Quasigroups and Related Systems **8** (2001), 73 – 85

Algebras of vector-valued functions

Valentin S. Trokhimenko

Abstract

Superpositions (compositions) of multiplace functions have various applications in the modern mathematics, especially in the algebraic theory of automata [1], [3], [4]. It is known that any automaton with n entrances and m exits can be defined by some functions of the form $f : A^n \to A^m$, which are called multiplace vector-valued functions. There are two types of compositions of such functions: serial $\circ$ and parallel $\star$ which were considered by B. Schweizer and A. Sklar in [5], [6], [7]. In this paper we find the abstract characterization of algebras of the form $(\Phi, \circ, \star, \Delta, F)$, where Φ is the set of multiplace vector-valued functions stable for compositions $\circ, \star$ and containing two functions $\Delta(x) = x$, $F(x, y) = y$. We also describe the case when Φ contains all vector-valued functions defined on a fixed set A. Automorphisms of such algebra are described too.

1. Introduction

Any mapping $f : A^n \to A^m$, where $n, m \in \mathbb{N}$ are fixed and A is a non-empty set, is called a *multiplace vector-valued function* (or simply *vector-function*) of *degree* n and *rank* m (cf. [5]). The degree and the rank of the multiplace vector-valued function f is denoted by αf and βf, respectively. $\gamma f = \alpha f - \beta f$ is called the *index* of f. The set of all multiplace vector-valued functions of degree n and rank m defined on a fixed set A is denoted by $\mathcal{T}(A^n, A^m)$.

According to [5], [6] and [7], on the set $\mathcal{T}(A) = \bigcup_{n,m \in \mathbb{N}} \mathcal{T}(A^n, A^m)$ we consider two binary operations: the serial composition $\circ$ and the parallel composition $\star$, which are defined in the following way:

2000 Mathematics Subject Classification: 20N15, 08N05
Keywords: vector valued function

Definition 1. The *serial composition* $f \circ g$ of vector-functions $f, g \in \mathcal{T}(A)$ is defined by

$$(f \circ g)(a_1, \ldots, a_d) = f(b_1, \ldots, b_{\alpha f}) b_{\alpha f+1} \ldots b_{d-\gamma g}\,, \tag{1}$$

where $a_1, \ldots, a_d \in A$, $d = \max\{\alpha f + \gamma g, \alpha g\}$, $b_1, \ldots, b_{d-\gamma g} \in A$ and $b_1 \ldots b_{d-\gamma g} = g(a_1, \ldots, a_{\alpha g}) a_{\alpha g+1} \ldots a_d$.

Definition 2. The *parallel composition* of vector-functions $f, g \in \mathcal{T}(A)$ is a vector-function $f \star g$ defined by

$$(f \star g)(a_1, \ldots, a_d) = f(a_1, \ldots, a_{\alpha f}) g(a_1, \ldots, a_{\alpha g})\,, \tag{2}$$

where $a_1, \ldots, a_d \in A$ and $d = \max\{\alpha f, \alpha g\}$.

It is easy to see that these operations are associative. Moreover, in the case $\alpha f = \beta g$, serial composition reduces to ordinary composition of functions.

Let I_i^n, where $n \in \mathbb{N}$, $1 \leqslant i \leqslant n$, be an *n-place i-th projection* of A, i.e. $I_i^n(a_1, \ldots, a_n) = a_i$ for all $a_1, \ldots, a_n \in A$. Obviously $\alpha I_i^n = n$, $\beta I_i^n = 1$ for all $1 \leqslant i \leqslant n \in \mathbb{N}$. Putting $\Delta(x) = I_1^1(x) = x$ and $F(x, y) = I_2^2(x, y) = y$, we can verify that

$$I_i^n = (F \circ (F \star \Delta))^{n-i} \circ F^{i-1}$$

for any $n \in \mathbb{N}$, $1 \leqslant i \leqslant n$ and $f \in \mathcal{T}(A)$, where $f^0 = \Delta$ and $f^{n+1} = f \circ f^n$.

If the subset Φ of $\mathcal{T}(A)$ contains Δ, F and is closed under operations $\circ, \star$, then a system $(\Phi, \circ, \star, \Delta, F)$ is called an *algebra of vector-functions*. In the case $\Phi = \mathcal{T}(A)$ we say that this algebra is *symmetrical*.

2. The main result

In this section we find an abstract characterization of algebras of vector valued-functions.

First we consider an algebra $(G, \circ, \star, e, f)$ of type $(2, 2, 0, 0)$ satisfying the following six axioms:

Axiom 1. *$(G, \circ)$ and $(G, \star)$ are semigroups and e is the unit of $(G, \circ)$.*

Let e_i^p denotes the expression $(f \circ (f \star e))^{p-i} \circ f^{i-1}$, where $p \in \mathbb{N}$, $1 \leqslant i \leqslant p$ and $(f \circ (f \star e))^0 = f^0 = e$.

Axiom 2. *For each* $g \in G$ *there exist* $m, n \in \mathbb{N}$ *such that*

$$g \circ (e^p \star \cdots \star e^p_p) = g\,, \quad (e^q_1 \star \cdots \star e^q_q) \circ g = g$$

for all $p \leqslant n$, $q \leqslant m$, $p, q \in \mathbb{N}$ *and*

$$g \circ (e^p \star \cdots \star e^p_p) \neq g\,, \quad (e^q_1 \star \cdots \star e^q_q) \circ g \neq g$$

for any $p > n$, $q > m$.

The numbers n and m are called *degree* and *rank* of g and are denoted by αg, βg, respectively.

Axiom 3. *For any* $g_1, g_2 \in G$ *the following conditions*

(a) $\alpha e = \beta e = \beta f = 1$, $\alpha f = 2$,

(b) $\alpha(g_1 \star g_2) = \max\{\alpha g_1, \alpha g_2\}$, $\quad \beta(g_1 \star g_2) = \beta g_1 + \beta g_2$,

(c) $\alpha(g_1 \circ g_2) = \max\{\alpha g_1 + \gamma g_2, \alpha g_2\}$, $\quad \beta(g_1 \circ g_2) = \max\{\beta g_1, \beta g_2 - \gamma g_1\}$,

where $\gamma g = \alpha g - \beta g$, *hold.*

Axiom 4. $f \circ (g_1 \star g_2) = g_2$ *for all* $g_1, g_2 \in G$ *such that* $\alpha g_1 = \alpha g_2$ *and* $\beta g_1 = \beta g_2 = 1$.

Axiom 5. *For all* $g_1, g_2, g_3 \in G$

(a) $g_1 \circ (g_2 \star g_3) = (g_1 \circ g_2) \star g_3$, *if* $\alpha g_1 \leqslant \beta g_2$,

(b) $(g_1 \star g_2) \circ g_3 = (g_1 \circ g_3) \star (g_2 \circ g_3)$, *if* $\beta g_3 \leqslant \min\{\alpha g_1, \alpha g_2\}$.

Axiom 6. *For all* $g_1, g_2, g_3, g_4 \in G$

(a) $(g_1 \star g_2) \circ (g_3 \star g_4) = (g_1 \circ g_3) \star (g_2 \circ (g_3 \star g_4))$, *if* $\alpha g_1 < \alpha g_2$, $\alpha g_1 = \beta g_3$, $\alpha g_2 = \beta(g_3 \star g_4)$,

(b) $(g_1 \star g_2) \circ (g_3 \star g_4) = (g_1 \circ (g_3 \star g_4)) \star (g_2 \circ g_3)$, *if* $\alpha g_1 > \alpha g_2$, $\alpha g_2 = \beta g_3$, $\alpha g_1 = \beta(g_3 \star g_4)$.

Now we can prove some auxiliary results on the algebra $(G, \circ, \star, e, f)$.

Proposition 1. *For all* $g_1, g_2 \in G$ *we have*

(a) $\gamma(g_1 \circ g_2) = \gamma g_1 + \gamma g_2$,

(b) $\gamma(g_1 \star g_2) = \gamma g_1 + \gamma g_2 - \min\{\alpha g_1, \alpha g_2\}$.

Proof. By simple application of the above Axiom 3(c). □

Proposition 2. *For each $n \in \mathbb{N}$ and all $1 \leqslant i \leqslant n$ the equations $\alpha e_i^n = n$, $\beta e_i^n = 1$ are true.*

Proof. Indeed, let g be an element of G such that $\beta g = 1$. Then, by Axiom 3(c), we obtain $\alpha g^n = n\alpha g - n + 1$ and $\beta g^n = 1$. Further

$$\alpha e_i^n = \alpha((f \circ (f \star e))^{n-i} \circ f^{i-1}) = \max\{\alpha((f \circ (f \star e))^{n-i}) + \gamma f^{i-1},\, \alpha f^{i-1}\}.$$

But $\alpha(f \circ (f \star e)) = 2$ and $\beta(f \circ (f \star e)) = 1$ by our Axiom 3. Thus $\alpha((f \circ (f \star e))^{n-i}) = n - i + 1$, $\beta((f \circ (f \star e))^{n-i}) = 1$, $\alpha f^{i-1} = i$, $\beta f^{i-1} = 1$. Hence $\alpha e_i^n = \max\{n, i\} = n$.

Similarly we can prove $\beta e_i^n = 1$. □

Proposition 2 implies that the equation

$$e_i^n \circ (e_1^n \star \cdots \star e_n^n) = e_i^n \tag{3}$$

is satisfied for all $n \in \mathbb{N}$ and $1 \leqslant i \leqslant n$.

Proposition 3. *For all $g_1, \ldots, g_n \in G$ such that $\alpha g_1 = \cdots = \alpha g_n$ and $\beta g_1 = \cdots = \beta g_n = 1$, the equation*

$$e_i^n \circ (g_1 \star \cdots \star g_n) = g_i \tag{4}$$

is satisfied for all $n \in \mathbb{N}$ and $1 \leqslant i \leqslant n$.

Proof. First let $n = 2$. If $i = 2$, then, according to Axiom 4, we have

$$e_2^2 \circ (g_1 \star g_2) = f \circ (g_1 \star g_2) = g_2\,.$$

If $i = 1$, then $e_1^2 \circ (g_1 \star g_2) = f \circ (f \star e) \circ (g_1 \star g_2)$. Hence by Axioms 6(b) and 4 we obtain

$$e_1^2 \circ (g_1 \star g_2) = f \circ \Big(\big(f \circ (g_1 \star g_2)\big) \star (e \circ g_1)\Big) = f \circ (g_2 \star g_1) = g_1.$$

Now let $n > 2$, $1 \leqslant i \leqslant n$. Then

$$\begin{aligned} e_i^n \circ (g_1 \star \cdots \star g_n) &= (e_1^2)^{n-i} \circ f^{i-1} \circ (g_1 \star \cdots \star g_n) \\ &= (e_1^2)^{n-i} \circ f^{i-2} \circ \Big(\big(f \circ (g_1 \star g_2)\big) \star g_3 \star \cdots \star g_n\Big) \\ &= (e_1^2)^{n-i} \circ f^{i-2} \circ (g_2 \star \cdots \star g_n). \end{aligned}$$

Repeating this procedure we obtain

$$\begin{aligned} e_i^n \circ (g_1 \star \cdots \star g_n) &= (e_1^2)^{n-i} \circ (g_i \star \cdots \star g_n) \\ &= (e_1^2)^{n-i-1} \circ \Big(\Big(e_1^2 \circ (g_i \star g_{i+1})\Big) \star g_{i+2} \star \cdots \star g_n\Big) \\ &= (e_1^2)^{n-i-1} \circ (g_i \star g_{i+2} \star \cdots \star g_n) = \cdots \\ &= e_1^2 \circ (g_i \star g_n) = g_i. \end{aligned}$$

This completes the proof. □

Proposition 4. *If* $x_1, \ldots, x_k \in G$ *are such that*

$$n = \beta x_1 + \cdots + \beta x_k \quad \text{and} \quad m = \max\{\alpha x_1, \ldots, \alpha x_k\},$$

then

$$e_i^n \circ (x_1 \star \cdots \star x_k) = e_s^{\beta x_p} \circ x_p \circ (e_1^m \star \cdots \star e_{\alpha x_p}^m) \tag{5}$$

for all $1 \leqslant i \leqslant n$, *where* $\sum_{j=1}^{p-1} \beta x_j < i \leqslant \sum_{j=1}^{p} \beta x_j$ *and* $s = i - \sum_{j=1}^{p-1} \beta x_j$.

Proof. Let $n_i = \beta x_i$ for all $x_i \in G$, $i = 1, \ldots, k$. By Axiom 3(b) we have $\alpha(x_1 \star \cdots \star x_k) = \max\{\alpha x_1, \ldots, \alpha x_k\} = m$. Applying Axiom 2 we obtain

$$e_i^n \circ (x_1 \star \cdots \star x_k) =$$
$$e_i^n \circ \Big(\Big((e_1^{n_1} \star \cdots \star e_{n_1}^{n_1}) \circ x_1\Big) \star \cdots \star \Big((e_1^{n_k} \star \cdots \star e_{n_k}^{n_k}) \circ x_k\Big)\Big) \circ (e_1^n \star \cdots \star e_m^m).$$

Further, by Axiom 5(b)

$$\begin{aligned} e_i^n \circ (x_1 \star \cdots \star x_k) = e_i^n \circ \Big((e_1^{n_1} \circ x_1) \star \cdots \star (e_{n_1}^{n_1} \circ x_1) \star \cdots \star \\ \star (e_1^{n_k} \circ x_k) \star \cdots \star (e_{n_k}^{n_k} \circ x_k)\Big) \circ (e_1^m \star \cdots \star e_m^m). \end{aligned}$$

This, together with Axiom 6 and Proposition 3, implies

$$\begin{aligned} e_i^n \circ (x_1 \star \cdots \star x_k) = e_i^n \circ \Big(\Big(e_1^{n_1} \circ x_1 \circ (e_1^m \star \cdots \star e_{\alpha x_1}^m)\Big) \star \cdots \star \\ \star \Big(e_{n_k}^{n_k} \circ x_1 \circ (e_1^m \star \cdots \star e_{\alpha x_k}^m)\Big)\Big) = e_s^{\beta x_p} \circ x_p \circ (e_1^m \star \cdots \star e_{\alpha x_p}^m), \end{aligned}$$

which completes the proof. □

Theorem 1. *An algebra* $(G, \circ, \star, e, f)$ *of type* $(2,2,0,0)$ *is isomorphic to some algebra of vector-functions if and only if it satisfies Axioms* $1-6$.

Proof. The necessity of Theorem is evident. We prove the sufficiency. For this let $(G, \circ, \star, e, f)$ be an algebra satisfying Axioms $1-6$ and let G_n be the set of all elements $g \in G$ such that $\alpha g = n$ and $\beta g = 1$. It is clear that $G_n \neq \emptyset$ for every $n \in \mathbb{N}$, because $e_i^n \in G_n$ for all $1 \leqslant i \leqslant n$. Note that $G_n \cap G_m = \emptyset$ for $n \neq m$. Let $\overline{G} = \bigtimes_{n\in\mathbb{N}} G_n$ be the Cartesian power of the family sets $(G_n)_{n\in\mathbb{N}}$.

For each $g \in G$ we define the vector-function $P_g : \overline{G}^{\,n} \to \overline{G}^{\,m}$, where $n = \alpha g$, $m = \beta g$, putting $P_g(\bar{x}_1, \ldots, \bar{x}_n) = \bar{y}_1 \ldots \bar{y}_m$ if and only if

$$\bar{y}_i(k) = e_i^m \circ g \circ \Big(\bar{x}_1(k) \star \cdots \star \bar{x}_n(k)\Big) \tag{6}$$

for every $1 \leqslant i \leqslant m$ and $k = 1, 2, \ldots$

We prove that the mapping $P : g \mapsto P_g$ is an isomorphism between algebras $(G, \circ, \star, e, f)$ and $(\Phi, \circ, \star, \Delta, F)$, where $\Phi = \{P_g \,|\, g \in G\}$.

First observe that $P_e = \Delta$ and $P_f = F$. Indeed, if $P_e(\bar{x}) = \bar{y}$ for some $\bar{x}, \bar{y} \in \overline{G}$, then $\bar{y}(k) = e_1^1 \circ e \circ \bar{x}(k) = \bar{x}(k)$ for all $k = 1, 2, \ldots$, because $e_1^1 = e$ is the unit of $(G, \circ)$. Thus $\bar{y}(k) = \bar{x}(k)$, $k = 1, 2, \ldots$ So, $P_e(\bar{x}) = \bar{x}$. Hence $P_e = \Delta$. Analogously, from Axiom 4, we deduce $P_f = F$.

Now prove that $P(g_1 \circ g_2) = P(g_1) \circ P(g_2)$ for all $g_1, g_2 \in G$, i.e.

$$P_{g_1 \circ g_2} = P_{g_1} \circ P_{g_2}\,. \tag{7}$$

Let $n_i = \alpha g_i$, $m_i = \beta g_i$, $i = 1, 2$, $n = \max\{n_1 + \gamma g_2, n_2\}$ and $m = \max\{m_1, m_2 - \gamma g_1\}$. By Axiom 3(c) $n = \alpha(g_1 \circ g_2)$, $m = \beta(g_1 \circ g_2)$. Thus the degree and the rank of the function $P_{g_1 \circ g_2}$ are equal n and m, respectively. Let

$$\bar{y}_1 \ldots \bar{y}_m = P_{g_1 \circ g_2}(\bar{x}_1, \ldots, \bar{x}_n)$$

for some $\bar{x}_1, \ldots, \bar{x}_n, \bar{y}_1, \ldots, \bar{y}_m \in \overline{G}$. If $n_1 > m_2$ then $m = m_1$. Therefore, by (6), we have

$$\bar{y}_i(k) = e_i^m \circ g_1 \circ g_2 \circ \Big(\bar{x}_1(k) \star \cdots \star \bar{x}_n(k)\Big)$$

for all $1 \leqslant i \leqslant m$ and $k = 1, 2, \ldots$ Since the equation

$$n_2 = \beta\Big(\bar{x}_1(k) \star \cdots \star \bar{x}_{n_2}(k)\Big)$$

is true, Axiom 5(a) gives

$$\bar{y}_i(k) = e_i^{m_1} \circ g_1 \circ \bigg(\Big(g_2 \circ \Big(\bar{x}_1(k) \star \cdots \star \bar{x}_{n_2}(k)\Big)\Big) \star \bar{x}_{n_2+1}(k) \star \cdots \star \bar{x}_n(k)\bigg).$$

Applying to this equation Axioms 2 and 6, we obtain

$$\begin{aligned}\bar{y}_i(k) &= e_i^{m_1} \circ g_1 \circ \Big(\Big(\Big(e_1^{m_2} \star \cdots \star e_{m_2}^{m_2}\Big) \circ g_2 \circ \Big(\bar{x}_1(k) \star \cdots \star \bar{x}_{n_2}(k)\Big)\Big) \star \\ &\qquad \star\ \bar{x}_{n_2+1}(k) \star \cdots \star \bar{x}_n(k)\Big) \\ &= e_i^{m_1} \circ g_1 \circ \Big(\Big(e_1^{m_2} \circ g_2 \circ \Big(\bar{x}_1(k) \star \cdots \star \bar{x}_{n_2}(k)\Big)\Big) \star \cdots \star \\ &\qquad \star \Big(e_{m_2}^{m_2} \circ g_2 \circ \Big(\bar{x}_1(k) \star \cdots \star \bar{x}_{n_2}(k)\Big)\Big) \star \bar{x}_{n_2+1}(k) \star \cdots \star \bar{x}_n(k)\Big).\end{aligned}$$

Let $\bar{z}_1 \dots \bar{z}_{m_2} = P_{g_2}(\bar{x}_1, \dots, \bar{x}_{n_2})$, i.e.

$$\bar{z}_i(k) = e_i^{m_2} \circ g_2 \circ (\bar{x}_1(k) \star \cdots \star \bar{x}_{n_2}(k))$$

for all $1 \leqslant i \leqslant m_2$ and $k = 1, 2, \dots$ Then

$$\bar{y}_i(k) = e_i^{m_1} \circ g_1 \circ \Big(\bar{z}_1(k) \star \cdots \star \bar{z}_{m_2}(k) \star \bar{x}_{n_2+1}(k) \star \cdots \star \bar{x}_n(k)\Big)$$

for all $1 \leqslant i \leqslant m_1$ and $k = 1, 2, \dots$ Thus

$$\bar{y}_1 \dots \bar{y}_{m_1} = P_{g_1}(\bar{z}_1, \dots, \bar{z}_{m_2}, \bar{x}_{n_2+1}, \dots, \bar{x}_n).$$

Therefore

$$\bar{y}_1 \dots \bar{y}_{m_1} = P_{g_1}(P_{g_2}(\bar{x}_1, \dots, \bar{x}_{n_2}), \bar{x}_{n_2+1}, \dots, \bar{x}_n),$$

i.e. $\bar{y}_1 \dots \bar{y}_m = (P_{g_1} \circ P_{g_2})(\bar{x}_1, \dots, \bar{x}_n)$, which proves (7) for $n_1 > m_2$, $m = m_1$.

Now let $n_1 \leqslant m_2$. Then $n = n_2$ and $m = m_2 - \gamma g_1$. Hence, for all $1 \leqslant i \leqslant m$, $k = 1, 2, \dots$ we have

$$\begin{aligned}\bar{y}_i(k) &= e_i^m \circ g_1 \circ g_2 \circ \Big(\bar{x}_1(k) \star \cdots \star \bar{x}_n(k)\Big) \\ &= e_i^m \circ g_1 \circ \Big(e_1^{m_2} \star \cdots \star e_{m_2}^{m_2}\Big) \circ g_2 \circ \Big(\bar{x}_1(k) \star \cdots \star \bar{x}_{n_2}(k)\Big) \\ &= e_i^m \circ g_1 \circ \Big(\Big(e_1^{m_2} \circ g_2 \circ \Big(\bar{x}_1(k) \star \cdots \star \bar{x}_{n_2}(k)\Big)\Big) \star \cdots \star \\ &\qquad \star\ \Big(e_{m_2}^{m_2} \circ g_2 \circ \Big(\bar{x}_1(k) \star \cdots \star \bar{x}_{n_2}(k)\Big)\Big)\Big) \\ &= e_i^m \circ g_1 \circ \Big(\bar{z}_1(k) \star \cdots \star \bar{z}_{m_2}(k)\Big).\end{aligned}$$

Now applying Axiom 5(a) we obtain

$$\bar{y}_i(k) = e_i^m \circ \Big(\Big(g_1 \circ \Big(\bar{z}_1(k) \star \cdots \star \bar{z}_{n_1}(k)\Big)\Big) \star \bar{z}_{n_1+1}(k) \star \cdots \star \bar{z}_{m_2}(k)\Big). \quad (8)$$

If $1 \leqslant i \leqslant m_1$, then applying Proposition 4 to (8) we get

$$\bar{y}_i(k) = e_i^{m_1} \circ g_1 \circ \Big(\bar{z}_1(k) \star \cdots \star \bar{z}_{n_1}(k)\Big)$$

for $k = 1, 2, \ldots$ Therefore

$$\bar{y}_1 \ldots \bar{y}_{m_1} = P_{g_1}(\bar{z}_1, \ldots, \bar{z}_{n_1}).$$

For $m_1 < i \leqslant m$ we have $\bar{y}_i(k) = \bar{z}_{i+\gamma g_1}(k)$, where $k = 1, 2, \ldots$ Whence $\bar{y}_{m_1+1} \ldots \bar{y}_m = \bar{z}_{n_1+1} \ldots \bar{z}_{m_2}$. So,

$$\bar{y}_1 \ldots \bar{y}_m = P_{g_1}(\bar{z}_1, \ldots, \bar{z}_{n_1})\bar{z}_{n_1+1} \ldots \bar{z}_{m_2}\,,$$

which, by Definition 1, gives $\bar{y}_1 \ldots \bar{y}_m = (P_{g_1} \circ P_{g_2})(\bar{x}_1, \ldots, \bar{x}_n)$. Thus

$$P_{g_1 \circ g_2}(\bar{x}_1, \ldots, \bar{x}_n) = (P_{g_1} \circ P_{g_2})(\bar{x}_1, \ldots, \bar{x}_n)$$

for all $\bar{x}_1, \ldots, \bar{x}_n \in \overline{G}$. This proves (7).

To verify $P(g_1 \star g_2) = P(g_1) \star P(g_2)$, i.e.

$$P_{g_1 \star g_2} = P_{g_1} \star P_{g_2} \tag{9}$$

for all $g_1, g_2 \in G$, assume that $n_i = \alpha g_i$, $m_i = \beta g_i$ for $i = 1, 2$, and $n = \max\{n_1, n_2\}$, $m = m_1 + m_2$. By Axiom 3(b), $n = \alpha(g_1 \star g_2) = \alpha(P_{g_1 \star g_2})$, $m = \beta(g_1 \star g_2) = \beta(P_{g_1 \star g_2})$. Let

$$\bar{y}_1 \ldots \bar{y}_m = P_{g_1 \star g_2}(\bar{x}_1, \ldots, \bar{x}_n)$$

for some $\bar{x}_1, \ldots, \bar{x}_n, \bar{y}_1, \ldots, \bar{y}_m \in \overline{G}$. Then, according to (6),

$$\bar{y}_i(k) = e_i^m \circ (g_1 \star g_2) \circ (\bar{x}_1(k) \star \cdots \star \bar{x}_n(k)) \tag{10}$$

for all $1 \leqslant i \leqslant m$ and $k = 1, 2, \ldots$

Assume that $n_1 \leqslant n_2$. Then $n = n_2$. Therefore, by Axiom 6, the equation (10) can be written in the form

$$\bar{y}_i(k) = e_i^m \circ \bigg(\Big(g_1 \circ \Big(\bar{x}_1(k) \star \cdots \star \bar{x}_{n_1}(k)\Big)\Big) \star \Big(g_2 \circ \Big(\bar{x}_1(k) \star \cdots \star \bar{x}_n(k)\Big)\Big)\bigg). \tag{11}$$

For $1 \leqslant i \leqslant m_1$ the above equation and Proposition 4 imply

$$\bar{y}_i(k) = e_i^{m_1} \circ g_1 \circ \Big(\bar{x}_1(k), \ldots, \bar{x}_{n_1}(k)\Big), \quad k = 1, 2, \ldots$$

Hence $\bar{y}_1 \dots \bar{y}_{m_1} = P_{g_1}(\bar{x}_1, \dots, \bar{x}_{n_1})$.

In the same manner, for $m_1 + 1 \leqslant i \leqslant m$, we obtain

$$\bar{y}_i(k) = e^{m_2}_{i-m_1} \circ g_2 \circ \Big(\bar{x}_1(k) \star \cdots \star \bar{x}_n(k)\Big), \quad k = 1, 2, \dots$$

and $\bar{y}_{m_1+1} \dots \bar{y}_m = P_{g_2}(\bar{x}_1, \dots, \bar{x}_n)$. Thus $\bar{y}_1 \dots \bar{y}_m = (P_{g_1} \star P_{g_2})(\bar{x}_1, \dots, \bar{x}_n)$. Hence

$$P_{g_1 \star g_2}(\bar{x}_1, \dots, \bar{x}_n) = (P_{g_1} \star P_{g_2})(\bar{x}_1, \dots, \bar{x}_n)$$

for all $\bar{x}_1, \dots, \bar{x}_n \in \overline{G}$. This proves (9) in the case $n_1 \leqslant n_2$.

The case $n_2 \leqslant n_1$ is analogous.

Now we prove that P is one-to-one. Let $P_{g_1} = P_{g_2}$ for some $g_1, g_2 \in G$. Then $\alpha g_1 = \alpha g_2$, $\beta g_1 = \beta g_2$. Therefore

$$e^m_i \circ g_1 \circ \Big(\bar{x}_1(k) \star \cdots \star \bar{x}_n(k)\Big) = e^m_i \circ g_2 \circ \Big(\bar{x}_1(k) \star \cdots \star \bar{x}_n(k)\Big) \qquad (12)$$

for all $1 \leqslant i \leqslant m = \beta g_1$, $\bar{x}_1, \dots, \bar{x}_n \in \overline{G}$, where $n = \alpha g_1$ and $k = 1, 2, \dots$

This for $\bar{x}_j = \bar{e}_j = (e^1_1, e^2_2, \dots, e^i_i, e^{i+1}_i, e^{i+2}_i, \dots) \in \overline{G}$, $j = 1, \dots, n$ and $k = n$, gives

$$e^m_i \circ g_1 \circ (e^n_1 \star \cdots \star e^n_n) = e^m_i \circ g_2 \circ (e^n_1 \star \cdots \star e^n_n)\,.$$

Thus $e^m_i \circ g_1 = e^m_i \circ g_2$ for all $1 \leqslant i \leqslant m$, and in the consequence

$$(e^m_1 \circ g_1) \star \cdots \star (e^m_m \circ g_1) = (e^m_1 \circ g_2) \star \cdots \star (e^m_m \circ g_2)\,.$$

Hence $(e^m_1 \star \cdots \star e^m_m) \circ g_1 = (e^m_1 \star \cdots \star e^m_m) \circ g_2$, which implies $g_1 = g_2$.

This completes the proof that $P : g \mapsto P_g$ is an isomorphism between algebras $(G, \circ, \star, e, f)$ and $(\Phi, \circ, \star, \Delta, F)$, where $\Phi = \{P_g \,|\, g \in G\}$. □

3. Symmetrical algebras

An algebra $(G, \circ, \star, e, f)$ of type $(2, 2, 0, 0)$ satisfying Axioms $1 - 6$ is called a *V-algebra*.

Let $\mathcal{G} = (G, \circ, \star, e, f)$ be a fixed V-algebra and let $\mathcal{G}' = (G', \circ, \star, e', f')$ be some other algebra of type $(2, 2, 0, 0)$.

Definition 3. A homomorphism $P : \mathcal{G} \to \mathcal{G}'$ is called a *v-homomorphism*, if $g \neq g \circ (e^n_1 \star \cdots \star e^n_n)$ implies $P(g) \neq P(g \circ (e^n_1 \star \cdots \star e^n_n))$ for any $g \in G$ and $n \in \mathbb{N}$.

It is easy to see that if P is a v-homomorphism of a V-algebra $\mathcal{G}$ onto an algebra $\mathcal{G}'$, then $\mathcal{G}'$ is a V-algebra too. In this case $\alpha g = \alpha P(g)$ and $\beta g = \beta P(g)$ for any $g \in G$. Conversely, if P is a homomorphism of a V-algebra $\mathcal{G}$ onto a V-algebra $\mathcal{G}'$ such that $\alpha g = \alpha P(g)$ and $\beta = \beta P(g)$ for all $g \in G$, then P is a v-homomorphism.

Definition 4. A subset H of a V-algebra $\mathcal{G}$ is called a *v-ideal*, if for all $x \in G$, $h_1, \dots, h_n \in H$, $1 \leqslant i \leqslant n$, where $n = \alpha x$ and $m = \beta x$, the condition $e_i^m \circ x \circ (h_1 \star \dots \star h_n) \in H$ is satisfied.

Generalizing the concept of dense ideals in semigroups (cf. [2]), we say that an ideal H of a V-algebra $\mathcal{G}$ is *dense* if and only if

(a) any v-homomorphism of $\mathcal{G}$, which is not an isomorphism, induces on H a homomorphism, which is not an isomorphism,

(b) if $\mathcal{G}$ is a V-subalgebra of V-algebra $\mathcal{G}' \neq \mathcal{G}$ and H is a v-ideal of $\mathcal{G}'$, then there exists a v-homomorphism of $\mathcal{G}'$, which is not an isomorphism, but induces on H an isomorphism.

Now consider the symmetrical algebra of vector-functions

$$\mathfrak{T} = (\mathcal{T}(A), \circ, \star, \Delta, F).$$

It is easy to verify that it satisfies Axioms $1-6$, i.e. it is a V-algebra.

By H_A we denote the set of all functions φ_a such that $a \in A$ and $\varphi_a(x) = a$ for all $x \in A$. Clearly, $\alpha\varphi_a = \beta\varphi_a = 1$ for all $a \in A$ and $(H_A, \circ)$ is a semigroup of left zeros.

The following three theorems are generalizations of similar results proved for transformation semigroups [2].

Theorem 2. *The set H_A is a dense v-ideal of $\mathfrak{T} = (\mathcal{T}(A), \circ, \star, \Delta, F)$.*

Proof. Let $\psi \in \mathcal{T}(A)$, $\varphi_{a_1}, \dots, \varphi_{a_n} \in H_A$, where $n = \alpha\psi$ and $a_1, \dots, a_n$ are elements of A. Suppose that

$$\psi(a_1, \dots, a_n) = b_1 \dots b_m$$

for some $b_1, \dots, b_m \in A$, where $m = \beta\psi$. We have $(I_i^m \circ \psi)(a_1, \dots, a_n) = b_i$ for $1 \leqslant i \leqslant m$, because $I_i^m(b_1, \dots, b_m) = b_i$. If $c \in A$, then

$$(I_i^m \circ \psi)(\varphi_{a_1}(c), \dots, \varphi_{a_n}(c)) = \varphi_{b_i}(c),$$

i.e. $(I_i^m \circ \psi \circ (\varphi_{a_1} \star \dots \star \varphi_{a_n}))(c) = \varphi_{b_i}(c)$. So,

$$I_i^m \circ \psi \circ (\varphi_{a_1} \star \dots \star \varphi_{a_n}) = \varphi_{b_i} \in H_A.$$

This proves that H_A is a v-ideal of $\mathfrak{T}$.

Now let P be a v-homomorphism of $\mathfrak{T}$, which is not an isomorphism. Hence, there are $\psi_1, \psi_2 \in \mathcal{T}(A)$ such that $\psi_1 \neq \psi_2$ and $P(\psi_1) = P(\psi_2)$. The last equation gives $\alpha P(\psi_1) = \alpha P(\psi_2)$ and $\beta P(\psi_1) = \beta P(\psi_2)$. So, there are elements $a_1, \dots, a_n \in A$ such that

$$\psi_1(a_1, \dots, a_n) \neq \psi_2(a_1, \dots, a_n).$$

Let $\psi_1(a_1, \dots, a_n) = b_1 \dots b_m$ and $\psi_2(a_1, \dots, a_n) = c_1 \dots c_m$, where n, m are degree and rank of functions ψ_1, ψ_2 respectively. Thus $b_1 \neq c_i$ for some $1 \leqslant i \leqslant m$, because $b_1 \dots b_m \neq c_1 \dots c_m$. Whence $\varphi_{b_i} \neq \varphi_{c_i}$. But

$$\begin{aligned} P(\varphi_{b_i}) &= P(I_i^m \circ \psi_1 \circ (\varphi_{a_1} \star \dots \star \varphi_{a_n})) \\ &= P(I_i^m) \circ P(\psi_1) \circ (P(\varphi_{a_1}) \star \dots \star P(\varphi_{a_n})) \\ &= P(I_i^m) \circ P(\psi_2) \circ (P(\varphi_{a_1}) \star \dots \star P(\varphi_{a_n})) \\ &= P(I_i^m \circ \psi_2 \circ (\varphi_{a_1} \star \dots \star \varphi_{a_n})) = P(\varphi_{c_i}). \end{aligned}$$

Thus, P induces on H_A a homomorphism, which is not isomorphism.

Now assume that H_A is a v-ideal of V-algebra $\mathcal{G} = (G, \circ, \star, \Delta, F)$ and $\mathfrak{T}$ is a proper subalgebra of $\mathcal{G}$. For each element $g \in G$ we consider the function $\lambda_g \in \mathcal{T}(A)$ defined in the following way:

$$b_1 \dots b_m = \lambda_g(a_1, \dots, a_n) \Longleftrightarrow \bigwedge_{i=1}^{m} I_i^m \circ g \circ (\varphi_{a_1} \star \dots \star \varphi_{a_n}) = \varphi_{b_i}\,, \quad (13)$$

where $n = \alpha g$, $m = \beta g$, $a_1, \dots, a_n, b_1, \dots, b_m \in A$. It is not difficult to see that the mapping $P : g \mapsto \lambda_g$ is a v-homomorphism of $\mathcal{G}$ into $\mathfrak{T}$. Since $\mathcal{T}(A) \subset G$ and $\mathcal{T}(A) \neq G$, for $g \in G \setminus \mathcal{T}(A)$ we have $g \neq P(g) = \lambda_g$. But $P(\lambda_g) = \lambda_g$, by (13). Therefore $P(g) = P(\lambda_g)$. Thus, P is a v-homomorphism, which is not an isomorphism, and which induces on H_A an identical isomorphism. □

Theorem 3. *A V-algebra $\mathcal{G} = (G, \circ, \star, e, f)$ is isomorphic to some symmetrical algebra of vector-functions if and only if it contains a dense v-ideal H, which is a semigroup of left zeros under the operation $\circ$ and $\alpha h = \beta h = 1$ for all $h \in H$.*

Proof. The necessity follows from Theorem 2. To prove the sufficiency we consider the mapping $P : G \to \mathcal{T}(H)$ defined by the formula

$$y_1 \dots y_m = P(g)(x_1, \dots, x_n) \Longleftrightarrow \bigwedge_{i=1}^{m} y_i = e_i^m \circ g \circ (x_1 \star \dots \star x_n) \qquad (14)$$

for all $g \in G$ and $x_1, \dots, x_n, y_1, \dots, y_m \in H$, where $n = \alpha g$, $m = \beta g$. From (14) it follows that $P(e) = \Delta$, $P(f) = F$. It is not difficult to verify that P is a v-homomorphism, which induces on H an isomorphism. But H is a dense v-ideal of $\mathcal{G}$, therefore, according to the definition of a dense v-ideal, P must be an isomorphism. Hence, a v-ideal H_H is a dense v-ideal of a homomorphic image of $(G, \circ, \star, e, f)$, i.e. $(P(G), \circ, \star, \Delta, F)$, because $(H_H, \circ)$ is isomorphic to $(H, \circ)$. But, by Theorem 2, a v-ideal H_H is a dense v-ideal of $(\mathcal{T}(H), \circ, \star, \Delta, F)$, therefore $P(G) \subset \mathcal{T}(H)$ implies $P(G) = \mathcal{T}(H)$. This proves that $\mathcal{G}$ is isomorphic to a symmetrical algebra of vector-functions. $\square$

Let $f : A \to A$ be some one-to-one mapping. By P_f we denote the mapping $\mathcal{T}(A) \to \mathcal{T}(A)$ defined by the condition

$$P_f(\varphi)(a_1, \dots, a_n) = b_1 \dots b_m \Longleftrightarrow$$
$$f^{-1}(b_1) \dots f^{-1}(b_m) = \varphi(f^{-1}(a_1), \dots, f^{-1}(a_n))$$

for all $\varphi \in \mathcal{T}(A)$ and $a_1, \dots, a_n, b_1, \dots, b_m \in A$, where $n = \alpha\varphi$, $m = \beta\varphi$. It is easy to see that P_f is an automorphism of $\mathfrak{T} = (\mathcal{T}(A), \circ, \star, \Delta, F)$. Such defined automorphism is called *inner*.

Theorem 4. *Every automorphism of* $\mathfrak{T} = (\mathcal{T}(A), \circ, \star, \Delta, F)$ *is inner.*

Proof. Let λ be some automorphism of $\mathfrak{T} = (\mathcal{T}(A), \circ, \star, \Delta, F)$, then $\lambda(\Delta) = \Delta$ and $\lambda(F) = F$. Therefore $\lambda(I_i^n) = I_i^n$ for $n \in \mathbb{N}$ and any $1 \leqslant i \leqslant n$. This implies $\alpha\varphi = \alpha\lambda(\varphi)$ and $\beta\varphi = \beta\lambda(\varphi)$ for every $\varphi \in \mathcal{T}(A)$.

We have also $\lambda(\varphi_a) \in H_A$ for all $a \in A$. Indeed, for any $\psi \in \mathcal{T}(A)$ such that $\alpha\psi = \beta\psi = 1$, holds $\varphi_1 \circ \psi = \varphi_a$, where $a \in A$. Therefore $\varphi_a \circ \lambda^{-1}(\varphi_b) = \varphi_a$, where $b \in A$. Thus, $\lambda(\varphi_a \circ \lambda^{-1}(\varphi_b)) = \lambda(\varphi_a)$, i.e. $\lambda(\varphi_a) \circ \varphi_b = \lambda(\varphi_a)$. Since H_A is a v-ideal of $\mathfrak{T}$, then $\lambda(\varphi_a) \circ \varphi_b \in H_A$, i.e. $\lambda(\varphi_a) \in H_A$.

Now consider the one-to-one correspondence $f_\lambda : A \to A$ such that

$$(a, b) \in f_\lambda \Longleftrightarrow (\varphi_a, \varphi_b) \in \lambda$$

for any $a, b \in A$.

Evidently $\lambda(\varphi_a) = \varphi_{f_\lambda(a)}$ and $\lambda^{-1}(\varphi_a) = \varphi_{f_\lambda^{-1}(a)}$ for each $a \in A$. Thus, for all $\varphi \in \mathcal{T}(A)$ and $a_1, \dots, a_n, b_1, \dots, b_m \in A$, where $n = \alpha\varphi$, $m = \beta\varphi$, we have

$$\begin{aligned} & b_1 \dots b_m = \lambda(\varphi)(a_1, \dots, a_n) \\ \Longleftrightarrow & \bigwedge_{i=1}^{m} \varphi_{b_i} = I_i^m \circ \lambda(\varphi) \circ (\varphi_{a_1} \star \dots \star \varphi_{a_n}) \\ \Longleftrightarrow & \bigwedge_{i=1}^{m} \varphi_{f_\lambda^{-1}(b_i)} = I_i^m \circ \varphi \circ (\varphi_{f_\lambda^{-1}(a_1)} \star \dots \star \varphi_{f_\lambda^{-1}(a_n)}) \\ \Longleftrightarrow & f_\lambda^{-1}(b_1) \dots f_\lambda^{-1}(b_m) = \varphi(f_\lambda^{-1}(a_1), \dots, f_\lambda^{-1}(a_n)) \\ \Longleftrightarrow & b_1 \dots b_m = P_{f_\lambda}(\varphi)(a_1, \dots, a_n). \end{aligned}$$

So, $\lambda = P_{f_\lambda}$, i.e. λ is an inner automorphism. □

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Department of Mathematics
Pedagogical University
21100 Vinnitsa
Ukraine
e-mail: vtrokhim@sovauma.com

Received September 12, 2001