Quasigroups and Related Systems **9** (2002), 67 – 84

Transversals in groups. 4. Derivation construction

Eugene A. Kuznetsov

Abstract

In the present work the derivation construction is studied by means of transversals in a group to a proper subgroup. It is shown that the method of derivation may be understood as a connection between different transversals in a group to a subgroup.

1. Introduction

The method of derivation has appeared in Dickson's works at the first time. It has been used to construct nearfields and quasifields from fields and skew-fields (see [14], [15]). Karzel [6] axiomatized and generalized this method for groups. Kiechle [7] gave a generalization of method of derivation which applied to construct loops with determined conditions by the help of groups.

In a present work the derivation construction is studied by means of transversals in a group to a proper subgroup. It is shown that the method of derivation may be understood as a connection between different transversals in a group to a subgroup. It give us a possibility to generalize the derivation construction for loops, i.e. to construct loops with some determined conditions by the help of some "good" loops.

2. Necessary definitions and notations

Definition 1. [2] A system $\langle E, \cdot\rangle$ is called a *right* (*left*) *quasigroup*, if for arbitrary $a, b \in E$ the equation $x \cdot a = b$ $\ (a \cdot y = b)$ has a unique solution in the set E. If in quasigroup $\langle E, \cdot\rangle$ there exists element $e \in E$ such that

$$x \cdot e = e \cdot x = e$$

2000 Mathematics Subject Classification: 20N15
Keywords: group, loop, transversal, derivation

for every $x \in E$, then system $\langle E, \cdot\rangle$ is called a *loop*.

Definition 2. [1] Let G be a group and H be a subgroup in G. A complete system $T = \{t_i\}_{i\in E}$ of representatives of the left (right) cosets of H in G $(e = t_1 \in H)$ is called a *left (right) transversal in G to H*.

Let $T = \{t_i\}_{i\in E}$ be a left transversal in G to H. We can define correctly (see [1, 9]) the following operation on the set E (E is an index set; left cosets of H in G are numbered by indexes from E):

$$x \overset{(T)}{\cdot} y = z \quad \overset{def}{\Longleftrightarrow} \quad t_x t_y = t_z h, \quad h \in H. \tag{1}$$

In [1] (and [9]) it is proved that $\langle E, \overset{(T)}{\cdot}\rangle$ is a left quasigroup with two-sided unit 1.

Below we shall consider (for simplicity) that $Core_G(H) = e$ (where

$$Core_G(H) = \underset{g\in G}{\cap} gHg^{-1}$$

is the maximal normal subgroup of the group G contained in the subgroup H) and shall study a permutation representation $\hat{G}$ of group G by left cosets on the subgroup H. According to [5], we have $\hat{G} \cong G$, where

$$\hat{g}(x) = y \quad \overset{def}{\Longleftrightarrow} \quad gt_xH = t_yH.$$

Note that $\hat{H} = St_1(\hat{G})$.

Lemma 1. ([9], Lemma 4) *Let T be an arbitrary left transversal in G to H. Then the following statements are true:*

1. $\hat{h}(1) = 1$ *for all* $h \in H$.
2. *For every $x, y \in E$ we have:* $\hat{t}_x(y) = x \overset{(T)}{\cdot} y, \quad \hat{t}_1(x) = \hat{t}_x(1) = x,$
$$\hat{t}_x^{-1}(y) = x \overset{(T)}{\backslash} y, \quad \hat{t}_x^{-1}(1) = x \overset{(T)}{\backslash} 1, \quad \hat{t}_x^{-1}(x) = 1$$
where $\overset{(T)}{\backslash}$ *is the left division in the system* $\langle E, \overset{(T)}{\cdot}, 1\rangle$
$(i.e.\ x \overset{(T)}{\backslash} y = z \Leftrightarrow x \overset{(T)}{\cdot} z = y)$. □

Let us denote
$$l_{a,b}^{(T)} = L^{-1}_{a \overset{(T)}{\cdot} b} L_a L_b$$

where $L_a(x) = a \overset{(T)}{\cdot} x$ is a left translation in the left loop $\langle E, \overset{(T)}{\cdot}, 1\rangle$. The group

$$LI(\,\langle E, \overset{(T)}{\cdot}, 1\rangle\,) = \langle\, l^{(T)}_{a,b} \mid a, b \in E\,\rangle$$

is called a *left inner permutation group*. It is easy to see that

$$l^{(T)}_{a,b}(x) = (a \overset{(T)}{\cdot} b) \overset{(T)}{\backslash} (a \overset{(T)}{\cdot} (b \overset{(T)}{\cdot} x)) = \hat{t}^{-1}_{a \overset{(T)}{\cdot} b}\, \hat{t}_a \hat{t}_b(x)\,. \tag{2}$$

Note that for every $a, b \in E \quad l^{(T)}_{a,b}(1) = 1$.

Definition 3. A left loop $\langle E, \cdot, 1\rangle$ is called

1. *left alternative*, if for every $x, y \in E: \quad x \cdot (x \cdot y) = (x \cdot x) \cdot y$,
2. *left IP-loop* (or *LIP-loop*), if for every $x \in E$ there exists the element $x' \in E$ such that $x' \cdot (x \cdot y) = y$ for every $y \in E$,
3. *left A_l-loop*, if for every $a, b \in E \;\; l_{a,b} \in Aut(\langle E, \cdot, 1\rangle)$.

Lemma 2. *Let a set $T = \{t_i\}_{i \in E}$ be a left transversal in the group G to its subgroup H. Then the following conditions are equivalent:*

1. *The system $\langle E, \overset{(T)}{\cdot}, 1\rangle$ is a left A_l-loop,*
2. *For every $\alpha \in LI(\langle E, \overset{(T)}{\cdot}, 1\rangle)$ we have $\alpha \hat{T} \alpha^{-1} \subseteq \hat{T}$.*

Proof. $1 \Longrightarrow 2$. Let the system $\langle E, \overset{(T)}{\cdot}, 1\rangle$ be a left A_l-loop, i.e. let

$$l^{(T)}_{a,b} = \hat{t}^{-1}_{a \cdot b} \hat{t}_a \hat{t}_b \in Aut(\langle E, \overset{(T)}{\cdot}, 1\rangle)$$

for every $a, b \in E$. Then

$$LI(\langle E, \overset{(T)}{\cdot}, 1\rangle) \subseteq Aut(\langle E, \overset{(T)}{\cdot}, 1\rangle). \tag{3}$$

Let $\alpha \in LI(\langle E, \overset{(T)}{\cdot}, 1\rangle)$. Because $\alpha \hat{t}_x \alpha^{-1} \in \hat{G}$ for every $x \in E$, then

$$\alpha \hat{t}_x \alpha^{-1} = \hat{t}_u \hat{h}_1 \tag{4}$$

for some $u \in E$ and $h_1 \in H$. In view of Lemma 1 we have

$$u = \hat{t}_u(1) = \hat{t}_u \hat{h}_1(1) = \alpha\, \hat{t}_x \alpha^{-1}(1) = \alpha\, \hat{t}_x(1) = \alpha(x),$$

i.e. the equation (4) may be rewritten in the following form

$$\alpha\hat{t}_x\alpha^{-1} = \hat{t}_{\alpha(x)}\hat{h}_1\,. \tag{5}$$

On the other hand, for every $x, y \in E$

$$\hat{t}_x\hat{t}_y = \hat{t}_{x \overset{(T)}{\cdot} y}\, l^{(T)}_{x,y},$$

$$\alpha\hat{t}_x\alpha^{-1}\alpha\hat{t}_y\alpha^{-1} = \alpha\hat{t}_{x \overset{(T)}{\cdot} y}\alpha^{-1}\alpha l^{(T)}_{x,y}\alpha^{-1},$$

which, by (5), gives

$$\hat{t}_{\alpha(x)}\hat{h}_1\hat{t}_{\alpha(y)}\hat{h}_2 = \hat{t}_{\alpha(x \overset{(T)}{\cdot} y)}\hat{h}_3\,\alpha l^{(T)}_{x,y}\alpha^{-1}$$

for some $h_2, h_3 \in H$. In view of Lemma 1 we have also

$$\hat{t}_{\alpha(x)}\hat{h}_1\hat{t}_{\alpha(y)}\hat{h}_2(1) = \hat{t}_{\alpha(x \overset{(T)}{\cdot} y)}\hat{h}_3\,\alpha l^{(T)}_{x,y}\alpha^{-1}(1),$$

$$\alpha(x) \overset{(T)}{\cdot} \hat{h}_1(\alpha(y)) = \alpha(x \overset{(T)}{\cdot} y). \tag{6}$$

But $\alpha \in LI(\langle E, \overset{(T)}{\cdot}, 1\rangle)$. Thus, by (3), we have $\alpha(x \overset{(T)}{\cdot} y) = \alpha(x) \overset{(T)}{\cdot} \alpha(y)$. So, by (6), we obtain:

$$\alpha(x) \overset{(T)}{\cdot} \hat{h}_1(\alpha(y)) = \alpha(x) \overset{(T)}{\cdot} \alpha(y)$$

for every $x, y \in E$. Since the system $\langle E, \overset{(T)}{\cdot}, 1\rangle$ is a left quasigroup with two-sided unit 1, for every $y \in E$ we get

$$\hat{h}_1(\alpha(y)) = \alpha(y).$$

Function α is a permutation on the set E, so for every $z \in E$ $\hat{h}_1(z) = z$, i.e. $h_1 = e$. Then in view of (5) we obtain:

$$\alpha\hat{t}_x\alpha^{-1} = \hat{t}_{\alpha(x)} \in \hat{T} \tag{7}$$

for every $\alpha \in LI(\langle E, \overset{(T)}{\cdot}, 1\rangle)$ and $x \in E$.

2 $\Longrightarrow$ 1. (See [8]) Let for every $\alpha \in LI(\langle E, \overset{(T)}{\cdot}, 1\rangle)$ and $x \in E$ exists an element $u \in E$ such that

$$\alpha\hat{t}_x\alpha^{-1} = \hat{t}_u \in \hat{T}. \tag{8}$$

Then, in view of Lemma 1, we have

$$u = \hat{t}_u(1) = \alpha\,\hat{t}_x\alpha^{-1}(1) = \alpha\,\hat{t}_x(1) = \alpha(x),$$

i.e. the equation (8) may be rewritten in the following way

$$\alpha\hat{t}_x\alpha^{-1} = \hat{t}_{\alpha(x)}. \tag{9}$$

But every $x, y \in E$ we have

$$\hat{t}_x\hat{t}_y = \hat{t}_{x\overset{(T)}{\cdot}y}\, l^{(T)}_{x,y}\,.$$

Then

$$\alpha\hat{t}_x\alpha^{-1}\alpha\hat{t}_y\alpha^{-1} = \alpha\hat{t}_{x\overset{(T)}{\cdot}y}\alpha^{-1}\alpha l^{(T)}_{x,y}\alpha^{-1},$$

which, by (9), gives

$$\hat{t}_{\alpha(x)}\hat{t}_{\alpha(y)} = \hat{t}_{\alpha(x\overset{(T)}{\cdot}y)}\alpha l^{(T)}_{x,y}\alpha^{-1}$$

and, in the consequence,

$$\alpha l^{(T)}_{x,y}\alpha^{-1} \in LI(\langle E, \overset{(T)}{\cdot}, 1\rangle) \subseteq \hat{H}\,,$$

because $\alpha, l^{(T)}_{x,y} \in LI(\langle E, \overset{(T)}{\cdot}, 1\rangle)$. Now, applying the definition of the left transversal T, we obtain

$$\alpha(x) \overset{(T)}{\cdot} \alpha(y) = \alpha(x \overset{(T)}{\cdot} y).$$

This means that $\alpha \in Aut(\langle E, \overset{(T)}{\cdot}, 1\rangle)$ and $\langle E, \overset{(T)}{\cdot}, 1\rangle$ is a left A_l-loop. □

Lemma 3. *Let $\langle E, \cdot, 1\rangle$ be a left loop. Then the following statements are true:*

1. *System $\langle E, \cdot, 1\rangle$ is LIP-loop if and only if for every $a \in E$: if $a \cdot a' = 1$ for some $a' \in E$ then $l_{a,a'} = id$.*
2. *System $\langle E, \cdot, 1\rangle$ is left alternative if and only if $l_{a,a} = id$ for every $a \in E$.*

Proof. See [7]. □

3. Derivation as a connection between transversals in a group by the same subgroup

Let us remind the general method of derivation used for construction of loops [7], section 7.

Let $\langle A, \cdot, 1\rangle$ be a group. The function $\varphi: A \to S_A$ such that $\varphi(a) \rightleftharpoons \varphi_a$, $\varphi_1 = id$ and $\varphi_a(1) = 1$ for any $a \in A$ is called a *weak derivation*. It is called a *derivation* if furthemore for all $a, b \in A$ there exists a unique $x \in A$ such that $x \cdot \varphi_x(a) = b$.

In [7], section 7, it was proved the following

Lemma 4. *Let $\langle A, \cdot, 1\rangle$ be a group with a weak derivation φ. Let us define the operation*

$$x \circ y \stackrel{def}{=} x \cdot \varphi_x(y). \tag{10}$$

Then:

1. *The system $\langle A, \circ, 1\rangle$ is a left loop with two-sided unit 1 (the identity elements of $\langle A, \cdot, 1\rangle$ and $\langle A, \circ, 1\rangle$ coincide). Moreover, for all $a \in A$ if $a \circ a' = 1$, then $a' = \varphi_a^{-1}(a^{-1})$.*
2. *If φ is a derivation, then system $\langle A, \circ, 1\rangle$ is a loop.*

The system $\langle A, \circ, 1\rangle$ is called a *derived (left) loop*. If $\varphi_a \in Aut\, A$ for every $a \in A$, then derivation is called *automorphic derivation*.

For the connection between two different left transversals in a group G by the same subgroup H see [9, 10].

Let $T = \{t_x\}_{x \in E}$ and $P = \{p_x\}_{x \in E}$ are two left transversals in a group G to its subgroup H. It is evident that for every $x \in E$ $p_x = t_x h_{(x)}$ for some collection $\{h_{(x)}\}_{x \in E}$, $h_{(x)} \in H$. As it was proved in [9], for systems $\langle E, \stackrel{(T)}{\cdot}, 1\rangle$ and $\langle E, \stackrel{(P)}{\cdot}, 1\rangle$ corresponding to the transversals T and P, by formula (1), we have:

$$x \stackrel{(P)}{\cdot} y = x \stackrel{(T)}{\cdot} \hat{h}_{(x)}(y). \tag{11}$$

It is easy to see that formulas (10) and (11) almost coincide; moreover, $\hat{h}_{(1)} = id$ and $\hat{h}_{(x)}(1) = 1$ for every $x \in E$. Note that unlike the derivation construction described above the system $\langle E, \stackrel{(T)}{\cdot}, 1\rangle$ is not a group; in a general case it is a left loop with two-sided unit. This means that the construction of new operations by the help of the connection between different left transversals in a group G to its subgroup H (formula (11)) generalize the derivation method (formula (10)). So the construction of weak derivation from formula (10) may be generalized up to the class of left loops which are corresponding to the left transversals in a some group G to its subgroup H. The system $\langle E, \overset{(P)}{\cdot}, 1\rangle$ from formula (11) will be called *derived left loop* and the set of permutations $\{\hat{h}_{(x)}\}_{x\in E}$ will be called a *deriving set.*

Lemma 5. *Let $\langle E, \overset{(P)}{\cdot}, 1\rangle$ be a derived left loop obtained by the method of weak derivation (formula (11)) from the left loop $\langle E, \overset{(T)}{\cdot}, 1\rangle$ by the help of derived set $\{\hat{h}_{(x)}\}_{x\in E}$. Then the following sentences are true:*

1. *Two-sided units of the left loops $\langle E, \overset{(P)}{\cdot}, 1\rangle$ and $\langle E, \overset{(T)}{\cdot}, 1\rangle$ coincide.*
2. *If $\overset{(P)}{a^{-1}}$ is a right inverse to the element a in $\langle E, \overset{(P)}{\cdot}, 1\rangle$, then $\overset{(P)}{a^{-1}} = \hat{h}^{-1}_{(a)}(\overset{(T)}{a^{-1}})$, where $\overset{(T)}{a^{-1}}$ is a right inverse to the element a in $\langle E, \overset{(T)}{\cdot}, 1\rangle$.*
3. *The left loop $\langle E, \overset{(P)}{\cdot}, 1\rangle$ is a loop (i.e. weak derivation is a derivation) if and only if the operations $\langle E, \overset{(T)}{\cdot}, 1\rangle$ and $B(x,y) = \hat{h}^{-1}_{(x)}(y)$ are orthogonal.*

Proof. 1. It is easy to see that if 1 is a unit in $\langle E, \overset{(T)}{\cdot}, 1\rangle$, then

$$1 \overset{(P)}{\cdot} x = 1 \overset{(T)}{\cdot} \hat{h}_{(1)}(x) = 1 \overset{(T)}{\cdot} x = x,$$

$$x \overset{(P)}{\cdot} 1 = x \overset{(T)}{\cdot} \hat{h}_{(x)}(1) = x \overset{(T)}{\cdot} 1 = x.$$

2. If $\overset{(P)}{a^{-1}}$ is a right inverse to a in $\langle E, \overset{(P)}{\cdot}, 1\rangle$, then $a \overset{(P)}{\cdot} \overset{(P)}{a^{-1}} = 1$. Thus $a \overset{(T)}{\cdot} \hat{h}_{(a)}(\overset{(P)}{a^{-1}}) = 1$ and $\overset{(P)}{a^{-1}} = \hat{h}^{-1}_{(a)}(\overset{(T)}{a^{-1}})$.

3. (See also [3], [10]) It is enough to prove that the equation $x \overset{(P)}{\cdot} a = b$ has a unique solution in E for any fixed $a, b \in E$ if and only if the operations $\langle E, \overset{(T)}{\cdot}, 1\rangle$ and $B(x,y) = \hat{h}^{-1}_{(x)}(y)$ are orthogonal, i.e. if and only if the system

$$\begin{cases} x \overset{(T)}{\cdot} y = a \\ B(x,y) = b \end{cases}$$

has a unique solution in the set $E \times E$ for any $a, b \in E$.

We have

$$\begin{cases} x \overset{(P)}{\cdot} a = b \\ x \overset{(T)}{\cdot} \hat{h}_{(x)}(a) = b \end{cases} \Longleftrightarrow \begin{cases} \hat{h}_{(x)}(a) = z \\ x \overset{(T)}{\cdot} z = b \end{cases} \Longleftrightarrow$$

$$\Longleftrightarrow \begin{cases} \hat{h}^{-1}_{(x)}(z) = a \\ x \overset{(T)}{\cdot} z = b \end{cases} \Longleftrightarrow \begin{cases} B(x,z) = a \\ x \overset{(T)}{\cdot} z = b \end{cases}$$

Last system has a unique solution if and only if the operations $\langle E, \overset{(T)}{\cdot}, 1\rangle$ and $B(x,y) = \hat{h}^{-1}_{(x)}(y)$ are orthogonal. $\square$

Remark 1. According to Cayley's Theorem (see [5], theorem 12.1.1, 12.1.3) every group K may be represented as a permutation group on the set K; this representation is regular. So any group K may be represented as a group transversal in S_K to $St_1(S_K)$. Then the construction of weak derivation of an arbitrary group $\langle A, \cdot, 1\rangle$ to the derived left loop $\langle A, \circ, 1\rangle$ may be represented as a construction of the left transversal $P = \{p_x\}_{x\in E}$ in the group S_A to $St_1(S_A)$ by the help of the group transversal $A^* = \{t_x\}_{x\in E}$ in the group S_A to $St_1(S_A)$. The corresponding system $\langle E, \overset{(A)}{\cdot}, 1\rangle$ is isomorphic to the group $\langle A, \cdot, 1\rangle$ and the system $\langle E, \overset{(P)}{\cdot}, 1\rangle$ is isomorphic to the derived left loop $\langle A, \circ, 1\rangle$.

Remark 2. The construction of weak derivation may also take place when there exists a group transversal T in the group G to its subgroup H. Then any other left transversal P in the group G to its subgroup H may be represented as a weak derivation of the group transversal T by the help of the deriving set $\{\hat{h}_{(x)}\}_{x\in E} \subset \hat{H}$.

Remark 3. The construction of automorphic derivation may be naturally represented as a connection between left transversals in the group G to its subgroup H, where G is a semidirect product (see [13], [11]) of a left loop $\langle E, \cdot, 1\rangle$ and group H, and $LI(\langle E, \cdot, 1\rangle) \subseteq H \subseteq Aut(\langle E, \cdot, 1\rangle)$.

4. Automorphic derivations

Let us investigate the case of weak automorphic derivation of left loops, i.e. the case of weak derivation with the condition

$$\{\hat{h}_{(x)}\}_{x\in E} \subseteq Aut(\langle E, \cdot, 1\rangle).$$

Lemma 6. *Let $\langle E, \overset{(P)}{\cdot}, 1\rangle$ be a derived left loop, which is obtained from a left loop $\langle E, \overset{(T)}{\cdot}, 1\rangle$ by means of weak automorphic derivation by the help of the deriving set $\{\hat{h}_{(x)}\}_{x\in E}$. We have*

1. *The following conditions are equivalent:*

(a) $\alpha \in Aut(\langle E, \overset{(T)}{\cdot}, 1\rangle)$ *is an automorphism of* $\langle E, \overset{(P)}{\cdot}, 1\rangle$,
(b) $\alpha \hat{h}_{(x)} \alpha^{-1} = \hat{h}_{(\alpha(x))}$ *for every* $x \in E$,
(c) $\alpha \in Aut(\langle E, \overset{(P)}{\cdot}, 1\rangle)$ *is an automorphism of* $\langle E, \overset{(T)}{\cdot}, 1\rangle$.

2. $l^{(P)}_{a,b} = \hat{h}^{-1}_{(a \overset{(P)}{\cdot} b)} l^{(T)}_{a,\hat{h}_{(a)}(b)} \hat{h}_{(a)} \hat{h}_{(b)}$ *for every* $a, b \in E$,

3. *The system* $\langle E, \overset{(P)}{\cdot}, 1\rangle$ *is left alternative if and only if for every* $a \in E$

$$\hat{h}_{(a \overset{(P)}{\cdot} a)} = l^{(T)}_{a,\hat{h}_{(a)}(a)} \hat{h}^2_{(a)}.$$

4. *The system* $\langle E, \overset{(P)}{\cdot}, 1\rangle$ *is a LIP-loop if and only if for every* $a \in E$

$$\hat{h}^{-1}_{(a)} = \hat{h}_{(\hat{h}^{-1}_{(a)}(a'))} l^{(T)}_{a,a'}, \quad \text{where } a' = \overset{(T)}{a^{-1}}.$$

5. *The system* $\langle E, \overset{(P)}{\cdot}, 1\rangle$ *is a left Bol loop if and only if for every* $a, b \in E$

$$\hat{h}_{(a \overset{(P)}{\cdot} (b \overset{(P)}{\cdot} a))} = l^{(T)}_{a,\hat{h}_{(a)}(b \overset{(P)}{\cdot} a)} \hat{h}_{(a)} l^{(T)}_{b,h_{(b)}(a)} \hat{h}_{(b)} \hat{h}_{(a)}.$$

6. *The system* $\langle E, \overset{(P)}{\cdot}, 1\rangle$ *is a group if and only if for every* $a, b \in E$

$$\hat{h}_{(a \overset{(P)}{\cdot} b)} = l^{(T)}_{a,\hat{h}_{(a)}(b)} \hat{h}_{(a)} \hat{h}_{(b)}.$$

Proof. 1. $(a) \Longleftrightarrow (b)$. If $\alpha \in Aut(\langle E, \overset{(T)}{\cdot}, 1\rangle)$, we have for every $x, y \in E$

$$\alpha(x) \overset{(P)}{\cdot} \alpha(y) = \alpha(x \overset{(P)}{\cdot} y),$$
$$\alpha(x) \overset{(T)}{\cdot} \hat{h}_{(\alpha(x))}\alpha(y) = \alpha(x \overset{(T)}{\cdot} \hat{h}_{(x)}(y)) = \alpha(x) \overset{(T)}{\cdot} \alpha\, \hat{h}_{(x)}(y),$$
$$\hat{h}_{(\alpha(x))}\alpha(y) = \alpha\, \hat{h}_{(x)}(y),$$
$$\hat{h}_{(\alpha(x))} = \alpha \hat{h}_{(x)} \alpha^{-1}.$$

$(c) \Longleftrightarrow (b)$. For every $x, y \in E$ we have $x \overset{(T)}{\cdot} y = x \overset{(P)}{\cdot} \hat{h}^{-1}_{(x)}(y)$. So the result follows as before.

2. Because of $\hat{h}_{(a)} \in Aut(\langle E, \overset{(T)}{\cdot}, 1\rangle)$ for every $a \in E$, so we have for every $a, b, x \in E$

$$a \overset{(P)}{\cdot} (b \overset{(P)}{\cdot} x) = (a \overset{(P)}{\cdot} b) \overset{(P)}{\cdot} l^{(P)}_{a,b}(x),$$
$$a \overset{(T)}{\cdot} \hat{h}_{(a)}(b \overset{(T)}{\cdot} \hat{h}_{(b)}(x)) = (a \overset{(T)}{\cdot} \hat{h}_{(a)}(b)) \overset{(T)}{\cdot} \hat{h}_{(a \overset{(P)}{\cdot} b)} l^{(P)}_{a,b}(x),$$
$$a \overset{(T)}{\cdot} (\hat{h}_{(a)}(b) \overset{(T)}{\cdot} \hat{h}_{(a)}\hat{h}_{(b)}(x)) = (a \overset{(T)}{\cdot} \hat{h}_{(a)}(b)) \overset{(T)}{\cdot} \hat{h}_{(a \overset{(P)}{\cdot} b)} l^{(P)}_{a,b}(x),$$

$$(a \overset{(T)}{\cdot} \hat{h}_{(a)}(b)) \overset{(T)}{\backslash} (a \overset{(T)}{\cdot} (\hat{h}_{(a)}(b) \overset{(T)}{\cdot} \hat{h}_{(a)}\hat{h}_{(b)}(x))) = \hat{h}_{(a \overset{(P)}{\cdot} b)} l^{(P)}_{a,b}(x).$$

In view of formula (2) we obtain for every $x \in E$

$$l^{(T)}_{a,\hat{h}_{(a)}(b)} \hat{h}_{(a)}\hat{h}_{(b)}(x) = \hat{h}_{(a \overset{(P)}{\cdot} b)} l^{(P)}_{a,b}(x),$$
$$l^{(P)}_{a,b} = \hat{h}^{-1}_{(a \overset{(P)}{\cdot} b)} l^{(T)}_{a,\hat{h}_{(a)}(b)} \hat{h}_{(a)}\hat{h}_{(b)}.$$

3. According to Lemma 3 the system $\langle E, \overset{(P)}{\cdot}, 1\rangle$ is left alternative if and only if $l^{(P)}_{a,a} = id$ for every $a \in E$. So by the condition 2) the result follows.

4. According to Lemma 3 the system $\langle E, \overset{(P)}{\cdot}, 1\rangle$ is a LIP-loop if and only if for every $a \in E$: if $a \overset{(P)}{\cdot} a'' = 1$ for some $a'' \in E$, then $l^{(P)}_{a,a''} = id$. So in view of the proposition 2) of present Lemma we obtain

$$id = \hat{h}^{-1}_{(a \overset{(P)}{\cdot} a'')} l^{(T)}_{a,\hat{h}_{(a)}(a'')} \hat{h}_{(a)}\hat{h}_{(a'')} = l^{(T)}_{a,\hat{h}_{(a)}(a'')} \hat{h}_{(a)}\hat{h}_{(a'')}.$$

Because of $a'' = \hat{h}^{-1}_{(a)}(a^{\overset{(T)}{-1}})$, then $\hat{h}^{-1}_{(a)} = \hat{h}_{(\hat{h}^{-1}_{(a)}(a'''))} l^{(T)}_{a,a'''}$, where $a''' = a^{\overset{(T)}{-1}}$.

5. It is easy to prove (see [7]) that the left Bol identity

$$(a \overset{(P)}{\cdot} (b \overset{(P)}{\cdot} a) \overset{(P)}{\cdot} x = a \overset{(P)}{\cdot} (b \overset{(P)}{\cdot} (a \overset{(P)}{\cdot} x)))$$

for every $a, b, x \in E$ is equivalent to the identity $l^{(P)}_{a,b \overset{(P)}{\cdot} a} = (l^{(P)}_{b,a})^{-1}$ for every $a, b \in E$. So in view of the condition 2) of the present Lemma we obtain

$$\hat{h}^{-1}_{(a \overset{(P)}{\cdot} (b \overset{(P)}{\cdot} a))} l^{(T)}_{a,\hat{h}_{(a)}(b \overset{(P)}{\cdot} a)} \hat{h}_{(a)}\hat{h}_{(b \overset{(P)}{\cdot} a)} = \hat{h}^{-1}_{(a)}\hat{h}^{-1}_{(b)} l^{(T)\,-1}_{b,\hat{h}_{(b)}(a)} \hat{h}_{(b \overset{(P)}{\cdot} a)},$$
$$\hat{h}_{(a \overset{(P)}{\cdot} (b \overset{(P)}{\cdot} a))} = l^{(T)}_{a,\hat{h}_{(a)}(b \overset{(P)}{\cdot} a)} \hat{h}_{(a)} l^{(T)}_{b,\hat{h}_{(b)}(a)} \hat{h}_{(b)}\hat{h}_{(a)}.$$

6. It is easy to prove that system $\langle E, \overset{(P)}{\cdot}, 1\rangle$ is a group if and only if $l^{(P)}_{a,b} = id$ for every $a, b \in E$. So in view of the condition 2) of present Lemma we obtain the result. □

Corollary 1. *Let the general assumptions of Lemma 6 hold. If the system $\langle E, \overset{(P)}{\cdot}, 1\rangle$ is a group, then the system $\langle E, \overset{(T)}{\cdot}, 1\rangle$ is an A_l-loop.*

Proof. In view of proposition 6) of Lemma 6 if the system $\langle E, \overset{(P)}{\cdot}, 1\rangle$ is a group then $\hat{h}_{(a \overset{(P)}{\cdot} b)} = l^{(T)}_{a,\hat{h}_{(a)}(b)} \hat{h}_{(a)}\hat{h}_{(b)}$. Because of $\hat{h}_{(a)} \in Aut(\langle E, \overset{(T)}{\cdot}, 1\rangle)$ for every $a \in E$, then $l^{(T)}_{a,\hat{h}_{(a)}(b)} \in Aut(\langle E, \overset{(T)}{\cdot}, 1\rangle)$ for every $a, b \in E$. But $\hat{h}_{(a)}$ is a permutation on the set E for every $a \in E$, then $l^{(T)}_{a,c} \in Aut(\langle E, \overset{(T)}{\cdot}, 1\rangle)$ for every $a, c \in E$. So $\langle E, \overset{(T)}{\cdot}, 1\rangle$ is an A_l-loop. □

4.1. An automorphic derivation of group

Let us apply the previous lemma to the case when $\langle E, \overset{(T)}{\cdot}, 1\rangle$ is a group.

Lemma 7. (See [7]) *Let $\langle E, \overset{(P)}{\cdot}, 1\rangle$ be a derived left loop, which is obtained from a group $\langle E, \overset{(T)}{\cdot}, 1\rangle$ by means of weak automorphic derivation by the help of deriving set $\{\hat{h}_{(x)}\}_{x\in E}$. Then the following statements are true:*

1. *The following conditions are equivalent:*
   (*a*) $\alpha \in Aut(\langle E, \overset{(T)}{\cdot}, 1\rangle)$ *is an automorphism of* $\langle E, \overset{(P)}{\cdot}, 1\rangle$,
   (*b*) $\alpha \hat{h}_{(x)} \alpha^{-1} = \hat{h}_{(\alpha(x))}$ *for every* $x \in E$,
   (*c*) $\alpha \in Aut(\langle E, \overset{(P)}{\cdot}, 1\rangle)$ *is an automorphism of the operation* $\langle E, \overset{(T)}{\cdot}, 1\rangle$.
2. $l^{(P)}_{a,b} = \hat{h}^{-1}_{(a \overset{(P)}{\cdot} b)} \hat{h}_{(a)}\hat{h}_{(b)} \in Aut(\langle E, \overset{(T)}{\cdot}, 1\rangle)$ *for every* $a, b \in E$,
3. *The system $\langle E, \overset{(P)}{\cdot}, 1\rangle$ is left alternative if and only if for every $a \in E$*
$$\hat{h}_{(a \overset{(P)}{\cdot} a)} = \hat{h}^2_{(a)}.$$
4. *The system $\langle E, \overset{(P)}{\cdot}, 1\rangle$ is LIP-loop if and only if for every $a \in E$*
$$\hat{h}^{-1}_{(a)} = \hat{h}_{(\hat{h}^{-1}_{(a)}(a'))}, \quad \text{where } a' = a^{\overset{(T)}{-1}}.$$
5. *The system $\langle E, \overset{(P)}{\cdot}, 1\rangle$ is a left Bol loop if and only if for every $a, b \in E$*
$$\hat{h}_{(a \overset{(P)}{\cdot} (b \overset{(P)}{\cdot} a))} = \hat{h}_{(a)}\hat{h}_{(b)}\hat{h}_{(a)}.$$
6. *The system $\langle E, \overset{(P)}{\cdot}, 1\rangle$ is a group if and only if for every $a, b \in E$*
$$\hat{h}_{(a \overset{(P)}{\cdot} b)} = \hat{h}_{(a)}\hat{h}_{(b)}.$$

Proof. It is the evident corollary of Lemma 6, because $l^{(T)}_{a,b} = id$ for every $a, b \in E$, if the system $\langle E, \overset{(T)}{\cdot}, 1\rangle$ is a group. See also [7]. □

Corollary 2. *Let the conditions of Lemma 7 hold. If for every $h \in H$ we have* $hh_{(x)}h^{-1} = h_{(\hat{h}(x))}$, *then the system* $\langle E, \overset{(P)}{\cdot}, 1\rangle$ *is an* A_l*-loop.*

Proof. It is evident because $LI(\langle E, \overset{(P)}{\cdot}, 1\rangle) \subseteq \hat{H}$ for every left transversal P in a group G to a subgroup H. □

Corollary 3. (See [15]) *Let the conditions of Lemma 7 hold. If* $\langle E, \overset{(P)}{\cdot}, 1\rangle$ *is a group, then* $\langle H_{(*)}, \cdot, h_{(1)}\rangle$, *where* $H_{(*)} \rightleftharpoons \{h_{(x)} | x \in E\}$, *is a subgroup of the group* H.

Proof. It is an evident corollary of 4) and 6) from Lemma 7. □

4.2. An automorphic derivation of A_l-loop

Let us apply Lemma 6 in a case when $\langle E, \overset{(T)}{\cdot}, 1\rangle$ is an A_l-loop.

Lemma 8. *Let* $\langle E, \overset{(P)}{\cdot}, 1\rangle$ *be a derived left loop, which is obtained from the* A_l*-loop* $\langle E, \overset{(T)}{\cdot}, 1\rangle$ *by means of weak automorphic derivation by the help of the deriving set* $\{\hat{h}_{(x)}\}_{x\in E}$. *Then:*

1. $l^{(P)}_{a,b} = \hat{h}^{-1}_{(a \overset{(P)}{\cdot} b)} l^{(T)}_{a,\hat{h}_{(a)}(b)} \hat{h}_{(a)} \hat{h}_{(b)} \in Aut(\langle E, \overset{(T)}{\cdot}, 1\rangle)$ *for every* $a, b \in E$,
2. $\langle E, \overset{(P)}{\cdot}, 1\rangle$ *is an* A_l*-loop if and only if for every* $\alpha \in LI(\langle E, \overset{(P)}{\cdot}, 1\rangle)$ *we have* $\alpha \hat{h}_{(x)} \alpha^{-1} = \hat{h}_{(\alpha(x))}$.

Proof. 1. Since the system $\langle E, \overset{(T)}{\cdot}, 1\rangle$ is an A_l-loop, $l^{(T)}_{a,b} \in Aut(\langle E, \overset{(T)}{\cdot}, 1\rangle)$ for every $a, b \in E$. But by the definition of a weak automorphic derivation $\hat{h}_{(x)} \in Aut(\langle E, \overset{(T)}{\cdot}, 1\rangle)$ for every $x \in E$. Thus, by the condition 2) of Lemma 6, we obtain our thesis.

2. If $\langle E, \overset{(P)}{\cdot}, 1\rangle$ is a A_l-loop, then in view of 1) from Lemma 6 and 1) from the present Lemma it is equivalent to $l^{(P)}_{a,b} \hat{h}_{(x)} l^{(P)\,-1}_{a,b} = \hat{h}_{(l^{(P)}_{a,b}(x))}$ for every $a, b \in E$. Since $LI(\langle E, \overset{(P)}{\cdot}, 1\rangle) = \langle l^{(P)}_{a,b} | a, b \in E\rangle$, for every $\alpha \in LI(\langle E, \overset{(P)}{\cdot}, 1\rangle)$ we obtain $\alpha \hat{h}_{(x)} \alpha^{-1} = \hat{h}_{(\alpha(x))}$. But according to the condition 2) of the present Lemma we have $LI(\langle E, \overset{(P)}{\cdot}, 1\rangle) \subseteq Aut(\langle E, \overset{(T)}{\cdot}, 1\rangle)$. □

Corollary 4. *Let* $\langle E, \overset{(T)}{\cdot}, 1\rangle$ *be an* A_l*-loop and* $G = LM(\langle E, \overset{(T)}{\cdot}, 1\rangle)$ *its left multiplication group. Then the system* $\langle E, \overset{(P)}{\cdot}, 1\rangle$ *is an* A_l*-loop if and only if* $\alpha h_{(x)}\alpha^{-1} = h_{(\alpha(x))}$ *for every* $\alpha \in LI(\langle E, \overset{(P)}{\cdot}, 1\rangle)$.

Proof. Since the system $\langle E, \overset{(T)}{\cdot}, 1\rangle$ is an A_l-loop and $H = LI(\langle E, \overset{(T)}{\cdot}, 1\rangle)$, every element $h \in H$ is an automorphism of $\langle E, \overset{(T)}{\cdot}, 1\rangle$. So in this case every weak derivation of the A_l-loop $\langle E, \overset{(T)}{\cdot}, 1\rangle$ is an automorphic weak derivation. In view of the condition 2) of Lemma 8 we obtain the necessity. □

Corollary 5. *Let the conditions of Corollary* 4 *hold. If for every* $h \in H$ *we have* $hh_{(x)}h^{-1} = h_{(h(x))}$*, then the system* $\langle E, \overset{(P)}{\cdot}, 1\rangle$ *is an* A_l*-loop.*

Proof. It is a consequence of our Corollary 4, because $LI(\langle E, \overset{(P)}{\cdot}, 1\rangle) \subseteq H$ for every left transversal P in a group G to a subgroup H. □

Definition 4. A transversal T in the group G by its subgroup H is called a *gyrotransversal* if $T^{-1} = T$ and $hTh^{-1} \subseteq T$ for every $h \in H$.

Lemma 9. *Let the set* $T = \{t_x\}_{x\in E}$ *be a gyrotransversal in the group* G *by the subgroup* H*. Then* $ht_xh^{-1} = t_{\hat{h}(x)}$ *and* $t_x^{-1} = t_{x\backslash 1}$ *for every* $h \in H$*,* $x \in E$*.*

Proof. As the set $T = \{t_x\}_{x\in E}$ is a gyrotransversal in the group G by the subgroup H, so for every $h \in H$ we have $hTh^{-1} \subseteq T$, i.e. $ht_xh^{-1} = t_v$, where $v = \hat{t}_v(1) = \hat{h}\,\hat{t}_x\hat{h}^{-1}(1) = \hat{h}\,\hat{t}_x(1) = \hat{h}\,(x)$. Thus $ht_xh^{-1} = t_{\hat{h}(x)}$.

By the definition we have also $T^{-1} = T$, i.e. $t_x^{-1} = t_z$ for every $x \in T$, where $z = \hat{t}_z(1) = \hat{t}_x^{-1}(1) = x\backslash 1$. Thus $t_x^{-1} = t_{x\backslash 1}$. □

Lemma 10. *Let the conditions of Lemma* 8 *hold. If* T *is a gyrotransversal in the group* G *by a subgroup* H*, then the following conditions are equivalent:*

1. *the set* P *is a gyrotransversal in the group* G *by a subgroup* H*,*
2. $hh_{(x)}h^{-1} = h_{(\hat{h}(x))}$ *and* $h_{(x)}^{-1} = h_{(x\backslash 1)}$ *for every* $h \in H$*,* $x \in E$*.*

Proof. 1. $\Longrightarrow$ 2. Let the conditions of the present Lemma hold and let P be a gyrotransversal in the group G by its subgroup H. Then, by the definition, for $h \in H$ we have $hPh^{-1} \subseteq P$ and $P^{-1} = P$, which implies $ht_xh_{(x)}h^{-1} = t_uh_{(u)}$ for all $h \in H$, $x \in E$ and some $u \in E$. Thus

$$u = \hat{t}_u(1) = \hat{t}_u\hat{h}_{(u)}(1) = \hat{h}\,\hat{t}_x\hat{h}_{(x)}\hat{h}^{-1}(1) = \hat{h}\,\hat{t}_x(1) = \hat{h}\,(x).$$

Thus the previous equation can be rewritten in the form $ht_xh_{(x)}h^{-1} = t_{\hat{h}(x)}h_{(\hat{h}(x))}$ and, in the consequence, in the form

$$t^{-1}_{\hat{h}(x)}ht_xh^{-1} = h_{(\hat{h}(x))}hh^{-1}_{(x)}h^{-1}. \tag{12}$$

As the set T is a gyrotransversal in the group G by the subgroup H, so, by Lemma 9, $ht_xh^{-1} = t_{\hat{h}(x)}$ for $h \in H$ and $x \in E$. Hence (12) has the form $e = h_{(\hat{h}(x))}hh^{-1}_{(x)}h^{-1}$, i.e. $hh_{(x)}h^{-1} = h_{(\hat{h}(x))}$, which proves the first condition of 2.

To prove the second, observe that $P^{-1} = P$ implies $(t_xh_{(x)})^{-1} = t_wh_{(w)}$ for $x \in E$, where

$$w = \hat{t}_w(1) = \hat{t}_w\hat{h}_{(w)}(1) = (\hat{t}_x\hat{h}_{(x)})^{-1}(1) = \hat{h}^{-1}_{(x)}\hat{t}^{-1}_x(1) = \hat{h}^{-1}_{(x)}(x\backslash 1).$$

This means that the equation $(t_xh_{(x)})^{-1} = t_wh_{(w)}$ can be written in the form $h^{-1}_{(x)}t^{-1}_x = t_{\hat{h}^{-1}_{(x)}(x\backslash 1)}h_{(\hat{h}^{-1}_{(x)}(x\backslash 1))}$, i.e. in the form

$$h_{(\hat{h}^{-1}_{(x)}(x\backslash 1))} = t^{-1}_{\hat{h}^{-1}_{(x)}(x\backslash 1)}h^{-1}_{(x)}t^{-1}_x = (t^{-1}_{\hat{h}^{-1}_{(x)}(x\backslash 1)}h^{-1}_{(x)}t_{x\backslash 1}h_{(x)})h^{-1}_{(x)}t^{-1}_{x\backslash 1}t^{-1}_x.$$

This together with $ht_xh^{-1} = t_{\hat{h}(x)}$ gives $t^{-1}_{\hat{h}^{-1}_{(x)}(x\backslash 1)}h^{-1}_{(x)}t_{x\backslash 1}h_{(x)} = e$, i.e. $h_{(\hat{h}^{-1}_{(x)}(x\backslash 1))} = h^{-1}_{(x)}t^{-1}_{x\backslash 1}t^{-1}_x$, which by $hh_{(x)}h^{-1} = h_{(\hat{h}(x))}$ implies the equation $h_{(\hat{h}^{-1}_{(x)}(x\backslash 1))} = h^{-1}_{(x)}h_{(x\backslash 1)}h_{(x)}$.

So we can write the equation $h_{(\hat{h}^{-1}_{(x)}(x\backslash 1))} = h^{-1}_{(x)}t^{-1}_{x\backslash 1}t^{-1}_x$ in the form $h^{-1}_{(x)}h_{(x\backslash 1)}h_{(x)} = h^{-1}_{(x)}t^{-1}_{x\backslash 1}t^{-1}_x$, i.e. $t_xt_{x\backslash 1}h_{(x\backslash 1)}h_{(x)} = e$.

Since the set T is a gyrotransversal in the group G by a subgroup H, by Lemma 9, for every $x \in E$ we have $t^{-1}_x = t_{x\backslash 1}$, which together with $t_xt_{x\backslash 1}h_{(x\backslash 1)}h_{(x)} = e$ implies $h_{(x\backslash 1)}h_{(x)} = e$. Hence $h^{-1}_{(x)} = h_{(x\backslash 1)}$. This proves the second condition of 2.

2. $\Longrightarrow$ 1. Let the conditions of Lemma 8 hold. If $hh_{(x)}h^{-1} = h_{(\hat{h}(x))}$ and $h^{-1}_{(x)} = h_{(x\backslash 1)}$ for all $h \in H$, $x \in E$, then, by Lemma 9, for every $h \in H$ and $x \in E$ we have

$$hp_xh^{-1} = ht_xh_{(x)}h^{-1} = (ht_xh^{-1})(hh_{(x)}h^{-1}) = t_{\hat{h}(x)}h_{(\hat{h}(x))} = p_{\hat{h}(x)},$$

i.e. $hPh^{-1} \subseteq P$. Moreover, by Lemma 9, for every $x \in E$ we have also

$$\begin{aligned} p_x^{-1} &= (t_x h_{(x)})^{-1} = h_{(x)}^{-1} t_x^{-1} = h_{(x\backslash 1)} t_{x\backslash 1} = \\ &= h_{(x\backslash 1)} t_{x\backslash 1} h_{(x\backslash 1)}^{-1} h_{(x\backslash 1)} = t_{\hat{h}_{(x\backslash 1)}(x\backslash 1)} h_{(x\backslash 1)} = \\ &= t_{\hat{h}_{(x\backslash 1)}(x\backslash 1)} h_{(\hat{h}_{(x\backslash 1)}(x\backslash 1))} h^{-1}_{(\hat{h}_{(x\backslash 1)}(x\backslash 1))} h_{(x\backslash 1)} = \\ &= t_{\hat{h}_{(x\backslash 1)}(x\backslash 1)} h_{(\hat{h}_{(x\backslash 1)}(x\backslash 1))} h_{(x\backslash 1)} h^{-1}_{(x\backslash 1)} h^{-1}_{(x\backslash 1)} h_{(x\backslash 1)} = \\ &= t_{\hat{h}_{(x\backslash 1)}(x\backslash 1)} h_{(\hat{h}_{(x\backslash 1)}(x\backslash 1))} = p_{\hat{h}_{(x\backslash 1)}(x\backslash 1)} \in P. \end{aligned}$$

This together with $hPh^{-1} \subseteq P$ proves that set P is a gyrotransversal in the group G by a subgroup H. □

5. Examples

Using propositions that are proved in the previous section, we will demonstrate some methods of construction of A_l-loops by the help of groups and A_l-loops.

Lemma 11. *If K is a group, $Inn\,(K)$ the group of its inner automorphisms, $M = K \times Inn\,(K)$ the semidirect product of K and $Inn\,(K)$, then the set $D = \{(x, \alpha_{a_x}) |\, x \in K\}$, where $a_x \in K$ are the indexes depended on $x \in K$, is a left transversal in the group M by a subgroup $H = Inn\,(K)$. Moreover, if $\alpha_u(a_x) = a_{\alpha_u(x)}$ for every $x, u \in K/Z(K)$, where $Z(K)$ is the center of K, then the system $\langle K, \overset{(D)}{\cdot}, 1\rangle$ is an A_l-loop.*

Proof. Let the conditions of the Lemma hold. Because $Inn\,(K) \subseteq Aut\,K$, then for the group transversal $K_0 = \{(x, id) |\, x \in K\}$ in the group M by subgroup $H = Inn\,(K)$ any weak derivation of group $\langle K, \overset{(K)}{\cdot}, 1\rangle$ is a weak automorphic derivation. According to Corollary 5, the system $\langle K, \overset{(D)}{\cdot}, 1\rangle$ is an A_l-loop if $\alpha_u \alpha_{a_x} \alpha_u^{-1} = \alpha_{a_{\alpha_u(x)}}$ for $x \in K$, $\alpha_u \in Inn\,(K)$.

This shows that for every $x, u, y \in K$ holds $\alpha_u \alpha_{a_x} \alpha_u^{-1}(y) = \alpha_{a_{\alpha_u(x)}}(y)$. Therefore

$$\begin{aligned} (ua_x u^{-1}) y (u a_x^{-1} u^{-1}) &= \alpha_u(a_x u^{-1} y u a_x^{-1}) = \alpha_u \alpha_{a_x}(u^{-1} y u) = \\ &= \alpha_u \alpha_{a_x} \alpha_u^{-1}(y) = \alpha_{a_{\alpha_u(x)}}(y) = a_{\alpha_u(x)} y\, a^{-1}_{\alpha_u(x)}\,. \end{aligned}$$

Hence $\alpha_u(a_x) y (\alpha_u(a_x))^{-1} = a_{\alpha_u(x)} y\, a^{-1}_{\alpha_u(x)}$ and $\alpha_{\alpha_u(a_x)} = \alpha_{a_{\alpha_u(x)}}$.

So, if $\alpha_u(a_x) = a_{\alpha_u(x)}$ holds, the system $\langle K, \overset{(D)}{\cdot}, 1\rangle$ is an A_l-loop. □

Corollary 6. *Let the conditions of Lemma* 11 *hold. If* $\alpha_u(a_x) = a_{\alpha_u(x)}$, *then* $a_x \in C_K(x)$ *for every* $x \in K$, *where* $C_K(x)$ *is the centralizer of* x *in the group* K.

Proof. It follows from $\alpha_u(a_x) = a_{\alpha_u(x)}$ for $u = x$. □

Remark 4. In the case $a_x = x$ we obtain so-called *diagonal transversal* $D = \{(x, \alpha_x) | \, x \in K\}$ investigated in [4].

Remark 5. For $a_x = x^m$, where $m \in Z - \{0, 1\}$ is fixed, we obtain the *generalized diagonal transversals* $D = \{(x, \alpha_x^m) | \, x \in K\}$ described in [12].

Now we will demonstrate the method of constructing of A_l-loops by the help of A_l-loop L with nontrivial right nucleus $N_r(L)$.

Lemma 12. *Let the conditions of Lemma* 8 *hold. If* $\hat{h}_{(x)} = l^{(T)}_{x, x \overset{(T)}{\cdot} c_0}$, *where* $c_0 \in N_r(\langle E, \overset{(T)}{\cdot}, 1\rangle)$, $c_0 \neq 1$, *then the system* $\langle E, \overset{(P)}{\cdot}, 1\rangle$ *is an* A_l*-loop.*

Proof. Since the system $\langle E, \overset{(T)}{\cdot}, 1\rangle$ is an A_l-loop, then

$$\hat{h}_{(x)} = l^{(T)}_{x, x \overset{(T)}{\cdot} c_0} \in LI(\langle E, \overset{(T)}{\cdot}, 1\rangle) \subseteq Aut(\langle E, \overset{(T)}{\cdot}, 1\rangle), \tag{13}$$

i.e. the weak derivation of A_l-loop $\langle E, \overset{(T)}{\cdot}, 1\rangle$ onto left loop $\langle E, \overset{(P)}{\cdot}, 1\rangle$ is a weak automorphic derivation. Moreover, in view of 1), Lemma 8 and (13) we obtain $l^{(P)}_{a,b} = \hat{h}^{-1}_{(a \overset{(P)}{\cdot} b)} l^{(T)}_{a, \hat{h}_{(a)}(b)} \hat{h}_{(a)} \hat{h}_{(b)} \in LI(\langle E, \overset{(T)}{\cdot}, 1\rangle)$ for every $a, b \in E$, i.e.

$$LI(\langle E, \overset{(P)}{\cdot}, 1\rangle) \subseteq LI(\langle E, \overset{(T)}{\cdot}, 1\rangle). \tag{14}$$

But $c_0 \in N_r(\langle E, \overset{(T)}{\cdot}, 1\rangle)$. Then $l^{(T)}_{a,b}(c_0) = c_0$ for every $a, b \in E$. So $\alpha(c_0) = c_0$ for every $\alpha \in LI(\langle E, \overset{(T)}{\cdot}, 1\rangle)$. Then, by (7) and (14), for every $x \in E$ and $\alpha \in LI(\langle E, \overset{(P)}{\cdot}, 1\rangle)$ we obtain:

$$\begin{aligned} \alpha \hat{h}_{(x)} \alpha^{-1} &= \alpha l^{(T)}_{x, x \overset{(T)}{\cdot} c_0} \alpha^{-1} = \alpha \hat{t}^{-1}_{x \overset{(T)}{\cdot} (x \overset{(T)}{\cdot} c_0)} \hat{t}_x \hat{t}_{x \overset{(T)}{\cdot} c_0} \alpha^{-1} = \\ &= (\alpha \hat{t}^{-1}_{x \overset{(T)}{\cdot} (x \overset{(T)}{\cdot} c_0)} \alpha^{-1})(\alpha \hat{t}_x \alpha^{-1})(\alpha \hat{t}_{x \overset{(T)}{\cdot} c_0} \alpha^{-1}) = \\ &= \hat{t}^{-1}_{\alpha(x \overset{(T)}{\cdot} (x \overset{(T)}{\cdot} c_0))} \hat{t}_{\alpha(x)} \hat{t}_{\alpha(x \overset{(T)}{\cdot} c_0)} = \hat{t}^{-1}_{\alpha(x) \overset{(T)}{\cdot} \alpha(x \overset{(T)}{\cdot} c_0)} \hat{t}_{\alpha(x)} \hat{t}_{\alpha(x \overset{(T)}{\cdot} c_0)} = \\ &= l^{(T)}_{\alpha(x), \alpha(x \overset{(T)}{\cdot} c_0)} = l^{(T)}_{\alpha(x), \alpha(x) \overset{(T)}{\cdot} \alpha(c_0)} = l^{(T)}_{\alpha(x), \alpha(x) \overset{(T)}{\cdot} c_0} = \hat{h}_{(\alpha(x))} \, . \end{aligned}$$

This, by 2) of Lemma 8, proves that $\langle E, \overset{(P)}{\cdot}, 1\rangle$ is an A_l-loop. □

Remark 6. The set $\{h_{(x)}\}_{x\in E}$ may be chosen in the another way. For example $\hat{h}_{(x)} = l^{(T)}_{x,c_0 \overset{(T)}{\cdot} x}$, or, in the general case $\hat{h}_{(x)} = l^{(T)}_{R_1(x,c_1),R_2(x,c_2)}$, where R_1, R_2 are terms of two variables on E. If $c_1, c_2 \in N_r(\langle E, \overset{(T)}{\cdot}, 1\rangle)$, then the system $\langle E, \overset{(P)}{\cdot}, 1\rangle$ is an A_l-loop.

References

[1] **R. Baer**: *Nets and groups.* 1. Trans. Amer. Math. Soc. **46** (1939), 110 – 141.

[2] **V.D. Belousov**: *Foundations of quasigroup and loop theory*, (Russian), Moscow, "Nauka", 1967.

[3] **V.D. Belousov**: *Crossed isotopies of quasigroups*, (Russian), Mat. Issled. **113** (1990), 14 – 20.

[4] **T. Foguel and A.A. Ungar**: *Gyrogroups and the decomposition of groups into twisted subgroups and subgroups*, Pasific J. Math. **197** (2001), 1 – 11.

[5] **M.I. Kargapolov and Yu.I. Merzlyakov**: *Foundations of group theory*, (Russian), Moscow, "Nauka", 1982.

[6] **H. Karzel**: *Unendliche Dicksonsche Fastkörper*, Arch. Math. (Basel), **16** (1965), 247 – 256.

[7] **H. Kiechle**: *Relatives of K-loops: Theory and examples*, Comm. Math. Univ. Carolinae **41** (2000), 301 – 324.

[8] **A. Kreuzer and H. Wefelscheid**: *On K-loops of finite order*, Resultate Math. **25** (1994), 79 – 102.

[9] **E.A. Kuznetsov**: *Transversals in groups.* 1. *Elementary properties*, Quasigroups Related Systems **1** (1994), 22 – 42.

[10] **E.A. Kuznetsov**: *Transversals in groups.* 2. *Loop transversals in a group by the same subgroup*, Quasigroups Related Systems **6** (1999), 1 – 12.

[11] **E.A. Kuznetsov**: *Transversals in groups.* 3. *Semidirect product of a transversal operation and subgroup*, Quasigroups Related Systems **8** (2001), 37 – 44.

[12] **E.A. Kuznetsov**: *Gyrogroups and left gyrogroups as a special transversals in groups*, (to appear).

[13] **L.V. Sabinin and O.I. Mikheev**: *Quasigroups and differential geometry*, Chapter XII in the book "*Quasigroups and loops: Theory and Applications*", Berlin, Helderman-Verlag, 1990, 357 – 430.

[14] **J. Timm**: *Zur Konstruktion von Fastringen I*, Abh. Math. Sem. Univ. Hamburg **35** (1970), 57 – 74.

[15] **H. Wähling**: *Theorie der Fastkörper*, Thales Verlag, Essen, 1987.

Institute of Mathematics
and Computer Science
Academy of Sciences
str. Academiei 5
MD-2028 Chisinau
Moldova

Received May 30, 2001
Revised March 20, 2002