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A note on Salem numbers and Golden mean

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Abstract

It is known that every Pisot number is a limit of Salem numbers. At present there are 47 known Salem numbers less than 1.3 and the list is known to be complete through degree 40. There is a well known relationship between Coxeter systems, Salem numbers, and Golden mean. In this short note, we have discovered the existence of Golden mean in the action of $PSL_2(Z)$ on $Q(\sqrt{5} \cup \{\infty\}$ and investigated some interesting properties of these.

1. Introduction

An algebraic integer $\lambda > 1$ is a *Pisot number* if its conjugates (other than λ itself) satisfy $|\lambda'| < 1$. Similarly, an algebraic integer $\lambda > 1$ is a *Salem number* if its conjugates (other than λ itself) satisfy $|\lambda'| \leqslant 1$ and include $\frac{1}{\lambda}$.

It is known that the Pisot numbers form a closed subset $P \subset R$, where R is a field of real numbers, and that every Pisot number is a limit of Salem numbers [4]. The smallest Pisot number λ_P, equivalent to 1.324717, is a root of $x^3 - x - 1 = 0$, while the smallest accumulation point in P is the Golden mean, $\lambda_G = \dfrac{1+\sqrt{5}}{2}$ equivalent to 1.61803. Note that $\lambda_G^2 = \dfrac{3+\sqrt{5}}{2}$ is equivalent to 2.61803...

2. Golden mean

Theorem. *In an action of the modular group on* $Q(\sqrt{5} \cup \{\infty\}$, λ_G *is the fixed point of the commutator of the modular group.*

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Proof. It is well known that the modular group $PSL_2(Z)$ is generated by the linear fractional transformations $x : z \mapsto \frac{-1}{z}$ and $y : z \mapsto \frac{z-1}{z}$ which obviously satisfy the relations $x^2 = y^3 = 1$.

Then $\lambda_G x = \frac{1-\sqrt{5}}{2}$, $\lambda_G xy = \frac{3+\sqrt{5}}{2}$, $\lambda_G xy^2 = \frac{-1+\sqrt{5}}{2}$, $\lambda_G xy^2 x = \frac{-1-\sqrt{5}}{2}$, $\lambda_G xy^2 xy^2 = \frac{3-\sqrt{5}}{2}$ and $\lambda_G xy^2 xy = \frac{1+\sqrt{5}}{2} = \lambda_G$. □

Corollary 1. $\lambda_G^2 - \lambda_G - 1 = 0$.

Proof. $\lambda_G xy^2 xy = (\lambda_G + 1)yxy = \left(\frac{\lambda_G + 1 - 1}{\lambda_G + 1}\right) xy = \frac{\lambda_G + 1 - 1}{\lambda_G + 1} + 1$. Therefore $\lambda_G xy^2 xy = \lambda_G$, and so $\frac{\lambda_G + 1 - 1}{\lambda_G + 1} + 1 = \lambda_G$ yields $\lambda_G^2 - \lambda_G - 1 = 0$. □

Corollary 2. *Let* $\overline{\lambda}_G$ *denote the algebraic conjugate of* λ_G. *Then:*

(*i*) $\lambda_G x = \overline{\lambda}_G$, $\lambda_G xy = \lambda_G^2$, $\lambda_G xy^2 = -\overline{\lambda}_G$,

(*ii*) $(\lambda_G xy^2)x = -\lambda_G$, $(\lambda_G xy^2)xy = \lambda_G$, $(\lambda_G xy^2)xy^2 = (\overline{\lambda}_G)^2$.

Proof. The proof follows directly from Corollary 1. □

All Pisot numbers $\lambda, \lambda_G + \epsilon$ are known [1]. The Salem numbers are less well understood. The catalog of 39 Salem numbers given in [1] includes all Salem numbers $\lambda < 1.3$ of degree less than or equal to 20 over the field of rationals. At present there are 47 known Salem numbers $\lambda < 1.3$, and the list of such is known to be complete through degree 40 [2] and [3].

Next we give approximation of the golden mean. The Golden mean $\lambda_G = \frac{1+\sqrt{5}}{2}$ is the quadratic irrationality, which is hardest to approximate by rational numbers, that is, $\lambda_G - \frac{p}{q} \neq 0$, where p and q are co-prime integers. We make $\left|\lambda_G - \frac{p}{q}\right|$ as small as possible for a fixed q, i.e., $\left|\lambda_G - \frac{p}{q}\right| < \varepsilon_q(\lambda_G)$, when $\varepsilon_q(\lambda_G)$ tends to zero as q tends to infinity. Trivially, $\varepsilon_q(\lambda_G) < \frac{1}{2q}$. We can, in fact, for any irrational α, choose a sequence $q_1, q_2, \ldots, q_n, \ldots$ tending to infinity such that $\varepsilon_{q_i}(\alpha) < \frac{1}{q_i^2}$. For the number $\lambda_G = \frac{1+\sqrt{5}}{2}$, we cannot do better than this. If $\beta = \dfrac{a\alpha + b}{c\alpha + d}$, where $ad - bc = \pm 1$ and a, b, c, d are integers then by Liouvelli's Theorem approximation by rational integers is roughly the same for α as for β. In other words, if α is nearly $\dfrac{p}{q}$ then $\dfrac{a\frac{p}{q} + b}{c\frac{p}{q} + d}$ is a good approximation to β. $\square$

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