = x$. So $(H, \odot)$ satisfies in condition (ii) of Definition 3.12.

Since $H^+ = \{0\}$ hence $0 \notin 0 \circ x$ for all $x \neq 0$. Therefore for all $0 \neq x \in H$ there exists $0 \neq x' \in H$ such that $0 \circ x = x'$. By Theorem 3.3 (vi) we have $0 \in (0 \circ x) \circ (0 \circ x) = x' \circ (0 \circ x) = x' \odot x = x \odot x'$. Thus the condition (iii) of Definition 3.12 holds. Therefore $(H, \odot)$ is a commutative semipolygroup. □

Theorem 3.18. *Let $(H, \circ, 0)$ be a hyper I-algebra such that $H^+ = \{0\}$. If $0 \circ (0 \circ x) = x$ and $x \circ x = 0$ hold for all $x \in H$, then $(H, \odot, 0)$ is an abelian group.*

Proof. By considering Theorem 3.17 it is sufficient to show that $x \circ y$ has only one element for all $x, y \in H$. On the contrary let $x_1 \neq x_2$ and $x_1, x_2 \in x \circ y$. Then by the proof of Theorem 3.17 we conclude that there are $x', y' \in H$ such that $0 \circ x = x'$, $0 \circ y = y'$, $0 \circ x' = x$ and $0 \circ y' = y$. By (HK2) and $x \circ x = 0$ we get that $y' = 0 \circ y = (x \circ x) \circ y = (x \circ y) \circ x$. Since $x_1, x_2 \in x \circ y$, hence $x_1 \circ x = y'$ and $x_2 \circ x = y'$. Thus $y' \circ x_1 = (x_1 \circ x) \circ x_1 = (x_1 \circ x_1) \circ x = 0 \circ x = x'$ and also $y' \circ x_2 = x'$. By (HK2) and hypothesis we get that $(y' \circ x') \circ x_1 = (y' \circ x_1) \circ x' = x' \circ x' = \{0\}$, similarly $(y' \circ x') \circ x_2 = \{0\}$. Since $0 \in (y' \circ x') \circ x_1$ so there exists $t \in y' \circ x'$ such that $0 \in t \circ x_1$. By (HK2) we have $(t \circ x_1) \circ t = (t \circ t) \circ x_1 = 0 \circ x_1$. Since $0 \in t \circ x_1$ hence $0 \circ t \subseteq 0 \circ x_1$. By the proof of Theorem 3.17 $0 \circ x_1$ has only one element so we get that $0 \circ t = 0 \circ x_1$. By hypothesis we have $t = 0 \circ (0 \circ t) = 0 \circ (0 \circ x_1) = x_1$. Therefore $x_1 \in y' \circ x'$. Since $(y' \circ x') \circ x_2 = 0$, then $x_1 \circ x_2 = 0$ and similarly $x_2 \circ x_1 = 0$. Thus (HK4) implies that $x_1 = x_2$, which is a contradiction. So $x \circ y$ has only one element. Therefore Theorem 3.17 implies that $(H, \odot, 0)$ is an abelian group. □

Since every group is a polygroup hence $(H, \odot)$ in Theorem 3.18 is a commutative polygroup. The following example shows that in Theorem 3.18 the condition $x \circ x = 0$ for all $x \in H$ is necessary.

Example 3.19. Let $H = \{0, 1, 2\}$ be a hyper I-algebra, in which the hyperopration $\circ$ is given by the following table:

$\circ$	0	1	2
0	$\{0\}$	$\{2\}$	$\{1\}$
1	$\{1\}$	$\{0,1\}$	$\{2\}$
2	$\{2\}$	$\{1,2\}$	$\{0,1\}$

Then $H^+ = \{0\}$, $0 \circ (0 \circ x) = x$ for all $x \in H$ and $1 \circ 1 \neq 0$. But $(H, \odot, 0)$ is not a group since $1 \odot 2 = \{0, 1\}$. □

Note that the above example also shows that if we omit the condition $x \circ x = 0$, in Theorem 3.18, then $(H, \odot)$ is not necessary to be a polygroup. Because the reversibility property does not hold. Indeed, in this example we have $1 \in 1 \odot 2 = 1 \circ (0 \circ 2) = 1 \circ 1 = \{0, 1\}$, but $1 \notin 1 \odot 2^{-1} = 1 \circ (0 \circ 2^{-1}) = 1 \circ (0 \circ 1) = 1 \circ 2 = 2$.

Theorem 3.20. *Let $(H, \circ, 0)$ be a hyper I-algebra. If $H^+ = \{0\}$ and $0 \circ (0 \circ x) = x$ for all $x \in H$, then $(H, \odot, 0)$ is a commutative hypergroup.*

Proof. The proof of Theorem 3.17 shows that $(H, \odot, 0)$ is commutative and associative. Let $a, b \in H$ be arbitrary. By the proof of Theorem 3.17 there exists $a' \in H$ such that $0 \in a' \odot a$ and $b \odot 0 = b$. Thus $b \in b \odot 0 \subseteq b \odot (a' \odot a) = (b \odot a') \odot a$. So there exists $t \in b \odot a'$ such that $b \in t \odot a = a \odot t$, namely $a \odot H = H \odot a = H$.

Hence $(H, \odot, 0)$ is a commutative hypergroup. □

Notation: Let $\mathcal{I}^+\mathcal{ALG}$ be a subcategory of $\mathcal{IALG}$ in which for every object H we have $H^+ = \{0\}$ and $0 \circ (0 \circ x) = 0$ for all $x \in H$. Similarly, let $\mathcal{CHG}$ be the category of commutative hypergroups with strong morphisms.

Theorem 3.21. *$G : \mathcal{I}^+\mathcal{ALG} \longrightarrow \mathcal{CHG}$ is a faithful functor, where $G(H, \circ, 0) = (H, \odot, 0)$ for $H \in \mathcal{I}^+\mathcal{ALG}$ and $G(f) = f$ for $f \in \mathcal{I}^+\mathcal{ALG}(H_1, H_2)$.*

Proof. Let $(H, \circ, 0)$ be an object in $\mathcal{I}^+\mathcal{ALG}$. Then by Theorem 3.20 we have $G(H) = (H, \odot, 0)$ is an object in $\mathcal{CHG}$.

Let $f \in \mathcal{I}^+\mathcal{ALG}(H_1, H_2)$. We prove that $Gf = f \in \mathcal{CHG}(G(H_1), G(H_2))$. By Theorem 3.20 we have

$$\begin{aligned} Gf(x \odot_1 y) &= f(x \odot_1 y) = f(x \circ_1 (0_1 \circ_1 y)) = f(x) \circ_2 (f(0_1) \circ_2 f(y)) \\ &= f(x) \circ_2 (0_2 \circ_2 f(y)) = f(x) \odot_2 f(y) = (Gf)(x) \odot_2 (Gf)(y). \end{aligned}$$

So it is easy to see that G satisfies to the other condition of a functor. Since G maps $\mathcal{I}^+\mathcal{ALG}(H_1, H_2)$ injectively to $\mathcal{CHG}(GH_1, GH_2)$, hence G is faithful. □

Remark 3.22. Let $F : \mathcal{CPG} \longrightarrow \mathcal{IALG}$ and $G : \mathcal{I}^+\mathcal{ALG} \longrightarrow \mathcal{CHG}$ be the functors which are defined in Theorem 3.11 and 3.21 respectively. By Theorem 3.9, we have $H^+ = \{0\}$ and $0 \diamond (0 \diamond x) = x$ for all $H \in F(\mathcal{CPG})$ and $x \in H$. Hence $F(\mathcal{CPG}) \subseteq \mathcal{I}^+\mathcal{ALG}$. Since $x \odot y = x \diamond (0 \diamond y) = x \diamond (0 \circ y^{-1}) = x \diamond (y^{-1}) = x \circ (y^{-1})^{-1} = x \circ y$. We get that $GF(H) = G(FH) = G(H) = H$ for all $H \in \mathcal{CPG}$ and $(GF)(f) = G(Ff) = G(f) = f$ for all $f \in \mathcal{CPG}(H_1, H_2)$. Therefore $GF = I$.

Let $\mathcal{CSPG}$ be the category of commutative semipolygroups. Then $f \in \mathcal{CSPG}((H_1, \circ_1, 0_1), (H_2, \circ_2, 0_2))$ if and only if $f(x \circ_1 y) = f(x) \circ_2 f(y)$ and $f(e_1) = e_2$.

Proposition 3.23. *$K : \mathcal{I}^+\mathcal{ALG} \longrightarrow \mathcal{CSPG}$ is a full embedding functor, where $K(H, \circ, 0) = (H, \odot, 0)$ for all $H \in \mathcal{I}^+\mathcal{ALG}$ and $K(f) = f$ for all $f \in \mathcal{I}^+\mathcal{ALG}(H_1, H_2)$.*

Proof. The proof of Theorem 3.21 shows that K is a faithful functor. Now we show that it is full, i.e. $K(\mathcal{I}^+\mathcal{ALG}(H_1, H_2)) = \mathcal{CSPG}(KH_1, KH_2)$. By the proof of Theorem 3.17, for all $y \in H$ there exists a unique $y' = y^{-1} \in H$ such that $0_1 \circ_1 y = y^{-1}$ and $0_1 \circ_1 y^{-1} = y$. Hence for all $f \in \mathcal{CSPG}(H_1, H_2)$ we get that

$$f(x \circ_1 y) = f(x \circ_1 (0_1 \circ_1 y^{-1})) = f(x \odot_1 y^{-1}) = f(x) \odot_2 f(y^{-1}).$$

Since $0_2 \in f(0_1) \subseteq f(y \odot_1 y^{-1}) = f(y) \odot f(y^{-1})$, hence by Definition 3.12 (iii) we get that $f(y^{-1}) = (f(y))^{-1}$. Thus we have

$$f(x \circ_1 y) = f(x) \odot_2 (f(y))^{-1} = f(x) \circ_2 (0_2 \circ_2 (f(y))^{-1}) = f(x) \circ_2 f(y).$$

Hence K is full functor. Since K maps $\mathcal{I}^+\mathcal{ALG}(H_1, H_2)$ injectively to $\mathcal{CSPG}(KH_1, KH_2)$, then K is faithful. Since K is full and faithful and one-to-one on objects so is full embedding. Thus $K(\mathcal{I}^+\mathcal{ALG})$ is a full subcategory of $\mathcal{CSPG}$. $\square$

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