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Dual positive implicative hyper K-ideals of type 3

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Abstract

In this note first we define the notion of *dual positive implicative hyper K-ideal of type* 3, where for simplicity is written by $DPIHKI-T3$. Then we determine all of the non-isomorphic hyper K-algebras of order 3, which have $D = \{0, 1\}$ as a $DPIHKI-T3$. To do this first we show that $D = \{1\}$ and $D = \{1, 2\}$ can not be $DPIHKI-T3$. Then we prove some lemmas which are needed for proving the main theorem. Finally we conclude that there are exactly 219 non-isomorphic hyper K-algebras of order 3 with the requested property.

1. Introduction

The hyperalgebraic structure theory was introduced by F. Marty [7] in 1934. Around the 40's, several authors worked on hypergroups, especially in France and in the United States, but also in Italy, Russia and Japan. Over the following decades, many important results appeared, but above all since 70's onwards the most luxuriant flourishing of hyperstructures has been seen (see for example [4]). Hyperstructures have many applications to several sectors of both pure and applied sciences.

Imai and Iéki [5] in 1966 introduced the notion of a BCK-algebra. Recently [2, 3, 9] Borzooei, Jun and Zahedi et .al. applied the hyperstructure to BCK-algebras and introduced the concept of hyper K-algebra which is a generalization of BCK-algebra. In [1], the authors have defined 8 types of positive implicative hyper K-ideals. Now in this note we define the notion of dual positive implicative hyper K-ideal of type 3, then we obtain some related results which have been mentioned in the abstract.

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2. Preliminaries

Definition 2.1. Let H be a non-empty set and " $\circ$ " be a *hyperoperation* on H, that is " $\circ$ " is a function from $H \times H$ to $\mathcal{P}^*(H) = \mathcal{P}(H) \setminus \{\emptyset\}$. Then H is called a *hyper K-algebra* if it contains a constant 0 and satisfies the following axioms:

(HK1) $(x \circ z) \circ (y \circ z) < x \circ y$,
(HK2) $(x \circ y) \circ z = (x \circ z) \circ y$,
(HK3) $x < x$,
(HK4) $x < y,\ y < x \Longrightarrow x = y$,
(HK5) $0 < x$,

for all $x, y, z \in H$, where $x < y$ is defined by $0 \in x \circ y$ and for every $A, B \subseteq H$, $A < B$ is defined by $\exists\, a \in A$, $\exists\, b \in B$ such that $a < b$.

Note that if $A, B \subseteq H$, then by $A \circ B$ we mean the set-theoretic union of all $a \circ b$ such that $a \in A$, $b \in B$.

The main properties of hyper K-algebras are described in [2] and [3]. For example in [2] the following theorem is proved.

Theorem 2.2. *Let $(H, \circ, 0)$ be a hyper K-algebra. Then for all $x, y, z \in H$ and for all non-empty subsets A, B and C of H we have:*

(i)	$x \circ y < z \Longleftrightarrow x \circ z < y$,	(vi)	$x \in x \circ 0$,
(ii)	$(x \circ z) \circ (x \circ y) < y \circ z$,	(vii)	$(A \circ C) \circ (A \circ B) < B \circ C$,
(iii)	$x \circ (x \circ y) < y$,	$(viii)$	$(A \circ C) \circ (B \circ C) < A \circ B$,
(iv)	$x \circ y < x$,	(ix)	$A \circ B < C \Longleftrightarrow A \circ C < B$,
(v)	$A \subseteq B \Longrightarrow A < B$,	(x)	$A \circ B < A$. □

Definition 2.3. Let $(H, \circ, 0)$ be a hyper K-algebra. If there exist an element $1 \in H$ such that $1 < x$ for all $x \in H$, then H is called a *bounded hyper K-algebra* and 1 is said to be the unit of H.

3. Dual positive implicative hyper K-algebras

From now H is a bounded hyper K-algebra with unit 1 and $1 \circ x = Nx$.

Definition 3.1. A non-empty subset D of H is called a *dual positive implicative hyper K-ideal type* 3 (shortly: DPIHKI-T3) if

(i) $1 \in D$,
(ii) $N((Nx \circ Ny) \circ Nz) < D$ and $N(Ny \circ Nz) < D$ imply
$N(Nx \circ Nz) \subseteq D$, $\forall\, x, y, z \in H$.

Example 3.2. Let $H = \{0, 1, 2\}$. Then the following table shows a hyper K-algebra structure on H with unit 1.

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0, 1\}$	$\{1\}$
2	$\{1, 2\}$	$\{0, 1\}$	$\{0, 1\}$

And $I = \{0, 1\}$ is a $DPIHKI - T3$. □

Theorem 3.3. *A non-empty subset D of H is a $DPIHKI - T3$ if and only if $N(Nx \circ Nz) \subseteq D$ for all $x, z \in H$.*

Proof. Let D be a $DPIHKI - T3$. Then by Definition 2.1 and Theorem 2.3 (x) we conclude that $N((Nx \circ Ny) \circ Nz) < D$ and $N(Ny \circ Nz) < D$ for all $x, y, z \in H$. So by hypothesis we get that $N(Nx \circ Nz) \subseteq D$ for all $x, z \in H$. The converse statement is obvious. □

To avoid repetitions let in the sequel $H = \{0, 1, 2\}$ be a bounded hyper K-algebra with unit 1.

Lemma 3.4. *In H we have* $1 \circ 0 = \{1\}$.

Proof. On the contrary let $1 \circ 0 \neq \{1\}$. Then we must have $1 \circ 0 = \{1, 2\}$. By $(HK2)$ we have $(1 \circ 0) \circ 2 = (1 \circ 2) \circ 0$, so $0 \in 2 \circ 2 \subseteq (1 \circ 0) \circ 2 = (1 \circ 2) \circ 0$. Thus there exists $x \in 1 \circ 2$ such that $0 \in x \circ 0$, which implies that $x < 0$, thus from (HK4) and $(HK5)$ we get that $x = 0$. Hence $0 \in 1 \circ 2$, that is $1 < 2$. Since $2 < 1$, thus $2 = 1$, which is a contradiction. □

Lemma 3.5. *For all $x \in H$ we have $NNx = x$ if and only if* $1 \circ 1 = \{0\}$ *and* $1 \circ 2 = \{2\}$.

Proof. Let $NNx = x$, i.e. $1 \circ (1 \circ x) = x$ for all x. Since $1 \circ (1 \circ 2) = 2$, we get that $0 \notin 1 \circ 2$ and $1 \notin 1 \circ 2$. So $1 \circ 2 = \{2\}$. Now since $1 \circ (1 \circ 1) = 1$, we conclude that $1 \notin 1 \circ 1$ and $2 \notin 1 \circ 1$. Thus $1 \circ 1 = \{0\}$.

The converse follows from Lemma 3.4 and hypothesis. □

Lemma 3.6. *Let $D_1 = \{1\}$ and $D_2 = \{1, 2\}$ in H. Then D_1 and D_2 are not $DPIHKI - T3$.*

Proof. Since $0 \in 1 \circ ((1 \circ 0) \circ (1 \circ 1)) = N(N0 \circ N1)$, $0 \notin D_1$ and $0 \notin D_2$, then by Theorem 3.3 D_1 and D_2 are not $DPIHKI-T3$. $\square$

Lemma 3.7. *Let $D=\{0,1\}$ in H. Then the following hold:*
(i) *if* $2\in 1\circ 2$, *then D is not a $DPIHKI-T3$,*
(ii) *if* $2\in 1\circ 1$, *then D is not a $DPIHKI-T3$.*

Proof. (i) Since $2\in 1\circ 2\subseteq 1\circ((1\circ 2)\circ(1\circ 1))=N(N2\circ N1)$ and $2\notin D$, then, by Theorem 3.3, D is not $DPIHKI-T3$.
(ii) The proof is similar as (i). $\square$

Theorem 3.8. *Let $D=\{0,1\}$ in H. Then D is a $DPIHKI-T3$ if and only if $2\notin 1\circ 2$ and $2\notin 1\circ 1$.*

Proof. Let $2\notin 1\circ 2$ and $2\notin 1\circ 1$. Thus $1\circ 2=\{1\}$ and $1\circ 1=\{0,1\}$ or $1\circ 1=\{0\}$. Now by some calculations we can get that $N(Nx\circ Nz)\subseteq D$, for all $x,z\in H$.

Conversely, on the contrary let $2\in 1\circ 2$ or $2\in 1\circ 1$. Then Lemma 3.7 $(i),(ii)$ gives a contradiction. Thus $2\notin 1\circ 2$ and $2\notin 1\circ 1$. $\square$

Remark 3.9. From now let $D=\{0,1\}$ be a $DPIHKI-T3$. Thus:
(i) From Theorem 3.8 we conclude that $1\circ 2=\{1\}$ and $1\circ 1=\{0,1\}$ or $1\circ 1=\{0\}$.
(ii) By (HK2) we have $(1\circ 1)\circ 0=(1\circ 0)\circ 1$ and $(1\circ 1)\circ 2=(1\circ 2)\circ 1$. Thus by (i) and Lemma 3.4 we conclude that $0\circ 0\subseteq\{0,1\}$ and $0\circ 2\subseteq\{0,1\}$. $\square$

Lemma 3.10. *If $2\circ 2=\{0\}$ and $0\circ 0=\{0\}$ in H, then $2\circ 0=\{2\}$.*

Proof. By (HK2) we have $(2\circ 0)\circ 2=(2\circ 2)\circ 0=0\circ 0=\{0\}$. If $1\in 2\circ 0$, then $1\circ 2\subseteq(2\circ 0)\circ 2=\{0\}$. Thus $1\circ 2=\{0\}$, which is a contradiction because $1\circ 2=\{1\}$, by Remark 3.9 (i). Thus $1\notin 2\circ 0$, hence $2\circ 0=\{2\}$. $\square$

Lemma 3.11. *If $2\circ 2=\{0\}$ and $0\circ 1=\{0\}$ in H, then $1\notin 2\circ 1$.*

Proof. On the contrary let $1\in 2\circ 1$. By (HK2) we have $(2\circ 1)\circ 2=(2\circ 2)\circ 1=0\circ 1=\{0\}$. Thus by Remark 3.9 (i) and hypothesis we have $1\in 1\circ 2\subseteq(2\circ 1)\circ 2=\{0\}$, which is a contradiction. $\square$

Lemma 3.12. If $2 \circ 1 = \{0, 2\}$ and $1 \circ 1 = \{0\}$ in H, then $2 \circ 0 = \{2\}$.

Proof. On the contrary let $2 \circ 0 \neq \{2\}$. Then we must have $2 \circ 0 = \{1, 2\}$. By (HK2) we have $(2 \circ 1) \circ 0 = (2 \circ 0) \circ 1$. By hypothesis we have $(2 \circ 1) \circ 0 = \{0, 1, 2\}$ and $(2 \circ 0) \circ 1 = \{0, 2\}$, which is a contradiction. □

Lemma 3.13. *Let* $0 \circ 1 = \{0, 1\}$ *and* $0 \circ 2 = \{0\}$ *in* H*.*

(i) *If* $2 \circ 2 \subseteq \{0, 2\}$*, then* $2 \circ 1 \not\subseteq \{0, 2\}$*.*

(ii) *If* $2 \circ 2 = \{0, 1\}$ *or* $2 \circ 2 = \{0, 1, 2\}$*, then* $2 \circ 1 \neq \{0\}$*.*

Proof. (i) On the contrary let $2 \circ 1 \subseteq \{0, 2\}$. If $2 \circ 1 = \{0, 2\}$ by (HK2) we have $(2 \circ 2) \circ 1 = (2 \circ 1) \circ 2$. By hypothesis we have $(2 \circ 2) \circ 1 = \{0, 1\}$ if $2 \circ 2 = \{0\}$ and $(2 \circ 2) \circ 1 = \{0, 1, 2\}$ if $2 \circ 2 = \{0, 2\}$. On the other hand $(2 \circ 1) \circ 2 = \{0\}$ if $2 \circ 2 = \{0\}$ and $(2 \circ 1) \circ 2 = \{0, 2\}$ if $2 \circ 2 = \{0, 2\}$, which is a contradiction. If $2 \circ 1 = \{0\}$, then the proof is similar as the case of $2 \circ 1 = \{0, 2\}$.

The proof of (ii) is similar as (i). □

Lemma 3.14. *Let* $0 \circ 1 = \{0, 2\}$ *in* H*. Then:*

(i) $2 \circ 2 \not\subseteq \{0, 1\}$,

(ii) $2 \circ 1 \not\subseteq \{0, 1\}$,

(iii) *if* $1 \circ 1 = \{0\}$*, then* $2 \circ 2 \neq \{0, 1, 2\}$*,*

(iv) *if* $0 \circ 0 = \{0\}$*, then* $2 \circ 0 = \{2\}$*,*

(v) *if* $2 \circ 2 = \{0, 1, 2\}$*, then* $0 \circ 2 = \{0, 1\}$.

Proof. (i) On the contrary let $2 \circ 2 \subseteq \{0, 1\}$. By (HK2) we have $(0 \circ 2) \circ 1 = (0 \circ 1) \circ 2$. If $2 \circ 2 = \{0\}$ by hypothesis and Remark 3.9 we get that $(0 \circ 1) \circ 2 \subseteq \{0, 1\}$ and $(0 \circ 2) \circ 1 = \{0, 2\}$ or $\{0, 1, 2\}$, which is a contradiction. If $2 \circ 2 = \{0, 2\}$, then the proof is similar as the case of $2 \circ 2 = \{0\}$.

The proof of the other cases are similar as above by considering the suitable modifications. □

Lemma 3.15. *Let* $0 \circ 1 = \{0, 1, 2\}$ *in* H*. Then:*

(i) $2 \circ 2 \not\subseteq \{0, 1\}$,

(ii) $2 \circ 1 \not\subseteq \{0, 1\}$,

(iii) *if* $2 \circ 2 = \{0, 2\}$ *and* $0 \circ 2 = \{0\}$*, then* $2 \circ 1 \neq \{0, 2\}$*,*

(iv) *if* $2 \circ 1 = \{0, 2\}$ *and* $1 \notin 2 \circ 2$*, then* $0 \circ 2 = \{0, 1\}$.

Proof. (i) On the contrary let $2 \circ 2 \subseteq \{0, 1\}$. By (HK2) we have $(0 \circ 2) \circ 1 =$

$(0 \circ 1) \circ 2$. If $2 \circ 2 = \{0\}$, then by hypothesis and Remark 3.9 we get that $(0 \circ 1) \circ 2 = \{0, 1\}$ and $(0 \circ 2) \circ 1 = \{0, 1, 2\}$, which is a contradiction. If $2 \circ 2 = \{0, 2\}$, then the proof is similar as the case of $2 \circ 2 = \{0\}$.

The proof of the other cases are similar as above by considering the suitable modifications. □

Lemma 3.16. *If* $2 \circ 1$, $2 \circ 2$ *and* $0 \circ 1 \subseteq \{0, 2\}$, *then* $0 \circ 2 = \{0\}$.

Proof. By (HK2) we have $(2 \circ 1) \circ 2 = (2 \circ 2) \circ 1 \subseteq \{0, 2\}$. Since $0 \circ 2 \subseteq (2 \circ 1) \circ 2 \subseteq \{0, 2\}$, and by Remark 3.9 (ii) $2 \notin 0 \circ 2$, we get that $0 \circ 2 = \{0\}$. □

Lemma 3.17. *Let* $2 \circ 1 = \{0\}$ *in* H. *Then:*

(i) *if* $1 \circ 1 = \{0, 1\}$ *and* $2 \circ 2 = \{0, 1\}$ *or* $\{0, 1, 2\}$, *then* $0 \circ 2 = \{0, 1\}$,

(ii) *if* $0 \circ 0 = \{0\}$ *and* $1 \circ 1 = \{0, 1\}$, *then* $2 \circ 0 = \{2\}$,

(iii) *if* $0 \circ 1 = \{0, 1\}$, *then* $0 \circ 2 = \{0, 1\}$,

(iv) *if* $0 \circ 0 = \{0, 1\}$, *then* $2 \circ 0 = \{1, 2\}$.

Proof. (i) By (HK2) we have $(2 \circ 2) \circ 1 = (2 \circ 1) \circ 2$. Now $(2 \circ 2) \circ 1 = \{0, 1\}$ and $(2 \circ 1) \circ 2 = 0 \circ 2$, therefore $0 \circ 2 = \{0, 1\}$.

(ii) On the contrary let $2 \circ 0 \neq \{2\}$. Then we must have $2 \circ 0 = \{1, 2\}$. By (HK2) we have $(2 \circ 1) \circ 0 = (2 \circ 0) \circ 1$, which is contradiction, because $1 \in (2 \circ 0) \circ 1$, while $1 \notin (2 \circ 1) \circ 0 = \{0\}$.

The proofs of (iii) and (iv) are similar. □

Lemma 3.18. *Let* $2 \circ 1 = \{0, 2\}$ *in* H. *Then:*

(i) *if* $1 \in 0 \circ 1$ *and* $1 \notin 2 \circ 2$, *then* $0 \circ 2 = \{0, 1\}$,

(ii) *if* $0 \circ 0 = \{0, 1\}$, *then* $2 \circ 0 = \{1, 2\}$.

Proof. (i) On the contrary let $0 \circ 2 \neq \{0, 1\}$. Then we must have $0 \circ 2 = \{0\}$, by Remark 3.9 (ii). By (HK2) we have $(2 \circ 1) \circ 2 = (2 \circ 2) \circ 1$. Now by hypothesis we have $(2 \circ 1) \circ 2 \subseteq \{0, 2\}$ and $1 \in (2 \circ 2) \circ 1$, which is a contradiction.

The proof of (ii) is similar as (i). □

Lemma 3.19. *If* $2 \circ 2 \subseteq \{0, 2\}$ *and* $0 \circ 0 = \{0, 1\}$, *then* $2 \circ 0 = \{1, 2\}$.

Proof. On the contrary let $2 \circ 0 \neq \{1, 2\}$. Then we must have $2 \circ 0 = \{2\}$. By (HK2) we have $(2 \circ 2) \circ 0 = (2 \circ 0) \circ 2$. Now by hypothesis we have

$1 \in (2 \circ 2) \circ 0$ and $1 \notin (2 \circ 0) \circ 2$, which is a contradiction. $\square$

Now we are ready to determine all of hyper K-algebras of order 3, in which $D = \{0, 1\}$ is a $DPIHKI - T3$.

Theorem 3.20 (Main theorem) *There are* 219 *non-isomorphic bounded hyper K-algebras of order* 3, *to have $D = \{0, 1\}$ as a $DPIHKI - T3$.*

Proof. Let $H = \{0, 1, 2\}$ and 1 be its unit. The following table shows a probable hyper K-algebra structure on H, in which $D = \{0, 1\}$ is a $DPIHKI - T3$:

$\circ$	0	1	2
0	a_{11}	a_{12}	a_{13}
1	a_{21}	a_{22}	a_{23}
2	a_{31}	a_{32}	a_{33}

By Remark 3.9 we have $a_{21} = 1 \circ 0 = \{1\}$, $a_{22} = 1 \circ 1 = \{0\}$ or $\{0, 1\}$, $a_{23} = 1 \circ 2 = \{1\}$, $a_{11} = 0 \circ 0 \subseteq \{0, 1\}$ and $a_{13} = 0 \circ 2 \subseteq \{0, 1\}$. Also since H is bounded, then by $(HK3)$ and $(HK5)$ we have $0 \in a_{12} \bigcap a_{32} \bigcap a_{33}$. There are two cases for $a_{22} = 1 \circ 1$. Let $1 \circ 1 = \{0\}$. Then by (HK2) we have $(1 \circ 1) \circ 0 = (1 \circ 0) \circ 1$, so $0 \circ 0 = \{0\}$. Similarly $(1 \circ 1) \circ 2 = (1 \circ 2) \circ 1$ implies that $0 \circ 2 = \{0\}$. We will show that in this case there exist exactly 40 non-isomorphic hyper K-algebras. In the other hand if $1 \circ 1 = \{0, 1\}$, then by Remark 3.9 (ii) we get that $0 \circ 0 \subseteq \{0, 1\}$ and $0 \circ 2 \subseteq \{0, 1\}$ and in this situation we will obtain exactly 179 non-isomorphic hyper K-algebras other than the previous 40 ones. So totally we have 219 different non-isomorphic bounded hyper K-algebras of order 3, to have $D = \{0, 1\}$ as a $DPIHKI - T3$. Now we give the details. To do this we consider two main cases $1 \circ 1 = \{0\}$ and $1 \circ 1 = \{0, 1\}$, and many subcases of them.

1. $1 \circ 1 = \{0\}$

We consider some subcases as follows:

1.1. $0 \circ 1 = \{0\}$

In this case also we consider 4 states as follows:

1.1.1. $2 \circ 2 = \{0\}$

By Lemmas 3.10 and 3.11 we must have $2 \circ 0 = \{2\}$ and $2 \circ 1 \subseteq \{0, 2\}$. So there exist 2 hyper K-algebras as follows:

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{1\}$
2	$\{2\}$	$\{0, 2\}$	$\{0\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{1\}$
2	$\{2\}$	$\{0\}$	$\{0\}$

1.1.2. $2 \circ 2 = \{0, 1\}$

By $(HK2)$ We have $(2 \circ 2) \circ 1 = (2 \circ 1) \circ 2$. We get that $(2 \circ 2) \circ 1 = \{0\}$. If $1 \in 2 \circ 1$ or $2 \in 2 \circ 1$, then $1 \in (2 \circ 1) \circ 2 = \{0\}$, which is a contradiction. Thus $1 \notin 2 \circ 1$ and $2 \notin 2 \circ 1$, hence $2 \circ 1 = \{0\}$. So there exists two hyper K-algebras as follows:

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{1\}$
2	$\{2\}$	$\{0\}$	$\{0,1\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{1\}$
2	$\{1,2\}$	$\{0\}$	$\{0,1\}$

1.1.3. $2 \circ 2 = \{0, 2\}$

If $2 \circ 1 = \{0, 2\}$, then by Lemma 3.12 we have $2 \circ 0 = \{2\}$. So there exists seven hyper K-algebras as follows:

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{1\}$
2	$\{2\}$	$\{0\}$	$\{0,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{1\}$
2	$\{2\}$	$\{0,1\}$	$\{0,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{1\}$
2	$\{2\}$	$\{0,2\}$	$\{0,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{1\}$
2	$\{2\}$	$\{0,1,2\}$	$\{0,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{1\}$
2	$\{1,2\}$	$\{0\}$	$\{0,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1\}$	$\{0,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1\}$	$\{0,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1,2\}$	$\{0,2\}$

1.1.4. $2 \circ 2 = \{0, 1, 2\}$

We prove that $2 \circ 1 \neq \{0, 2\}$. On the contrary, let $2 \circ 1 = \{0, 2\}$. By $(HK2)$ we have $(2 \circ 2) \circ 1 = (2 \circ 1) \circ 2$, while $(2 \circ 2) \circ 1 = \{0, 2\}$ and $(2 \circ 1) \circ 2 = \{0, 1, 2\}$, which is a contradiction. So there exist six hyper K-algebras:

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{1\}$
2	$\{2\}$	$\{0\}$	$\{0,1,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{1\}$
2	$\{2\}$	$\{0,1\}$	$\{0,1,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{1\}$
2	$\{2\}$	$\{0,1,2\}$	$\{0,1,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1,2\}$	$\{0,1,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{1\}$
2	$\{1,2\}$	$\{0\}$	$\{0,1,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1\}$	$\{0,1,2\}$

1.2. $0 \circ 1 = \{0,1\}$

In this case also we consider four states as follows:

1.2.1. $2 \circ 2 = \{0\}$

By Lemmas 3.10 and 3.13(i) we have $2 \circ 0 = \{2\}$ and $2 \circ 1 \not\subseteq \{0,2\}$. So there exist two hyper K-algebras as follows:

$\circ$	0	1	2
0	$\{0\}$	$\{0,1\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{1\}$
2	$\{2\}$	$\{0,1,2\}$	$\{0\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,1\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{1\}$
2	$\{2\}$	$\{0,1\}$	$\{0\}$

1.2.2. $2 \circ 2 = \{0,1\}$

By Lemma 3.13 (ii) we have $2 \circ 1 \neq \{0\}$. If $2 \circ 1 = \{0,2\}$, then by Lemma 3.12 we have $2 \circ 0 = \{2\}$. So there exist five hyper K-algebras:

$\circ$	0	1	2
0	$\{0\}$	$\{0,1\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{1\}$
2	$\{2\}$	$\{0,2\}$	$\{0,1\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,1\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{1\}$
2	$\{2\}$	$\{0,1\}$	$\{0,1\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,1\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{1\}$
2	$\{2\}$	$\{0,1,2\}$	$\{0,1\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,1\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1\}$	$\{0,1\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,1\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1,2\}$	$\{0,1\}$

1.2.3. $2 \circ 2 = \{0,2\}$

By Lemma 3.13 (i) we have $2 \circ 1 \not\subseteq \{0,2\}$. So there exist four hyper K-algebras as follows:

$\circ$	0	1	2
0	$\{0\}$	$\{0,1\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{1\}$
2	$\{2\}$	$\{0,1\}$	$\{0,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,1\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{1\}$
2	$\{2\}$	$\{0,1,2\}$	$\{0,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,1\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1\}$	$\{0,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,1\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1,2\}$	$\{0,2\}$

1.2.4. $2 \circ 2 = \{0,1,2\}$

This case is similar to 1.2.2. So there exist five hyper K-algebras:

$\circ$	0	1	2
0	$\{0\}$	$\{0,1\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{1\}$
2	$\{2\}$	$\{0,2\}$	$\{0,1,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,1\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{1\}$
2	$\{2\}$	$\{0,1\}$	$\{0,1,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,1\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{1\}$
2	$\{2\}$	$\{0,1,2\}$	$\{0,1,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,1\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1\}$	$\{0,1,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,1\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1,2\}$	$\{0,1,2\}$

1.3. $0\circ 1=\{0,2\}$

In this case we have only one state, since by Lemma 3.14 $(i),(ii)$ we have $2\circ 2\not\subseteq\{0,1\}$ and $2\circ 2\neq\{0,1,2\}$.

1.3.1. $2\circ 2=\{0,2\}$

By Lemma 3.14 $(i),(iv)$ we have $2\circ 1\not\subseteq\{0,1\}$ and $2\circ 0=\{2\}$. So there exist two hyper K-algebras as follows:

$\circ$	0	1	2
0	$\{0\}$	$\{0,2\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{1\}$
2	$\{2\}$	$\{0,1,2\}$	$\{0,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,2\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{1\}$
2	$\{2\}$	$\{0,2\}$	$\{0,2\}$

1.4. $0\circ 1=\{0,1,2\}$

We have two states, since $2\circ 2\not\subseteq\{0,1\}$, by Lemma 3.15 (i).

1.4.1. $2\circ 2=\{0,2\}$

By Lemma 3.15 $(ii),(iii)$ we have $2\circ 1\not\subseteq\{0,1\}$ and $2\circ 1\neq\{0,2\}$. So there exist two hyper K-algebras as follows:

$\circ$	0	1	2
0	$\{0\}$	$\{0,1,2\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{1\}$
2	$\{2\}$	$\{0,1,2\}$	$\{0,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,1,2\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1,2\}$	$\{0\}$

1.4.2. $2\circ 2=\{0,1,2\}$

By Lemma 3.15 (ii) we have $2\circ 1\not\subseteq\{0,1\}$. If $2\circ 1=\{0,2\}$, then by Lemma 3.12 we have $2\circ 0=\{2\}$. So there exist three hyper K-algebras:

$\circ$	0	1	2
0	$\{0\}$	$\{0,1,2\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{1\}$
2	$\{2\}$	$\{0,1,2\}$	$\{0,1,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,1,2\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{1\}$
2	$\{2\}$	$\{0,2\}$	$\{0,1,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,1,2\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1,2\}$	$\{0,1,2\}$

Now we consider the following case:

2. $1\circ 1=\{0,1\}$

This case has two subcases $0\circ 0=\{0\}$ or $\{0,1\}$.

2.1. $0\circ 0=\{0\}$

We consider the following subcases as:

2.1.1. $0\circ 1=\{0\}$

In this case also we consider four states as follows:

2.1.1.1. $2\circ 2=\{0\}$

By Lemmas 3.10 and 3.11 we have $2 \circ 0 = \{2\}$ and $1 \notin 2 \circ 1$. If $2 \circ 1 \subseteq \{0,2\}$, then by Lemma 3.16 we have $0 \circ 2 = \{0\}$. So there exist two hyper K-algebras as follows:

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,2\}$	$\{0\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0\}$	$\{0\}$

2.1.1.2. $2 \circ 2 = \{0,1\}$

If $2 \circ 1 = \{0\}$, then by Lemma 3.17 $(i),(ii)$ we have $0 \circ 2 = \{0,1\}$ and $2 \circ 0 = \{2\}$. So there exist 13 hyper K-algebras as follows:

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,1\}$	$\{0,1\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,2\}$	$\{0,1\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,1,2\}$	$\{0,1\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,1\}$	$\{0,1\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,2\}$	$\{0,1\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,1,2\}$	$\{0,1\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0\}$	$\{0,1\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,2\}$	$\{0,1\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1,2\}$	$\{0,1\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1\}$	$\{0,1\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1\}$	$\{0,1\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,2\}$	$\{0,1\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1,2\}$	$\{0,1\}$

2.1.1.3. $2 \circ 2 = \{0,2\}$

If $2 \circ 1 \subseteq \{0,2\}$, then $0 \circ 2 = \{0\}$ by Lemma 3.16. If $2 \circ 1 = \{0\}$, then $2 \circ 0 = \{2\}$ by Lemma 3.17 (i). So there exist 11 hyper K-algebras:

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,2\}$	$\{0,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,1,2\}$	$\{0,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0\}$	$\{0,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,1\}$	$\{0,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1\}$	$\{0,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1,2\}$	$\{0,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,2\}$	$\{0,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,1,2\}$	$\{0,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1\}$	$\{0,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1,2\}$	$\{0,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,1\}$	$\{0,2\}$

2.1.1.4. $2 \circ 2 = \{0,1,2\}$

If $2 \circ 1 = \{0\}$, then by Lemma 3.17 $(i),(ii)$ we have $0 \circ 2 = \{0,1\}$ and $2 \circ 0 = \{2\}$. So there exist 13 hyper K-algebras as follows:

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,2\}$	$\{0,1,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,1,2\}$	$\{0,1,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1\}$	$\{0,1,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,2\}$	$\{0,1,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1,2\}$	$\{0,1,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,1\}$	$\{0,1,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,1\}$	$\{0,1,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,1,2\}$	$\{0,1,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1\}$	$\{0,1,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1,2\}$	$\{0,1,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,2\}$	$\{0,1,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0\}$	$\{0,1,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,2\}$	$\{0,1,2\}$

2.1.2. $0 \circ 1 = \{0,1\}$

In this case also we consider four states as follows:

2.1.2.1. $2 \circ 2 = \{0\}$

By Lemma 3.10 we have $2 \circ 0 = \{2\}$. If $0 \circ 2 = \{0\}$, then by Lemma 3.13 (i) we have $2 \circ 1 \not\subseteq \{0, 2\}$. So there exist six hyper K-algebras as follows:

$\circ$	0	1	2
0	$\{0\}$	$\{0,1\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,1\}$	$\{0\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,1\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,1,2\}$	$\{0\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,1\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0\}$	$\{0\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,1\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,1\}$	$\{0\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,1\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,2\}$	$\{0\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,1\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,1,2\}$	$\{0\}$

2.1.2.2. $2 \circ 2 = \{0, 1\}$

If $2 \circ 1 = \{0\}$, then by Lemma 3.17 $(i), (ii)$ we have $0 \circ 2 = \{0, 1\}$ and $2 \circ 0 = \{2\}$. So there exist 13 hyper K-algebras as follows:

$\circ$	0	1	2
0	$\{0\}$	$\{0,1\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,2\}$	$\{0,1\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,1\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1\}$	$\{0,1\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,1\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1,2\}$	$\{0,1\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,1\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0\}$	$\{0,1\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,1\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,1\}$	$\{0,1\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,1\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,2\}$	$\{0,1\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,1\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1\}$	$\{0,1\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,1\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,2\}$	$\{0,1\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,1\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1,2\}$	$\{0,1\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,1\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,1\}$	$\{0,1\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,1\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,2\}$	$\{0,1\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,1\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,1,2\}$	$\{0,1\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,1\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,1,2\}$	$\{0,1\}$

2.1.2.3. $2 \circ 2 = \{0, 2\}$

If $2 \circ 1 = \{0\}$, then by lemma 3.17 $(i), (ii)$ we have $0 \circ 2 = \{0, 1\}$ and $2 \circ 0 = \{2\}$. If $2 \circ 1 = \{0, 2\}$, then by Lemma 3.18 (i) we have $0 \circ 2 = \{0, 1\}$.

So there exist 11 hyper K-algebras as follows:

$\circ$	0	1	2
0	$\{0\}$	$\{0,1\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,1\}$	$\{0,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,1\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,1,2\}$	$\{0,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,1\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1\}$	$\{0,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,1\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1,2\}$	$\{0,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,1\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0\}$	$\{0,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,1\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,1\}$	$\{0,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,1\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,2\}$	$\{0,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,1\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,1,2\}$	$\{0,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,1\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1\}$	$\{0,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,1\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,2\}$	$\{0,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,1\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1,2\}$	$\{0,2\}$

2.1.2.4. $2 \circ 2 = \{0,1,2\}$

This case is similar as the case of 2.1.2.2. So there exist 13 hyper K-algebras as follows:

$\circ$	0	1	2
0	$\{0\}$	$\{0,1\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,1,2\}$	$\{0,1,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,1\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1\}$	$\{0,1,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,1\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,2\}$	$\{0,1,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,1\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1,2\}$	$\{0,1,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,1\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,2\}$	$\{0,1,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,1\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1\}$	$\{0,1,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,1\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1,2\}$	$\{0,1,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,1\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0\}$	$\{0,1,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,1\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,1\}$	$\{0,1,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,1\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,2\}$	$\{0,1,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,1\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,2\}$	$\{0,1,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,1\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,1\}$	$\{0,1,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,1\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,1,2\}$	$\{0,1,2\}$

2.1.3. $0 \circ 1 = \{0,2\}$

In this case we have two states since by Lemma 3.14 (i) we obtain $2 \circ 2 \not\subseteq \{0,1\}$.

2.1.3.1. $2 \circ 2 = \{0,2\}$

If $2 \circ 1 = \{0,2\}$, then by Lemma 3.16 we have $0 \circ 2 = \{0\}$. By Lemma 3.14 $(i),(iv)$ we have $2 \circ 1 \not\subseteq \{0,1\}$ and $2 \circ 0 = \{2\}$. So there exist three hyper K-algebras as follows:

$\circ$	0	1	2
0	$\{0\}$	$\{0,2\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,1,2\}$	$\{0,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,2\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,2\}$	$\{0,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,2\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,1,2\}$	$\{0,2\}$

2.1.3.2. $2 \circ 2 = \{0,1,2\}$

By Lemma 3.14 $(i),(iv),(v)$ we must have $2 \circ 1 \not\subseteq \{0,1\}$, $2 \circ 0 = \{2\}$ and $0 \circ 2 = \{0,1\}$. So there exist two hyper K-algebras as follows:

$\circ$	0	1	2
0	$\{0\}$	$\{0,2\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,2\}$	$\{0,1,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,2\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,1,2\}$	$\{0,1,2\}$

2.1.4. $0 \circ 1 = \{0,1,2\}$

By Lemma 3.15 (i) we have $2 \circ 2 \not\subseteq \{0,1\}$, thus in this case we have only two states as follows:

2.1.4.1. $2 \circ 2 = \{0,2\}$

By Lemma 3.15 (i) we have $2 \circ 1 \not\subseteq \{0,1\}$. If $2 \circ 1 = \{0,2\}$, then by Lemma 3.18 (i) we have $0 \circ 2 = \{0,1\}$. So there exist six hyper K-algebras as follows:

$\circ$	0	1	2
0	$\{0\}$	$\{0,1,2\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,1,2\}$	$\{0,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,1,2\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1,2\}$	$\{0,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,1,2\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,1,2\}$	$\{0,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,1,2\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,2\}$	$\{0,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,1,2\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1,2\}$	$\{0,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,1,2\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,2\}$	$\{0,2\}$

2.1.4.2. $2 \circ 2 = \{0, 1, 2\}$

By Lemma 3.15 (ii) we have $2 \circ 1 \not\subseteq \{0, 1\}$. So there exist eight hyper K-algebras as follows:

$\circ$	0	1	2
0	$\{0\}$	$\{0,1,2\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,2\}$	$\{0,1,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,1,2\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,1,2\}$	$\{0,1,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,1,2\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,2\}$	$\{0,1,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,1,2\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1,2\}$	$\{0,1,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,1,2\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,2\}$	$\{0,1,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,1,2\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,1,2\}$	$\{0,1,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,1,2\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,2\}$	$\{0,1,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,1,2\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1,2\}$	$\{0,1,2\}$

Now we consider the following case:

2.2. $0 \circ 0 = \{0, 1\}$

We consider some subcases as follows:

2.2.1. $0 \circ 1 = \{0\}$

In this case also we consider four states as follows:

2.2.1.1. $2 \circ 2 = \{0\}$

By Lemmas 3.19 and 3.11 we have $2 \circ 0 = \{1, 2\}$ and $1 \notin 2 \circ 1$. Since $1 \notin 2 \circ 1$, hence $2 \circ 1 \subseteq \{0, 2\}$ and by Lemma 3.16 we have $0 \circ 2 = \{0\}$. So there exist two hyper K-algebras as follows:

$\circ$	0	1	2
0	$\{0,1\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,2\}$	$\{0\}$

$\circ$	0	1	2
0	$\{0,1\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0\}$	$\{0\}$

2.2.1.2. $2 \circ 2 = \{0, 1\}$

If $2 \circ 1 \subseteq \{0, 2\}$, then by Lemmas 3.17 (iv) and 3.18 (ii) we have $2 \circ 0 = \{1, 2\}$. If $2 \circ 1 = \{0\}$, then by Lemma 3.17 (i) we have $0 \circ 2 = \{0, 1\}$. So there exist 11 hyper K-algebras as follows:

$\circ$	0	1	2
0	$\{0,1\}$	$\{0\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,2\}$	$\{0,1\}$

$\circ$	0	1	2
0	$\{0,1\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,2\}$	$\{0,1\}$

$\circ$	0	1	2
0	$\{0,1\}$	$\{0\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0\}$	$\{0,1\}$

$\circ$	0	1	2
0	$\{0,1\}$	$\{0\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,1\}$	$\{0,1\}$

$\circ$	0	1	2
0	$\{0,1\}$	$\{0\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1\}$	$\{0,1\}$

$\circ$	0	1	2
0	$\{0,1\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,1\}$	$\{0,1\}$

$\circ$	0	1	2
0	$\{0,1\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1\}$	$\{0,1\}$

$\circ$	0	1	2
0	$\{0,1\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1,2\}$	$\{0,1\}$

$\circ$	0	1	2
0	$\{0,1\}$	$\{0\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,1,2\}$	$\{0,1\}$

$\circ$	0	1	2
0	$\{0,1\}$	$\{0\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1,2\}$	$\{0,1\}$

$\circ$	0	1	2
0	$\{0,1\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,1,2\}$	$\{0,1\}$

2.2.1.3. $2 \circ 2 = \{0,2\}$

By Lemma 3.19 we have $2 \circ 0 = \{1,2\}$. If $2 \circ 1 \subseteq \{0,2\}$, then by Lemma 3.16 we have $0 \circ 2 = \{0\}$. So there exist six hyper K-algebras as follows:

$\circ$	0	1	2
0	$\{0,1\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0\}$	$\{0,2\}$

$\circ$	0	1	2
0	$\{0,1\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,2\}$	$\{0,2\}$

$\circ$	0	1	2
0	$\{0,1\}$	$\{0\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1\}$	$\{0,2\}$

$\circ$	0	1	2
0	$\{0,1\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1\}$	$\{0,2\}$

$\circ$	0	1	2
0	$\{0,1\}$	$\{0\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1,2\}$	$\{0,2\}$

$\circ$	0	1	2
0	$\{0,1\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1,2\}$	$\{0,2\}$

2.2.1.4. $2 \circ 2 = \{0,1,2\}$

This case is similar to 2.2.1.2. So there exist 11 hyper K-algebras:

$\circ$	0	1	2
0	$\{0,1\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,1,2\}$	$\{0,1,2\}$

$\circ$	0	1	2
0	$\{0,1\}$	$\{0\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0\}$	$\{0,1,2\}$

$\circ$	0	1	2
0	$\{0,1\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,2\}$	$\{0,1,2\}$

$\circ$	0	1	2
0	$\{0,1\}$	$\{0\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,2\}$	$\{0,1,2\}$

$\circ$	0	1	2
0	$\{0,1\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1,2\}$	$\{0,1,2\}$

$\circ$	0	1	2
0	$\{0,1\}$	$\{0\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1,2\}$	$\{0,1,2\}$

$\circ$	0	1	2
0	$\{0,1\}$	$\{0\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,1,2\}$	$\{0,1,2\}$

$\circ$	0	1	2
0	$\{0,1\}$	$\{0\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1\}$	$\{0,1,2\}$

$\circ$	0	1	2
0	$\{0,1\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1\}$	$\{0,1,2\}$

$\circ$	0	1	2
0	$\{0,1\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,1\}$	$\{0,1,2\}$

$\circ$	0	1	2
0	$\{0,1\}$	$\{0\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,1\}$	$\{0,1,2\}$

2.2.2. $0 \circ 1 = \{0,1\}$

Consider the following four states:

2.2.2.1. $2 \circ 2 = \{0\}$

By Lemma 3.19 we have $2 \circ 0 = \{1,2\}$. If $2 \circ 1 \subseteq \{0,2\}$, then by Lemmas 3.17 (iii) and 3.18 (i) we have $0 \circ 2 = \{0,1\}$. So there exist six hyper K-algebras as follows:

$\circ$	0	1	2
0	$\{0,1\}$	$\{0,1\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1,2\}$	$\{0\}$

$\circ$	0	1	2
0	$\{0,1\}$	$\{0,1\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0\}$	$\{0\}$

$\circ$	0	1	2
0	$\{0,1\}$	$\{0,1\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1\}$	$\{0\}$

$\circ$	0	1	2
0	$\{0,1\}$	$\{0,1\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,2\}$	$\{0\}$

$\circ$	0	1	2
0	$\{0,1\}$	$\{0,1\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1,2\}$	$\{0\}$

$\circ$	0	1	2
0	$\{0,1\}$	$\{0,1\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1\}$	$\{0\}$

2.2.2.2. $2 \circ 2 = \{0,1\}$

If $2 \circ 1 \subseteq \{0,2\}$, then by Lemmas 3.17 (iv) and 3.18 (ii) we have $2 \circ 0 = \{1,2\}$. If $2 \circ 1 = \{0\}$, then by Lemma 3.17 (iii) we have $0 \circ 2 = \{0,1\}$. So there exist 11 hyper K-algebras as follows:

$\circ$	0	1	2
0	$\{0,1\}$	$\{0,1\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,1\}$	$\{0,1\}$

$\circ$	0	1	2
0	$\{0,1\}$	$\{0,1\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,1\}$	$\{0,1\}$

$\circ$	0	1	2
0	$\{0,1\}$	$\{0,1\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1\}$	$\{0,1\}$

$\circ$	0	1	2
0	$\{0,1\}$	$\{0,1\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1\}$	$\{0,1\}$

$\circ$	0	1	2
0	$\{0,1\}$	$\{0,1\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0\}$	$\{0,1\}$

$\circ$	0	1	2
0	$\{0,1\}$	$\{0,1\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,2\}$	$\{0,1\}$

$\circ$	0	1	2
0	$\{0,1\}$	$\{0,1\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,2\}$	$\{0,1\}$

$\circ$	0	1	2
0	$\{0,1\}$	$\{0,1\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,1,2\}$	$\{0,1\}$

$\circ$	0	1	2
0	$\{0,1\}$	$\{0,1\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,1,2\}$	$\{0,1\}$

$\circ$	0	1	2
0	$\{0,1\}$	$\{0,1\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1,2\}$	$\{0,1\}$

$\circ$	0	1	2
0	$\{0,1\}$	$\{0,1\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1,2\}$	$\{0,1\}$

2.2.2.3. $2 \circ 2 = \{0,2\}$

By Lemma 3.19 we have $2 \circ 0 = \{1,2\}$. If $2 \circ 1 \subseteq \{0,2\}$, then by Lemmas 3.18 (i) and 3.17 (i) we have $0 \circ 2 = \{0,1\}$. So there exist six hyper K-algebras as follows:

$\circ$	0	1	2
0	$\{0,1\}$	$\{0,1\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1\}$	$\{0,2\}$

$\circ$	0	1	2
0	$\{0,1\}$	$\{0,1\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1,2\}$	$\{0,2\}$

$\circ$	0	1	2
0	$\{0,1\}$	$\{0,1\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1\}$	$\{0,2\}$

$\circ$	0	1	2
0	$\{0,1\}$	$\{0,1\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1,2\}$	$\{0,2\}$

$\circ$	0	1	2
0	$\{0,1\}$	$\{0,1\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,2\}$	$\{0,2\}$

$\circ$	0	1	2
0	$\{0,1\}$	$\{0,1\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0\}$	$\{0,2\}$

2.2.2.4. $2 \circ 2 = \{0,1,2\}$

If $2 \circ 1 \subseteq \{0,2\}$, then by Lemmas 3.17 (iv) and 3.18 (ii) we have $2 \circ 0 = \{1,2\}$. If $2 \circ 1 = \{0\}$, then by Lemma 3.17 (i) we have $0 \circ 2 = \{0,1\}$. So there exist 11 hyper K-algebras as follows:

$\circ$	0	1	2
0	$\{0,1\}$	$\{0,1\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,1\}$	$\{0,1,2\}$

$\circ$	0	1	2
0	$\{0,1\}$	$\{0,1\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,1,2\}$	$\{0,1,2\}$

$\circ$	0	1	2
0	$\{0,1\}$	$\{0,1\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0\}$	$\{0,1,2\}$

$\circ$	0	1	2
0	$\{0,1\}$	$\{0,2\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1,2\}$	$\{0,1,2\}$

$\circ$	0	1	2
0	$\{0,1\}$	$\{0,1\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,2\}$	$\{0,1,2\}$

$\circ$	0	1	2
0	$\{0,1\}$	$\{0,1\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1\}$	$\{0,1,2\}$

$\circ$	0	1	2
0	$\{0,1\}$	$\{0,1\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1\}$	$\{0,1,2\}$

$\circ$	0	1	2
0	$\{0,1\}$	$\{0,1\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,2\}$	$\{0,1,2\}$

$\circ$	0	1	2
0	$\{0,1\}$	$\{0,1\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1,2\}$	$\{0,1,2\}$

$\circ$	0	1	2
0	$\{0,1\}$	$\{0,1\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,1,2\}$	$\{0,1,2\}$

$\circ$	0	1	2
0	$\{0,1\}$	$\{0,1\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,1\}$	$\{0,1,2\}$

2.2.3. $0 \circ 1 = \{0, 2\}$

By Lemma 3.14 (i) we have $2 \circ 2 \not\subseteq \{0, 1\}$. Thus this case has only two stats as follows:

2.2.3.1. $2 \circ 2 = \{0, 2\}$

By Lemmas 3.14 (ii) and 3.19 we have $2 \circ 1 \not\subseteq \{0, 1\}$ and $2 \circ 0 = \{1, 2\}$. If $2 \circ 1 = \{0, 2\}$, then by Lemma 3.16 we have $0 \circ 2 = \{0\}$. So there exist three hyper K-algebras as follows:

$\circ$	0	1	2
0	$\{0,1\}$	$\{0,2\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,2\}$	$\{0,2\}$

$\circ$	0	1	2
0	$\{0,1\}$	$\{0,2\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1,2\}$	$\{0,2\}$

$\circ$	0	1	2
0	$\{0,1\}$	$\{0,2\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1,2\}$	$\{0,2\}$

Case 2.2.3.2. $2 \circ 2 = \{0, 1, 2\}$.

By Lemmas 3.14 $(ii), (v)$ we have $2 \circ 1 \not\subseteq \{0, 1\}$ and $0 \circ 2 = \{0, 1\}$. If $2 \circ 1 = \{0, 2\}$, then by Lemma 3.18 (ii) we have $2 \circ 0 = \{1, 2\}$. So there exist three hyper K-algebras as follows:

$\circ$	0	1	2
0	$\{0,1\}$	$\{0,2\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,1,2\}$	$\{0,1,2\}$

$\circ$	0	1	2
0	$\{0,1\}$	$\{0,2\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,2\}$	$\{0,1,2\}$

$\circ$	0	1	2
0	$\{0,1\}$	$\{0,2\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1,2\}$	$\{0,1,2\}$

2.2.4. $0 \circ 1 = \{0, 1, 2\}$

By Lemma 3.15 (i) we have $2 \circ 2 \not\subseteq \{0, 1\}$. Thus this case has only two states as follows:

2.2.4.1. $2 \circ 2 = \{0, 2\}$

By Lemmas 3.15 (i) and 3.19 we have $2 \circ 1 \not\subseteq \{0, 1\}$ and $2 \circ 0 = \{1, 2\}$. If $2 \circ 1 = \{0, 2\}$, then by Lemma 3.18 (i) we have $0 \circ 2 = \{0, 1\}$. So there exist three hyper K-algebras as follows:

$\circ$	0	1	2
0	$\{0,1\}$	$\{0,1,2\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1,2\}$	$\{0,2\}$

$\circ$	0	1	2
0	$\{0,1\}$	$\{0,1,2\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,2\}$	$\{0,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,1,2\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1,2\}$	$\{0,2\}$

2.2.4.2. $2 \circ 2 = \{0, 1, 2\}$

By Lemma 3.15 (ii) we have $2 \circ 1 \not\subseteq \{0, 1\}$. If $2 \circ 1 = \{0, 2\}$, then by Lemma 3.18 (ii) we have $2 \circ 0 = \{1, 2\}$. So there exist six hyper K-algebras as follows:

$\circ$	0	1	2
0	$\{0,1\}$	$\{0,1,2\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,2\}$	$\{0,1,2\}$

$\circ$	0	1	2
0	$\{0,1\}$	$\{0,1,2\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,2\}$	$\{0,1,2\}$

$\circ$	0	1	2
0	$\{0,1\}$	$\{0,1,2\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1,2\}$	$\{0,1,2\}$

$\circ$	0	1	2
0	$\{0,1\}$	$\{0,1,2\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1,2\}$	$\{0,1,2\}$

$\circ$	0	1	2
0	$\{0,1\}$	$\{0,1,2\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,1,2\}$	$\{0,1,2\}$

$\circ$	0	1	2
0	$\{0,1\}$	$\{0,1,2\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,1,2\}$	$\{0,1,2\}$

Now we show that each pair of the above 219 hyper K-algebras are not isomorphic together. On the contrary let $(H_1, \circ_1, 0)$ and $(H_2, \circ_2, 0)$ be isomorphic. then there exists an isomorhpism $f : H_1 \to H_2$. So $f(x \circ_1 y) = f(x) \circ_2 f(y)$, for all $x, y \in H$, thus we have $f(0_1) = 0_2$, $f(1) = 2$, $f(2) = 1$. But $f(1 \circ_1 2) = f(\{1\}) = \{2\}$ and $f(1) \circ_2 f(2) = 2 \circ 1 \supseteq \{0\}$, which is a contradiction, since $0 \notin f(1 \circ_1 2) = \{2\}$. □

References

[1] **R.A. Borzooei, P. Corsini, M.M. Zahedi**: *Some kinds of positive implicative hyper K-ideals*, J. Discrete Math. Sci. & Cryptography, (to appear).

[2] **R.A. Borzooei, A. Hasankhani, M.M. Zahedi and Y.B. Jun**: *On hyper K-algebras*, Math. Japon. **52** (2000), 113 – 121.

[3] **R.A. Borzooei and M.M. Zahedi**: *Positive implicative hyper K-ideals*, Sci. Math. Japon. **53** (2001), 525 – 533.

[4] **P. Corsini**: *Prolegomena of hypergroup theory*, Aviani Editore, 1993.

[5] **Y. Imai and K. Iséki**: *On axiom systems of propositional calculi XIV*, Proc. Japan Academy **42** (1966), 19 – 22.

[6] **K. Iséki and S. Tanaka**: *An introduction to the theory of BCK-algebras*, Math. Japon. **23** (1978), 1 – 26.

[7] **F. Marty**: *Sur une generalization de la notion de groups*, 8th Congress Math. Scandinaves, Stockholm 1934, 45 – 49.

[8] **J. Meng and Y.B. Jun**: *BCK-algebras*, Kyung Moonsa, Seoul, Korea 1994.

[9] **M.M. Zahedi, R.A. Borzooei, Y.B. Jun and A. Hasankhani**: *Some results on hyper K-algebra*, Scientiae Math. **3** (2000), 53 − 59.

[10] **M.M. Zahedi, R.A. Borzooei and H. Rezaei**: *Some classifications of hyper K-algebras of order* 3, Sci. Math. Japon. **53** (2001), 133 − 142.

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