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On B-algebras and quasigroups

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Abstract

In this paper we discuss further relations between B-algebras and quasigroups.

1. Introduction

Y. Imai and K. Iséki introduced two classes of abstract algebras: BCK-algebras and BCI-algebras ([2, 3]). It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras. In [4, 5] Q. P. Hu and X. Li introduced a wide class of abstract algebras: BCH-algebras. They have shown that the class of BCI-algebras is a proper subclass of the class of BCH-algebras. J. Neggers and H. S. Kim introduced in [8] the notion of d-algebras, i.e. algebras satisfying (1) $xx = 0$, (5) $0x = 0$, (6) $xy = 0$ and $yx = 0$ imply $x = y$, which is another useful generalization of BCK-algebras, and then they investigated several relations between d-algebras and BCK-algebras as well as some other interesting relations between d-algebras and oriented digraphs. Recently, Y. B. Jun, E. H. Roh and H. S. Kim introduced in [6] a new notion, called an BH-algebra, determined by (1), (2) $x0 = x$ and (6), which is a generalization of $BCH/BCI/BCK$-algebras. They also defined the notions of ideals and boundedness in BH-algebras, and showed that there is a maximal ideal in bounded BH-algebras. J. Neggers and H. S. Kim introduced in [9] and investigated a class of algebras which is related to several classes of algebras of interest such as $BCH/BCI/BCK$-algebras and which seems to have rather nice properties without being excessively complicated otherwise. In this paper we discuss further relations between B-algebras and other topics, especially quasigroups. This is a continuation of [9].

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2. Preliminaries

A *B-algebra* is a non-empty set X with a constant 0 and a binary operation "$\cdot$" (denoted by juxtaposition) satisfying the following axioms:

(1) $xx = 0$,

(2) $x0 = x$,

(3) $(xy)z = x(z(0y))$

for all $x, y, z \in X$.

Example 2.1. It is easy to see that $X = \{0, 1, 2, 3, 4, 5\}$ with the multiplication:

$\cdot$	0	1	2	3	4	5
0	0	2	1	3	4	5
1	1	0	2	4	5	3
2	2	1	0	5	3	4
3	3	4	5	0	2	1
4	4	5	3	1	0	2
5	5	3	4	2	1	0

is a B-algebra.

The following result is proved in [9].

Proposition 2.2. *If* $(X; \cdot, 0)$ *is a B-algebra, then*

(i) $x(yz) = (x(0z))y$,

(ii) $(xy)(0y) = x$,

(iii) $xz = yz$ *implies* $x = y$

for all $x, y, z \in X$.

A B-algebra $(X; \cdot, 0)$ is said to be 0-*commutative* if $x(0y) = y(0x)$ for any $x, y \in X$.

The B-algebra from the above example is not 0-commutative, since we have $3 \cdot (0 \cdot 4) = 2 \neq 1 = 4 \cdot (0 \cdot 3)$. A simple example of a 0-commutative B-algebra is a Boolean group. It is not difficult to see that a B-algebra is a Boolean group iff it satisfies one from the following identities: $0x = x$, $xy = yx$, $(xy)z = x(yz)$.

3. B-algebras and quasigroups

Lemma 3.1. *Let* $(X;\cdot,0)$ *be a* B*-algebra. Then for all* $x,y\in X$

- (i) $xy=0$ *implies* $x=y$,
- (ii) $0x=0y$ *implies* $x=y$,
- (iii) $0(0x)=x$.

Proof. (i) Trivially follows from Proposition 2.2 (iii) and the fact that $0=yy$.
(ii) If $0x=0y$, then

$$0=xx=(xx)0=x(0(0x))=x(0(0y))=(xy)0=xy\,,$$

and hence $x=y$ by (i).
(iii) For any $x\in X$, since $0x=(0x)0=0(0(0x))$ by (ii), we have $x=0(0x)$. □

Theorem 3.2. *In any* B*-algebra the left cancellation law holds.*

Proof. Assume that $xy=xz$. Then $0(xy)=0(xz)$. By Proposition 2.2 (i), we obtain that $(0(0y))x=(0(0z))x$. By Lemma 3.1 (iii) we have $yx=zx$. Hence $y=z$ by Proposition 2.2 (iii). □

Let L_a and R_a be the *left* and *right* translation of X (respectively), i.e. let $L_a(x)=ax$ and $R_a(x)=xa$ for all $x\in X$.

Lemma 3.3. *If* $(X;\cdot,0)$ *is a* B*-algebra, then*

- (i) L_0 *is a bijection,*
- (ii) $R_0=R_0^{-1}=id_X$,
- (iii) L_a *and* R_a *are injective for all* $a\in X$,
- (iv) $L_0^{-1}(0\cdot x)=L_0^{-1}(L_0(x))=x$ *and* $0\cdot(L_0^{-1}(x))=L_0(L_0^{-1}(x))=x$ *for* $x\in X$.

Proof. (i) Since $0(0x)=x$, $L_0^2=id_X$ and so L_0 is a bijection.
(ii) is a consequence of (2).
(iii) follows from Proposition 2.2 (iii) and Theorem 3.2. □

Lemma 3.4. L_a *and* R_a *are surjective for all* $a\in X$.

Proof. Let $c \in X$. Putting $b = (L_0^{-1}(c)) \cdot (0 \cdot a)$, we obtain

$$\begin{aligned} L_a(b) &= L_a(L_0^{-1}(c) \cdot (0 \cdot a)) = a \cdot (L_0^{-1}(c) \cdot (0 \cdot a)) \\ &= (a \cdot a) \cdot (L_0^{-1}(c))) = 0 \cdot (L_0^{-1}(c)) = c. \end{aligned}$$

Thus L_a is surjective.

Similarly, for $b = c \cdot (L_0^{-1}(a))$ we have

$$\begin{aligned} R_a(b) &= R_a((c \cdot (L_0^{-1}(a)) = (c \cdot (L_0^{-1}(a))) \cdot a \\ &= (c \cdot (L_0^{-1}(a))) \cdot (0 \cdot (L_0^{-1}(a))) = c. \end{aligned}$$

by Proposition 2.2 (ii). Hence R_a is surjective. □

Theorem 3.5. *Every B-algebra is a quasigroup.*

Proof. By Lemma 3.3 (iii) and Lemma 3.4. □

Proposition 3.6. *A B-algebra $(X; \cdot, 0)$ satisfies the identity $(yx)x = y$ if and only if it is a loop and* 0 *is its neutral element.*

Proof. If a B-algebra $(X; \cdot, 0)$ satisfies the identity $(yx)x = y$, then putting $y = 0$ in this identity we have $(0x)x = 0$, which by Lemma 3.1 (i) gives $0x = x$. Hence 0 is the neutral element of $(X; \cdot, 0)$. By Theorem 3.5 $(X; \cdot, 0)$ is a loop.

Conversely, if 0 is the neutral element of a B-algebra $(X; \cdot, 0)$, then

$$(yx)x = y(x(0x)) = y(xx) = y0 = y$$

for all $x, y \in X$. This proves the proposition. □

Theorem 3.7. *A B-algebra satisfies the identity $x(xy) = y$ if and only if it is* 0*-commutative.*

Proof. If a B-algebra $(X; \cdot, 0)$ satisfies the identity $x(xy) = y$, then

$$\begin{aligned} (x(0y))y &= x(y(0(0y))) = x(yy) = x0 = x = y(yx) \\ &= y(y(0(0x))) = (y(0x))y\,. \end{aligned}$$

Hence we have $(x(0y))y = (y(0x))y$. Then, by the right cancellation law, we obtain $x(0y) = y(0x)$.

The converse statement is proved in [9]. □

Remark. A B-algebra satisfying the identity $x(xy) = y$ is not, in general, a loop. Indeed, if $(G, +, 0)$ is an abelian group, then G with the operation $x \cdot y = x - y$ is an example of a 0-commutative B-algebra, which satisfies this identity but it is not a loop.

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