Abel-Grassmann's bands

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Abstract

Abel-Grassmann's groupoids or shortly *AG*-groupoids have been considered in a number of papers, although under the different names. In some papers they are named *LA*-semigroups [3] in others left invertive groupoids [2]. In this paper we deal with *AG*-bands, i.e., *AG*-groupoids whose all elements are idempotents. We introduce a few congruence relations on *AG*-band and consider decompositions of Abel-Grassmann's bands induced by these congruences. We also give the natural partial order on Abel-Grassmann's band.

1. Introduction

A groupoid S in which the following

$$(\forall a, b, c \in S) \ \ ab \cdot c = cb \cdot a, \tag{1}$$

is true is called an *Abel-Grassmann's* groupoid, [5]. It is easy to verify that in every *AG*-groupoid the *medial law* $ab \cdot cd = ac \cdot bd$ holds.

Abell-Grassmann's groupoids are not associative in general, however they have a close relation with semigroups and with commutative structures. Introducing a new operation on *AG*-groupoid makes it a commutative semigroup. On the other hand introducing a new operation on a commutative inverse semigroup turns it into an *AG*-groupoid.

Abel-Grassmann's groupoid satisfying $(\forall a, b, c \in S)$ $ab \cdot c = b \cdot ca$ (called weak associative law in [4]) is an AG^*-groupoid. It is easy to prove that any AG^*-groupoid satisfies the permutation identity of a next type

$$a_1 a_2 \cdot a_3 a_4 = a_{\pi(1)} a_{\pi(2)} \cdot a_{\pi(3)} a_{\pi(4)},$$

2000 Mathematics Subject Classification: 20N02
Keywords: AG-groupoid, antirectangular *AG*-band, AG-band decompositions
Supported by Grant 1379 of Ministry of Science through Math. Inst.SANU

where π is any permutation on a set $\{1, 2, 3, 4\}$, [5].

Let $(S, \cdot)$ be AG-groupoid, $a \in S$ be a fixed element. We can define the "sandwich" operation on S as follows:

$$x \circ y = xa \cdot y, \quad x, y \in S.$$

It is easy to verify that $x \circ y = y \circ x$ for any $x, y \in S$, in other words $(S, \circ)$ is a commutative groupoid. If S is AG^*-groupoid and $x, y, z \in S$ are arbitrary elements, then

$$(x \circ y) \circ z = ((xa \cdot y)a)z = za \cdot (xa \cdot y)$$

and

$$x \circ (y \circ z) = xa \cdot (y \circ z) = xa \cdot (ya \cdot z) = za \cdot (ya \cdot x) = za \cdot (xa \cdot y),$$

whence $(x \circ y) \circ z = x \circ (y \circ z)$. Consequently $(S, \circ)$ is a commutative semigroup.

Let S be the commutative inverse semigroup. We define a new operation on S as follows:

$$a \bullet b = ba^{-1}, \quad a, b \in S.$$

It has been shown in [3] that $(S, \bullet)$ is Abel-Grassmann's groupoid. Connections mentioned above makes AG-groupoid to be among the most interesting nonassociative structures. Same as in Semigroup Theory bands and band decompositions appears as the most fruitful methods for research on AG-groupoids.

If in AG-groupoid S every element is an idempotent, then S is an AG-band.

An AG-groupoid S is an *AG-band Y of AG-groupoids S_α* if

$$S = \bigcup_{\alpha \in Y} S_\alpha,$$

Y is an AG-band, $S_\alpha \cap S_\beta = \emptyset$ for $\alpha, \beta \in Y$, $\alpha \neq \beta$ and $S_\alpha S_\beta \subseteq S_{\alpha\beta}$.

A congruence ρ on S is called *band congruence* if S/ρ is a band.

2. Some decompositions of AG-bands

Let S be a semigroup and for each $a \in S$, $a^2 = a$. That is, let S be an associative band. If for all $a, b \in S$, $ab = ba$, then S is a *semilattice*. If for all $a, b \in S$, $a = aba$, then S is the *rectangular band*. It is a well known result in Semigroup Theory that the associative band S is a semilattice of rectangular bands. It is not hard to prove that a commutative AG-band is a semilattice.

Let us now introduce the following notion.

Definition 2.1. Let S be an AG-band, we say that S is an *antirectangular AG-band* if for every $a, b \in S$, $a = ba \cdot b$.

Let us remark that in that case it holds

$$a = ba \cdot b = ba \cdot bb = bb \cdot ab = b \cdot ab. \tag{2}$$

From above it follows that each antirectangular AG-band is a quasigroup.

Example 2.1. Let a groupoid S be a given by the following table.

$\cdot$	1	2	3	4
1	1	4	2	3
2	3	2	4	1
3	4	1	3	2
4	2	3	1	4

Then S is an antirectangular AG-band and a quasigroup. Let us remark that S is the unique AG-band of order 4 and we shall see below that it appears frequently in band decompositions both as an AG-band into which other bands can be decomposed and like a component. For this reasons from now on we shall call this band *Traka* 4 or simply $T4$. We also remark that nonassociative AG-bands of order $\leqslant 3$ do not exist.

An AG-band is *anticommutative* if for all $a, b \in S$, $ab = ba$ impies that $a = b$.

Lemma 2.1. *Every antirectangular AG-band is anticommutative.*

Proof. Let S be an antirectangular band, $a, b \in S$ and $ab = ba$. Then

$$a = ba \cdot b = ab \cdot b = bb \cdot a = ba = ab = aa \cdot b = ba \cdot a = ab \cdot a = b. \quad \square$$

Theorem 2.1. *If S is an AG-band, then S is an AG-band Y of, in general case nontrivial, antirectangular AG-bands S_α, $\alpha \in Y$.*

Proof. Let S be an AG-band. Then we define the relation ρ on S as

$$a \rho b \Longleftrightarrow a = ba \cdot b, \; b = ab \cdot a. \tag{3}$$

Clearly, the relation ρ is reflexive and symmetric. If $a\rho b$, $b\rho c$, then by (2) and (3) we have

$$ac \cdot a = ac \cdot (ba \cdot b) = ((ba \cdot b)c)a = (cb \cdot ba)a$$
$$= (a \cdot ba) \cdot cb = b \cdot cb = c.$$

Similarly, $a = ca \cdot c$ thus the relation ρ is transitive. Hence, ρ is an equivalence relation.

Let $a\rho b$ and $c \in S$. Then by (1) and the medial law we have

$$ac = (ba \cdot b)c = cb \cdot ba = (cc \cdot b) \cdot ba = (bc \cdot c) \cdot ba$$
$$= (ba \cdot c) \cdot bc = (ba \cdot cc) \cdot bc = (bc \cdot ac) \cdot bc.$$

Dually, $bc = (ac \cdot bc) \cdot ac$ and so $ac\rho bc$. Also,

$$ca = cc \cdot a = ac \cdot c = ((ba \cdot b)c)c = (cb \cdot ba)c = (c \cdot ba) \cdot cb$$
$$= (cc \cdot ba) \cdot cb = (cb \cdot ca) \cdot cb.$$

Dually, $cb = (ca \cdot cb) \cdot ca$ and so $ca\rho cb$. Hence, ρ is a congruence on S.

Since S is a band we have that ρ is a band congruence on S. From $a\rho b$ we have $a = a^2\rho ab$, whence it follows that ρ-classes are closed under the operation. By the definition of ρ it follows that ρ-classes are antirectangular AG-bands. By Lemma 2.1, ρ classes are anticommutative AG-bands. □

In Example 2.1. we have $\rho = S \times S$.

Example 2.2. Let AG-band S be given by the following table.

$\cdot$	1	2	3	4	5	6
1	1	2	2	5	6	4
2	2	2	2	5	6	4
3	2	2	3	5	6	4
4	6	6	6	4	2	5
5	4	4	4	6	5	2
6	5	5	5	2	4	6

Now, $S = S_\alpha \cup S_\beta \cup S_\gamma$ where $S_\alpha = \{1\}$, $S_\beta = \{3\}$, $S_\gamma = \{2, 4, 5, 6\}$ are equivalence classes $mod\, \rho$ and $Y = \{\alpha, \beta, \gamma\}$ is a semilattice. Obviously, S_α, S_β are trivial AG-bands and S_γ is anti-isomorphic with AG-band $T4$ (as is Example 2.1.).

Lemma 2.2. *Let S be an AG-band and $e, a, b \in S$. Then $ea = eb$ implies that $ae = be$ and conversely.*

Proof. Suppose that $ea = eb$, then

$$\begin{aligned} ae &= aa\cdot e = ea\cdot a = eb\cdot a = eb\cdot aa = ea\cdot ba = eb\cdot ba \\ &= (ee\cdot b)\cdot ba = (be\cdot e)\cdot ba = (ba\cdot e)\cdot be = (ea\cdot b)\cdot be \\ &= (eb\cdot b)\cdot be = (bb\cdot e)\cdot be = be\cdot be = be. \end{aligned}$$

Conversely, suppose that $ae = be$, then

$$ea = ee\cdot a = ae\cdot e = be\cdot e = ee\cdot b = eb.$$ □

Remark 2.1. As a consequence of Lemma 2.2, $e = ef$ and so $e = fe$ and conversely.

Theorem 2.2. *Let S be an AG-band. Then the relation ν defined on S by*

$$a\nu b \Longleftrightarrow (\exists e\in S)\ ea = eb$$

is a band congruence relation on S.

Proof. Reflexivity and symmetry is obvious. Suppose that $a\nu b$ and $b\nu c$ for some $a, b, c\in S$. Then there exist elements $e, f\in S$ such that $ea = eb$ and $fb = fc$. According to the Lemma 2.2 we also have $ae = be$, $bf = cf$. Now

$$\begin{aligned} fe\cdot a &= ae\cdot f = be\cdot f = be\cdot ff = bf\cdot ef = cf\cdot ef \\ &= ce\cdot ff = ce\cdot f = fe\cdot c, \end{aligned}$$

implies that ν is transitive.

It remains to prove compatibility. Suppose $a\nu b$ and let $c\in S$ be an arbitrary element. Then there exists $e\in S$ such that $ea = eb$. We have, now

$$c\cdot ea = c\cdot eb \Longrightarrow cc\cdot ea = cc\cdot eb \Longrightarrow ce\cdot ca = ce\cdot cb,$$

so $a\nu cb$. Similarly

$$ea\cdot c = eb\cdot c \Longrightarrow ea\cdot cc = eb\cdot cc \Longrightarrow ec\cdot ac = ec\cdot bc,$$

so $ac\nu bc$. □

In Example 2.1 we have $\nu \equiv \triangle$, since S is a quasigroup. In Example 2.2, $S = S_\alpha\cup S_\beta\cup S_\gamma\cup S_\delta$, where $S_\alpha = \{1,2,3\}$, $S_\beta = \{4\}$, $S_\gamma = \{5\}$, $S_\delta = \{6\}$ are the equivalence classes $mod\,\nu$. Let us remark that *AG*-band $Y = \{\alpha,\beta,\gamma,\delta\}$ is anti-isomorphic with $T4$.

Lemma 2.3. *Let S be an AG-groupoid. Then the relation σ on S defined by the formula*

$$a\sigma b \Longleftrightarrow ab = ba$$

is reflexive, symmetric and compatible.

Proof. Clearly σ is reflexive and symmetric. If $a\sigma b$ and $c \in S$, then by medial law we have

$$ac \cdot bc = ab \cdot cc = ba \cdot cc = bc \cdot ac,$$
$$ca \cdot cb = cc \cdot ab = cc \cdot ba = cb \cdot ca.$$

Hence $ac\sigma bc$, $ca\sigma cb$, and so σ is left and right compatible. This means that σ is compatible. □

Definition 2.2. Let S be an AG-band. Then S is *transitively commutative* if for every $a, b, c \in S$ from $ab = ba$ and $bc = cb$ it follows that $ac = ca$.

It is easy to verify that AG-bands in examples 2.1 and 2.2 are transitively commutative.

Theorem 2.3. *Let S be a transitively commutative AG-band. Then S is an AG-band Y of, in general case nontrivial, semilattices S_α, $\alpha \in Y$.*

Proof. In this way the relation σ defined by (3) is transitive. Now, by Lemma 2.3 we have that relation σ is a band congruence on S. Clearly, σ-classes are commutative AG-bands, i.e., semilattices. □

In Example 2.2 we have that $S = S_\alpha \cup S_\beta \cup S_\gamma \cup S_\delta$, AG-band $Y = \{\alpha, \beta, \gamma, \delta\}$ is anti-isomorphic with AG-band $T4$, $S_\alpha = \{1, 2, 3\}$ is nontrivial semilattice and $S_\beta = \{4\}$, $S_\gamma = \{5\}$, $S_\delta = \{6\}$ are trivial semilattices.

Now, let S be a transitively commutative AG-band, and let $a\sigma b \iff ab = ba$. Then from

$$ab \cdot a = ba \cdot a = aa \cdot b = aa \cdot bb = ab \cdot ab,$$
$$ab \cdot b = bb \cdot a = bb \cdot aa = ba \cdot ba = ab \cdot ab$$

it follows that $ab \cdot a = ab \cdot b$, and so $a\nu b$. Hence, if S is an transitively commutative AG-band, then $\sigma \subseteq \nu$.

3. The natural partial order of AG-band

Theorem 3.1. *If S is AG-band, then the relation $\leqslant$ defined on $E(S)$*

$$e \leqslant f \iff e = ef$$

is a (natural) partial order relation and $\leqslant$ is compatible with the right and with the left with multiplication.

Proof. Clearly, $e \leqslant e$ and relation $\leqslant$ is reflexive. Let $e \leqslant f$, $f \leqslant e$, then $e = ef$, $f = fe$ and by the Remark 2.1 we have $e = f$ so relation $\leqslant$ is antisymmetric. If $e \leqslant f$, $f \leqslant g$ then $e = ef$, $f = fg$ also by the Remark 2.1 it holds that $f = gf$. Now by (1) it follows that

$$eg = ef \cdot g = gf \cdot e = fe = e.$$

Hence, $e \leqslant g$ and relation $\leqslant$ is transitive thus $\leqslant$ is a partial order relation. Now, $e \leqslant f \Longleftrightarrow e = ef$ and $g \in S$ yields

$$eg = ef \cdot g = ef \cdot gg = eg \cdot fg,$$
$$ge = g \cdot ef = gg \cdot ef = ge \cdot gf$$

so $eg \leqslant fg$, $ge \leqslant gf$. Hence, the relation $\leqslant$ is left and right compatible with multiplication. □

In Example 2.1, $\leqslant \equiv \triangle$. In Example 2.2 we have $2 < 1$, $2 < 3$ while other elements are uncomparable.

References

[1] **J. Dénes and A. D. Keedwell**: *Latin squares and their applications*, Akadémia Kiadó, Budapest (1974).

[2] **P. Holgate**: *Groupoids satisfying simple invertive law*, Math. Student **61** (1992), 101 – 106.

[3] **Q. Mushtaq and S. M. Yusuf**: *On LA-semigroup defined by a commutative inverse semigroup*, Matematički vesnik **40** (1988), 59 – 62.

[4] **Q. Mushtaq and M. S. Kamran**: *On LA-semigroups with weak associative law*, Scientific Khyber **1** (1989), 69 – 71.

[5] **P. V. Protić and N. Stevanović**: *On Abel-Grassmann's groupoids* (*review*), Proc. math. conf. in Pristina (1994), 31 – 38.

[6] **P. V. Protić and N. Stevanović**: *AG-test and some general properties of Abel-Grasmann's groupoids*, PU.M.A. **6** (1995), 371 – 383.

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Received August 28, 2002
Revised February 26, 2003