Quasigroups and Related Systems **8** (2001), 67 – 72

On quadratic B-algebras

Hee Kon Park and Hee Sik Kim

Abstract

In this paper we introduce the notion of quadratic B-algebra which is a medial quasigroup, and obtain that every quadratic B-algebra on a field X with $|X| \geqslant 3$, is a BCI-algebra.

1. Introduction

Y. Imai and K. Iséki introduced two classes of abstract algebras: BCK-algebras and BCI-algebras ([6, 7]). It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras. In [4, 5] Q. P. Hu and X. Li introduced a wide class of abstract algebras: BCH-algebras. They have shown that the class of BCI-algebras is a proper subclass of the class of BCH-algebras. J. Neggers and H. S. Kim ([10]) introduced the notion of d-algebras, i.e. algebras $(X;*,e)$ defined by (i) $x*x=e$, (v) $e*x=e$, (vi) $x*y=e$ and $y*x=e$ imply $x=y$, which is another useful generalization of BCK-algebras, and then they investigated several relations between d-algebras and BCK-algebras as well as some other interesting relations between d-algebras and oriented digraphs. Y. B. Jun, E. H. Roh and H. S. Kim introduced in [8] a new notion, called an BH-algebra, i.e. algebras $(X;*,e)$ satisfying (i), (ii) $x*e=x$ and (vi), which is a generalization of $BCH/BCI/BCK$-algebras. They also defined the notions of ideals and boundedness in BH-algebras, and showed that there is a maximal ideal in bounded BH-algebras. J. Neggers, S. S. Ahn and H. S. Kim (cf. [10]) introduced the notion of a Q-algebra, and generalized some theorems discussed in BCI-algebras. Recently, J. Neggers and H. S. Kim introduced and investigated a class of algebras, called a B-algebra ([12, 13]), which is related to several classes of algebras of interest such as $BCH/BCI/BCK$-algebras and which seems to have rather nice

2000 Mathematics Subject Classification: 06F35
Keywords: B-algebra, Q-algebra, BCI-algebra

properties without being excessively complicated otherwise. B-algebras are also unipotent quasigroups which plays an important role in the theory of Latin squares (cf. [3]).

In this paper we introduce the notion of quadratic B-algebra which is a medial quasigroup, and obtain that every quadratic B-algebra on a field X with $|X| \geqslant 3$, is a BCI-algebra.

2. B-algebras

A *B-algebra* (cf. [12]) is a non-empty set X with a constant e and a binary operation $*$ satisfying the following axioms:

(i) $x * x = e$,

(ii) $x * e = x$,

(iii) $(x * y) * z = x * (z * (e * y))$

for all $x, y, z \in X$.

Example 2.1. (cf. [12]) Let X be the set of all real numbers except for a negative integer $-n$. Define a binary operation $*$ on X by

$$x * y = \frac{n(x-y)}{n+y}.$$

Then $(X; *, 0)$ is a B-algebra with $e = 0$.

Example 2.2. (cf. [13]) Let $X = \{0, 1, 2, 3, 4, 5\}$ be a set with the following table:

$*$	0	1	2	3	4	5
0	0	2	1	3	4	5
1	1	0	2	4	5	3
2	2	1	0	5	3	4
3	3	4	5	0	2	1
4	4	5	3	1	0	2
5	5	3	4	2	1	0

Then $(X; *, 0)$ is a B-algebra with $e = 0$.

In [2] the following result is proved.

Lemma 2.3. *Let* $(X;*,e)$ *be a* B*-algebra. Then we have the following statements.*

(a) *If* $x*y=e$ *then* $x=y$ *for any* $x,y\in X$.

(b) *If* $e*x=e*y$, *then* $x=y$ *for any* $x,y\in X$.

(c) $e*(e*x)=x$ *for any* $x\in X$.

J. Neggers, S. S. Ahn and H. S. Kim introduced in [10] the notion of Q-algebra, as an algebra $(X,;*,e)$ satisfying (i), (ii) and

(iv) $(x*y)*z=(x*z)*y$

for any $x,y,z\in X$. It is easy to see that B-algebras and Q-algebras are different notions. For example, $X=\{0,1,2,3\}$ with $*$ defined by the following table:

$*$	0	1	2	3
0	0	0	0	0
1	1	0	0	0
2	2	0	0	0
3	3	3	3	0

is a Q-algebra ([10]), but not a B-algebra, since $(3*2)*1=0\neq 3=3*(1*(0*2))$. Example 2.2 is a B-algebra ([13]), but not a Q-algebra, since $(5*3)*4=3\neq 4=(5*4)*3$.

Theorem 2.4. (cf. [10]) *Every* Q*-algebra satisfying the conditions* (iv) *and*

(vii) $(x*y)*(x*z)=z*y$

for any $x,y,z\in X$, *is a* BCI*-algebra.*

3. Quadratic B-algebras

Let X be a field with $|X|\geqslant 3$. An algebra $(X;*)$ is said to be *quadratic* if $x*y$ is defined by

$$x*y=a_1x^2+a_2xy+a_3y^2+a_4x+a_5y+a_6\,,$$

where $a_1,\dots,a_6\in X$ are fixed.

A quadratic algebra $(X;*)$ is said to be a *quadratic* B*-algebra* if for some fixed $e\in X$ it satisfies the conditions (i), (ii) and (iii). Similarly, a quadratic algebra $(X;*)$ is said to be a *quadratic* Q*-algebra* if for some fixed $e\in X$ it satisfies the conditions (i), (ii) and (iv).

In [10] it is proved that in every quadratic Q-algebra $(X;*,e)$ the operation $*$ has the form $x*y=x-y+e$.

We prove that the similar result is true for quadratic B-algebras.

Theorem 3.1. *Let X be a field with $|X|\geqslant 3$. Then every quadratic B-algebra $(X;*,e)$, $e\in X$, has the form $x*y=x-y+e$, where $x,y\in X$.*

Proof. Let

$$x*y=Ax^2+Bxy+Cy^2+Dx+Ey+F \tag{1}$$

for some fixed $A,B,C,D,E,F\in X$.

Consider (i). Then

$$e=x*x=(A+B+C)x^2+(D+E)x+F. \tag{2}$$

Let $x=0$ in (2). Then we obtain $F=e$. Hence (1) turns out to be

$$x*y=Ax^2+Bxy+Cy^2+Dx+Ey+e \tag{3}$$

If $y=x$ in (3), then

$$e=x*x=(A+B+C)x^2+(D+E)x+e,$$

for any $x\in X$, and hence we obtain $A+B+C=0=D+E$, i.e. $E=-D$ and $B=-A-C$. Hence (3) turns out to be

$$x*y=(x-y)(Ax-Cy+D)+e. \tag{4}$$

Let $y=e$ in (4). Then by (ii) we have

$$x=x*e=(x-e)(Ax-Ce+D)+e,$$

i.e. $(Ax-Ce+D-1)(x-e)=0$. Since X is a field, either $x-e=0$ or $Ax-Ce+D-1=0$. Since $|X|\geqslant 3$, we have $Ax-Ce+D-1=0$, for any $x\in X$. This means that $A=0$, $1-D+Ce=0$. Thus (4) turns out to be

$$x*y=(x-y)+C(x-y)(e-y)+e. \tag{5}$$

To satisfy the condition (iv) we need to determine the constant C, but its computation is so complicated that we use Lemma 2.3 (iii) instead. If we replace e by x, and x by y respectively in (5), then

$$e * x = (e - x) + C(e - x)(e - x) + e. \tag{6}$$

It follows that

$$\begin{aligned} e * (e * x) &= e * [(e - x) + C(e - x)^2 + e] \\ &= x - C(e - x)^2 + C(e - x)\{1 + C(e - x)\}^2 \\ &= x + C^3(e - x)^4 + 2C^2(e - x)^3 . \end{aligned}$$

Since $x = e * (e * x)$, we obtain

$$C^2(e - x)^3\{-Cx + 2 + Ce\} = 0.$$

Since X is a field with $|X| \geqslant 3$, we obtain $C = 0$. This means that every quadratic B-algebra $(X; *, e)$ has the form $x * y = x - y + e$, where $x, y \in X$, completing the proof. □

It follows from Theorem 3.1 that the quadratic B-algebras are medial quasigroups (cf. [1]).

Example 3.2. Let $\mathcal{R}$ be the set of all real numbers. Define $x * y = x - y + \sqrt{2}$. Then $(\mathcal{R}; *, \sqrt{2})$ is a quadratic B-algebra.

Example 3.3. Let $\mathcal{K} = GF(p^n)$ be a Galois field. Define $x * y = x - y + e$, $e \in \mathcal{K}$. Then $(\mathcal{K}; *, e)$ is a quadratic B-algebra.

As a simple consequence of Theorem 3.1 and results proved in [10] we obtain:

Proposition 3.4. *Let X be a field with $|X| \geqslant 3$. Then every quadratic B-algebra on X is a Q-algebra, and conversely.*

Proposition 3.5. *Let X be a field with $|X| \geqslant 3$. If $(X; *, e)$ is a quadratic B-algebra, then $(x * y) * (x * z) = z * y$ for any $x, y, z \in X$.*

Proof. Straightforward. □

Theorem 3.6. *Let X be a field with $|X| \geqslant 3$. Then every quadratic B-algebra on X is a BCI-algebra.*

Proof. It is an immediate consequence of Proposition 3.5 and Theorem 2.4. □

References

[1] **V. D. Belousov**: *Foundations of the theory of quasigroups and loops*, (Russian), Nauka, Moscov 1967.

[2] **J. R. Cho and H. S. Kim**: *On B-algebras and quasigroups*, Quasigroups and Related Systems **8** (2001), 1 − 6.

[3] **J. Dénes and A. D. Keedwell**: *Latin squares and their applications*, Académai Kiadó, Budapest 1974.

[4] **Q. P. Hu and X. Li**: *On BCH-algebras*, Math. Seminar Notes **11** (1983), 313 − 320.

[5] **Q. P. Hu and X. Li**: *On proper BCH-algebras*, Math. Japonica **30** (1985), 659 − 661.

[6] **K. Iséki and S. Tanaka**: *An introduction to the theory of BCK-algebras*, Math. Japonica **23** (1978), 1 − 26.

[7] **K. Iséki**: *On BCI-algebras*, Math. Seminar Notes **8** (1980), 125−130.

[8] **Y. B. Jun, E. H. Roh and H. S. Kim**: *On BH-algebras*, Scientiae Math. **1** (1998), 347 − 354.

[9] **J. Meng and Y. B. Jun**: *BCK-algebras*, Kyung Moon Sa Co., Seoul 1994.

[10] **J. Neggers, S. S. Ahn and H. S. Kim**: *On Q-algebras*, Int. J. Math. and Math. Sci. **27** (2001), 749 − 757.

[11] **J. Neggers and H. S. Kim**: *On d-algebras*, Math. Slovaca **49** (1999), 19 − 26.

[12] **J. Neggers and H. S. Kim**: *On B-algebras*, (submitted)

[13] **J. Neggers and H. S. Kim**: *A fundamental theorem of B-homomorphism for B-algebras*, Intern. Math. J. **2** (2002), (to appear)

Received April 19, 2001

Hee Kon Park, Department of Engineering Sciences, Hanyang University, Seoul 133-791, Korea

Hee Sik Kim, Department of Mathematics, Hanyang University, Seoul 133-791, Korea, e-mail: heekim@hanyang.ac.kr