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Fuzzification of n-ary groupoids

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Abstract

Our work in this paper is concerned with the fuzzification of subgroupoid and ideals in n-ary groupoids.

1. Introduction

After the introduction of the concept of fuzzy sets by Zadeh [13], several researches were conducted on the generalizations of the notion of fuzzy set and application to many algebraic structures such as: groups [11], quasigroups [7], semirings [9], BCC-algebras [8] et cetera. All these applications are connected with binary operations.

Now we extend this concept to the set with one n-ary operation, i.e., to the set G with one operation $f : G^n \to G$, where $n \geqslant 2$. Such defined groupoid will be denoted by $\mathcal{G}$.

According to the tradition (cf. [2] and [6]) the sequence of elements $x_i, \ldots, x_j$ will be denoted by x_i^j (for $j < i$ it is empty symbol). This means that $f(x_1, x_2, \ldots, x_n)$ will be written as $f(x_1^n)$.

An n-ary groupoid $\mathcal{G}$ is called *unipotent* (cf. [5]) if it contains an element θ such that $f(x, x, \ldots, x) = \theta$ for all $x \in G$. Such groupoid is obviously an *n-ary semigroup*, i.e., it satisfies

$$f(x_1^{i-1}, f(x_i^{n+i-1}), x_{n+i}^{2n-1}) = f(x_1^{j-1}, f(x_j^{n+j-1}), x_{n+j}^{2n-1})$$

for all $x_1, \ldots, x_{2n-1} \in G$ and $i, j \in \{1, 2, \ldots, n\}$.

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A non-empty subset S of $\mathcal{G}$ is an *n-ary subgroupoid* (briefly: *subgroupoid*) if it is closed with respect to the operation f. A subset S is called a *k-ideal* of $\mathcal{G}$ if $f(x_1^{k-1}, a, x_{k+1}^n) \in S$ for all $x_1, \ldots, x_n \in G$ and all $a \in S$ (cf. [12]). If S is a k-ideal for every $k = 1, 2, \ldots, n$, then it is called an *ideal*.

2. Fuzzy subgroupoids

By a *fuzzy set* μ in a set G we mean a function $\mu : G \to [0,1]$. The set

$$L(\mu, t) = \{x \in G : \mu(x) \geqslant t\},$$

where $t \in [0,1]$ is fixed, is called a *level subset of* μ. $Im(\mu)$ denotes the image set of μ.

Definition 2.1. A fuzzy set μ defined on G is called a *fuzzy subgroupoid* of an n-ary groupoid $\mathcal{G} = (G, f)$ if

$$\mu(f(x_1^n)) \geqslant \min\{\mu(x_1), \ldots, \mu(x_n)\}$$

for all $x_1, \ldots, x_n \in G$.

Lemma 2.2. *If μ is a fuzzy subgroupoid of a unipotent groupoid (G, f, θ), then $\mu(\theta) \geqslant \mu(x)$ for all $x \in G$.*

Proof. Indeed, $f(x, x, \ldots, x) = \theta$ for all $x \in G$ implies

$$\mu(\theta) = \mu(f(x, \ldots, x)) \geqslant \min\{\mu(x), \ldots, \mu(x)\} = \mu(x),$$

which completes the proof. □

Theorem 2.3. *A fuzzy set μ of an n-ary groupoid $\mathcal{G}$ is a fuzzy subgroupoid iff for every $t \in [0,1]$, $L(\mu, t)$ is either empty or a subgroupoid of $\mathcal{G}$.*

Proof. Straightforward. □

Theorem 2.4. *Any subgroupoid of $\mathcal{G}$ can be realized as a level subgroupoid of some fuzzy subgroupoid of $\mathcal{G}$.*

Proof. Let (T,f) be a subgroupoid of a given n-ary groupoid $\mathcal{G}$ and let μ be a fuzzy set in $\mathcal{G}$ defined by

$$\mu(x)=\begin{cases} t & if \ \ x\in T, \\ s & if \ \ x\notin T, \end{cases}$$

where $0\leqslant s<t\leqslant 1$ is fixed.

It is not difficult to see that μ is a fuzzy subgroupoid of $\mathcal{G}$ such that $L(\mu,t)=T$. □

Proposition 2.5. *Two level subgroupoids $L(\mu,s)$, $L(\mu,t)$ $(s<t)$ of a fuzzy subgroupoid μ are equal iff there is no $x\in G$ such that $s\leqslant\mu(x)<t$.*

Proof. See the proof of Proposition 3.7 in [7]. □

Corollary 2.6. *Let μ be a fuzzy subgroupoid of $\mathcal{G}$. If $Im(\mu)=\{t_1,t_2,...,t_m\}$, where $t_1<t_2<...<t_m$, then the family of levels $L(\mu,t_i)$, $1\leqslant i\leqslant m$, constitutes all the level subgroupoids of μ.* □

Corollary 2.7. *Let μ be a fuzzy subgroupoid with finite $Im(\mu)$. If $\mu_s=\mu_t$ for some $s,t\in Im(\mu)$, then $s=t$.* □

Corollary 2.8. *Let μ and ρ be two fuzzy subgroupoids of $\mathcal{G}$ with the same family of levels. If $Im(\mu)=\{t_1,\ldots,t_m\}$ and $Im(\rho)=\{s_1,\ldots,s_p\}$, where $t_1>t_2>\ldots>t_m$ and $s_1>s_2>\ldots>s_p$, then*

a) $m=p$,

b) $L(\mu,t_i)=L(\rho,s_i)$ *for* $i=1,\ldots,m$,

c) *if* $\mu(x)=t_i$, *then* $\rho(x)=s_i$ *for* $x\in G$ *and* $i=1,\ldots,m$.

Proof. (a) and (b) are obvious. To prove (c) let $x\in G$ be such that $\mu(x)=t_i$ and $\rho(x)=s_j$. From (b) and $\mu(x)=t_i$ follows $x\in L(\rho,s_i)$. Thus $\rho(x)\geqslant s_i$ and $s_j\geqslant s_i$, i.e. $\rho_{s_j}\subseteq\rho_{s_i}$. Since $x\in L(\rho,s_j)=L(\mu,t_j)$, we obtain $t_i=\mu(x)\geqslant t_j$. This gives $L(\mu,t_i)\subseteq L(\mu,t_j)$, and, in the consequence (by (b)) $L(\rho,s_i)=L(\mu,t_i)\subseteq L(\mu,t_j)=L(\rho,s_j)$.

Thus $L(\rho, s_i) = L(\rho, s_j)$. But, by Corollary 2.7, $s_i = s_j$. Therefore $\rho(x) = s_i$. $\square$

Corollary 2.9. *If fuzzy subgroupoids μ and ρ defined on $\mathcal{G}$ have the same finite family of levels, then $\mu = \rho$ iff $Im(\mu) = Im(\rho)$.* $\square$

Theorem 2.10. *Let $\{S_\lambda : \lambda \in \Lambda\}$, where $\emptyset \neq \Lambda \subseteq [0,1]$, be a collection of subgroupoids of $\mathcal{G}$ such that*

(i) $G = \bigcup\limits_{\lambda\in\Lambda} S_\lambda$,

(ii) $\alpha > \beta \Longleftrightarrow S_\alpha \subset S_\beta$ *for all* $\alpha, \beta \in \Lambda$.

Then μ defined by

$$\mu(x) = \sup\{\lambda \in \Lambda : x \in S_\lambda\}$$

is a fuzzy subgroupoid of $\mathcal{G}$.

Proof. By Theorem 2.3, it is sufficient to show that every non-empty level $L(\mu, \alpha)$ is a subgroupoid of $\mathcal{G}$.

Let $L(\mu, \alpha) \neq \emptyset$ for some fixed $\alpha \in [0,1]$. Then

$$\alpha = \sup\{\lambda \in \Lambda : \lambda < \alpha\} = \sup\{\lambda \in \Lambda : S_\alpha \subset S_\lambda\}$$

or

$$\alpha \neq \sup\{\lambda \in \Lambda : \lambda < \alpha\} = \sup\{\lambda \in \Lambda : S_\alpha \subset S_\lambda\}.$$

In the first case we have $L(\mu, \alpha) = \bigcap\limits_{\lambda<\alpha} S_\lambda$, because

$$x \in L(\mu, \alpha) \Longleftrightarrow x \in S_\lambda \quad \text{for all} \quad \lambda < \alpha \Longleftrightarrow x \in \bigcap_{\lambda<\alpha} S_\lambda\,.$$

In the second, there exists $\varepsilon > 0$ such that $(\alpha - \varepsilon,\, \lambda) \cap \Lambda = \emptyset$. In this case $L(\mu, \alpha) = \bigcup\limits_{\lambda\geqslant\alpha} S_\lambda$. Indeed, if $x \in \bigcup\limits_{\lambda\geqslant\alpha} S_\lambda$, then $x \in S_\lambda$ for some $\lambda \geqslant \alpha$, which gives $\mu(x) \geqslant \lambda \geqslant \alpha$. Thus $x \in L(\mu, \alpha)$, i.e., $\bigcup\limits_{\lambda\geqslant\alpha} S_\lambda \subseteq L(\mu, \alpha)$.

Conversely, if $x \notin \bigcup\limits_{\lambda\geqslant\alpha} S_\lambda$, then $x \notin S_\lambda$ for all $\lambda \geqslant \alpha$, which implies $x \notin S_\lambda$ for all $\lambda > \alpha - \varepsilon$, i.e., if $x \in S_\lambda$ then $\lambda \leqslant \alpha - \varepsilon$. Thus $\mu(x) \leqslant \alpha - \varepsilon$. Therefore $x \notin L(\mu, \alpha)$. Hence $L(\mu, \alpha) \subseteq \bigcup\limits_{\lambda\geqslant\alpha} S_\lambda$, and in the consequence $L(\mu, \alpha) = \bigcup\limits_{\lambda\geqslant\alpha} S_\lambda$. This completes our proof because (as it is not difficult to see) $\bigcup\limits_{\lambda\geqslant\alpha} S_\lambda$ and $\bigcap\limits_{\lambda<\alpha} S_\lambda$ are subgroupoids. $\square$

Theorem 2.11. *Let μ be a fuzzy set defined in an n-ary groupoid $\mathcal{G}$ and let $Im(\mu)=\{t_0,t_1,\ldots,t_m\}$, where $t_0>t_1>\ldots>t_m$. If $S_0\subset S_1\subset\ldots\subset S_m=G$ are subgroupoids of $\mathcal{G}$ such that $\mu(S_k\setminus S_{k-1})=t_k$ for $k=0,1,\ldots,m$, where $S_{-1}=\emptyset$, then μ is a fuzzy subgroupoid.*

Proof. For arbitrary elements $x_1,\ldots,x_n\in G$ there exists only one k such that $f(x_1^n)\in S_k\setminus S_{k-1}$ and only one k_i such that $x_i\in S_{k_i}\setminus S_{k_i-1}$. Thus $\mu(f(x_1^n))=t_k$, $\mu(x_i)=t_{k_i}$.

Suppose $t_{k_i}>t_k$ for all $i=1,2,\ldots,n$. Then, by the assumption, $k_i<k$ and $S_{k_i}\subseteq S_s\subseteq S_{k-1}\subset S_k$, where $s=\max\{k_1,\ldots,k_n\}$. Hence $x_1,\ldots,x_n\in S_{k-1}$ and, in the consequence, $f(x_1^n)\in S_{k-1}$ because S_{k-1} is a subgroupoid. This is a contradiction. Therefore there is at least one $t_{k_i}<t_k$.

In this case obviously $\mu(f(x_1^n))=t_k>\min\{\mu(x_1),\ldots,\mu(x_n)\}$, which completes the proof. $\square$

Corollary 2.12. *Let μ be a fuzzy set defined in an n-ary groupoid $\mathcal{G}$ and let $Im(\mu)=\{t_0,t_1,\ldots,t_m\}$, where $t_0>t_1>\ldots>t_m$. If $S_0\subset S_1\subset\ldots\subset S_m=G$ are subgroupoids of $\mathcal{G}$ such that $\mu(S_k)=t_k$ for $k=0,1,\ldots,m$, then μ is a fuzzy subgroupoid.* $\square$

Corollary 2.13. *If $Im(\mu)=\{t_0,t_1,...,t_n\}$, where $t_0>t_1>...>t_n$, is the image of a fuzzy subgroupoid μ in $\mathcal{G}$, then all levels $L(\mu,t_k)$ are subgroupoid of $\mathcal{G}$, $\mu(L(\mu,t_0))=t_0$ and $\mu(L(\mu,t_k)\setminus L(\mu,t_{k-1}))=t_k$ for $k=1,2,...,n$.*

Proof. All levels $L(\mu,t_k)$ are subgroupoids by Theorem 2.3. Obviously $\mu(L(\mu,t_0))=t_0$. Moreover $\mu(L(\mu,t_1))\geqslant t_1$ implies $\mu(x)=t_0$ for $x\in L(\mu,t_0)$ and $\mu(x)=t_1$ for $x\in L(\mu,t_0)\setminus L(\mu,t_1)$.

Repeating this procedure, we conclude that $\mu(L(\mu,t_k)\setminus L(\mu,t_{k-1}))$ $=t_k$ for all $k=1,2,...,n$. $\square$

Proposition 2.14. *Let $\mathcal{G}$ be a unipotent n-ary groupoid. If μ is a fuzzy subgroupoid in $\mathcal{G}$ with the image $Im(\mu)=\{t_i:i\in I\}$ and*

$\Omega = \{L(\mu, t_i) : t_i \in Im(\mu)\}$, *then*

(a) *there exists a unique* $t_0 \in Im(\mu)$ *such that* $t_0 \geqslant t_i$ *for all* $t_i \in Im(\mu)$,

(b) G *is the set-theoretic union of all* $L(\mu, t_i) \in \Omega$,

(c) *the members of* Ω *form a chain,*

(d) Ω *contains all level subgroupoids of* μ *iff* μ *attains its infimum on all subgroupoids of* $\mathcal{G}$.

Proof. (a) Follows from the fact that in a unipotent n-ary groupoid $t_0 = \mu(\theta) \geqslant \mu(x)$ for all $x \in G$ (see Lemma 2.2).

(b) If $x \in G$, then $\mu(x) = t_x \in Im(\mu)$. Thus $x \in L(\mu, t_x) \subseteq \bigcup L(\mu, t_i) \subseteq G$, where $t_i \in Im(\mu)$, which proves (b).

(c) Since $L(\mu, t_i) \subseteq L(\mu, t_j) \iff t_i \geqslant t_j$ for $i, j \in I$, then the set Ω is totally ordered by inclusion.

(d) Suppose that Ω contains all level subgroupoids of μ. Let S be a subgroupoid of $\mathcal{G}$. If μ is constant on S, then we are done. Assume that μ is not constant on S. We consider two cases: (1) $S = G$ and (2) $S \subset G$. For $S = G$ let $\beta = \inf Im(\mu)$. Then $\beta \leqslant t \in Im(\mu)$, i.e. $L(\mu, \beta) \supseteq L(\mu, t)$ for all $t \in Im(\mu)$. But $L(\mu, 0) = G \in \Omega$ because Ω contains all level subgroupoids of μ. Hence there exists $t' \in Im(\mu)$ such that $L(\mu, t') = G$. It follows that $L(\mu, \beta) \supset L(\mu, t') = G$ so that $L(\mu, \beta) = L(\mu, t') = G$ because every level subgroupoid of μ is a subgroupoid of $\mathcal{G}$.

Now it sufficient to show that $\beta = t'$. If $\beta < t'$, then there exists $t'' \in Im(\mu)$ such that $\beta \leqslant t'' < t'$. This implies $L(\mu, t'') \supset L(\mu, t') = G$, which is a contradiction. Therefore $\beta = t' \in Im(\mu)$.

In the case $S \subset G$ we consider the fuzzy set μ_S defined by

$$\mu_S(x) = \begin{cases} \alpha & for \quad x \in S, \\ 0 & for \quad x \in G \setminus S. \end{cases}$$

From the proof of our Theorem 2.4 it follows that μ_S is a fuzzy subgroupoid of $\mathcal{G}$.

Let

$$J = \{i \in I : \mu(x) = t_i \quad for\ some \quad x \in S\}.$$

Then $\Omega_S = \{L(\mu, t_i) : i \in J\}$ contains (by the assumption) all level subgroupoids of μ_S. This means that there exists $x_0 \in S$ such that

$\mu(x_0) = \inf\{\mu_S(x) : x \in S\}$, i.e., $\mu(x_0) = \mu_S(x)$ for some $x \in S$. Hence μ attains its infimum on all subgroupoids of $\mathcal{G}$.

To prove the converse let $L(\mu, \alpha)$ be a level subgroupoid of μ. If $\alpha = t$ for some $t \in Im(\mu)$, then $L(\mu, \alpha) \in \Omega$. If $\alpha \neq t$ for all $t \in Im(\mu)$, then there does not exist $x \in G$ such that $\mu(x) = \alpha$.

Let $S = \{x \in G : \mu(x) > \alpha\}$. Obviously $\theta \in S$ and $\mu(x_i) > \alpha$ for all $x_1, x_2, \dots, x_n \in S$. From the fact that μ is a fuzzy subgroupoid we obtain

$$\mu(f(x_1^n)) \geqslant \min\{\mu(x_1), \mu(x_2), \dots, \mu(x_n)\} > \alpha\,,$$

which proves $f(x_1^n) \in S$. Hence (S, f) is a subgroupoid. By hypothesis, there exists $y \in S$ such that $\mu(y) = \inf\{\mu(x) : x \in S\}$. But $\mu(y) \in Im(\mu)$ implies $\mu(y) = t'$ for some $t' \in Im(\mu)$. Hence $\inf\{\mu(x) : x \in S\} = t' > \alpha$.

Note that there does not exist $z \in G$ such that $\alpha \leqslant \mu(z) < t'$. This gives $L(\mu, \alpha) = L(\mu, t')$. Hence $L(\mu, \alpha) \in \Omega$. Thus Ω contains all level subgroupoids of μ. □

Proposition 2.15. *Let $\mathcal{G}$ be a groupoid such that every descending chain $S_1 \supset S_2 \supset \dots$ of subgroupoids of $\mathcal{G}$ terminates at finite step. If μ is a fuzzy subgroupoid in $\mathcal{G}$ such that a sequence of elements of $Im(\mu)$ is strictly increasing, then μ has finite number of values.*

Proof. Analogously as the proof of Proposition 3.17 in [7]. □

Theorem 2.16. *If every fuzzy subgroupoid μ defined on $\mathcal{G}$ has the finite image, then every descending chain of subgroupoids of $\mathcal{G}$ terminates at finite step.*

Proof. Suppose there exists a strictly descending chain

$$S_0 \supset S_1 \supset S_2 \supset \dots$$

of subgroupoids of G which does not terminate at finite step. We prove that μ defined by

$$\mu(x) = \begin{cases} \frac{k}{k+1} & for \quad x \in S_k \setminus S_{k+1}, \\ 1 & for \quad x \in \bigcap S_k, \end{cases}$$

where $k = 0, 1, 2, ...$ and $S_0 = G$, is a fuzzy subgroupoid with an infinite number of values.

If $f(x_1^n) \in \bigcap S_k$, then obviously

$$\mu(f(x_1^n)) = 1 \geqslant \min\{\mu(x_1),\, \mu(x_2), \dots, \mu(x_n)\}\,.$$

If $f(x_1^n) \notin \bigcap S_k$, then $f(x_1^n) \in S_p \setminus S_{p+1}$ for some $p \geqslant 0$ and there exists at least one $i = 1, 2, \dots, n$ such that $x_i \notin \bigcap S_k$, because $x_1, x_2, \dots, x_n \in \bigcap S_k$ implies $f(x_1^n) \in \bigcap S_k$.

Let S_m be a maximal subgroupoid of $\mathcal{G}$ such that at least one of $x_1, x_2, \dots, x_n$ belongs to $S_m \setminus S_{m+1}$. Obviously $m \leqslant p$. Indeed, for $m > p$ we have $x_1, x_2, \dots, x_n \in S_m \subseteq S_{p+1} \subset S_p$ and, in the consequence, $f(x_1^n) \in S_{p+1}$, which is impossible. Thus $m \leqslant p$ and

$$\mu(f(x_1^n)) = \frac{p}{p+1} \geqslant \min\{\mu(x_1),\, \dots\, , \mu(x_n)\} = \frac{m}{m+1}\,.$$

This proves that μ is a fuzzy subgroupoid and has an infinite number of different values. Obtained contradiction completes our proof. $\square$

Theorem 2.17. *Every ascending chain of subgroupoids of a groupoids $\mathcal{G}$ terminates at finite step iff the set of values of any fuzzy groupoid in $\mathcal{G}$ is a well-ordered subset of* $[0, 1]$.

Proof. If the set of values of a fuzzy subgroupoid μ is not well-ordered, then there exists a strictly decreasing sequence $\{t_i\}$ such that $t_i = \mu(x_i)$ for some $x_i \in G$. But in this case subgroupoids $B_i = \{x \in G : \mu(x) \geqslant t_i\}$ form a strictly ascending chain, which is a contradiction.

To prove the converse suppose that there exist a strictly ascending chain $S_1 \subset S_2 \subset S_3 \subset ...$ of subgroupoids. Then $M = \bigcup S_i$ is a subgroupoid of $\mathcal{G}$ and μ defined by

$$\mu(x) = \begin{cases} 0 & \mathit{for}\ x \notin M\,, \\ \frac{1}{k} & \mathit{where}\ k = \min\{i : x \in A_i\} \end{cases}$$

is a fuzzy subgroupoid on $\mathcal{G}$.

Indeed, if all $x_1, x_2, \dots, x_n$ are in M, then for every x_i there exists a minimal number k_i such that $x_i \in S_{k_i}$, and a minimal number p

such that $f(x_1^n) \in S_p$. Obviously all $x_1, x_2, \ldots, x_n, f(x_1^n)$ are in S_k, where $k = \max\{k_1, k_2, \ldots, k_n\}$. Thus $k \geqslant p$ and

$$\mu(f(x_1^n) = \frac{1}{p} \geqslant \frac{1}{k} = \min\{\mu(x_1), \mu(x_2), \ldots, \mu(x_n)\}\,.$$

The case when at least one of $x_1, x_2, \ldots, x_n$ is not in M is obvious.

This proves that μ is a fuzzy subgroupoid. Since the chain of subgroupoids $S_1 \subset S_2 \subset S_3 \subset \ldots$ is not terminating, μ has a strictly descending sequence of values. This contradicts that the set of values of any fuzzy subgroupoid is well-ordered. The proof is complete. □

3. Normal fuzzy subgroupoids

Definition 3.1. A fuzzy set μ of G is said to be *normal* if there exists $x \in G$ such that $\mu(x) = 1$.

A simple example of a normal fuzzy set is a characteristic function χ_S, where S is a fixed subset of G.

Note that if μ is a normal fuzzy subgroupoid of a unipotent groupoid $\mathcal{G}$, then $\mu(\theta) = 1$, and hence μ is normal in a unipotent groupoid iff $\mu(\theta) = 1$.

According to [13] we say that a fuzzy set μ is contained in a fuzzy set ρ (and denote this fact by $\mu \subseteq \rho$) iff $\mu(x) \leqslant \rho(x)$ for all $x \in G$. Obviously $\mu = \rho$ iff $\mu(x) = \rho(x)$ for all $x \in G$.

Proposition 3.2. *Let μ be a fuzzy subgroupoid of a unipotent groupoid $\mathcal{G}$. Then a fuzzy set μ^+ defined on $\mathcal{G}$ by $\mu^+(x) = \mu(x) + 1 - \mu(\theta)$, is a normal fuzzy subgroupoid of $\mathcal{G}$ such that $\mu \subseteq \mu^+$.*

Proof. If $x_1, x_2, \ldots, x_n \in G$, then

$$\begin{aligned}
\mu^+(f(x_1^n)) &= \mu(f(x_1^n)) + 1 - \mu(\theta) \\
&\geqslant \min\{\mu(x_1), \mu(x_2), \ldots, \mu(x_n)\} + 1 - \mu(\theta) \\
&= \min\{\mu(x_1) + 1 - \mu(\theta), \ldots, \mu(x_n) + 1 - \mu(\theta)\} \\
&= \min\{\mu^+(x_1), \mu^+(x_2), \ldots, \mu^+(x_n)\}\,,
\end{aligned}$$

which proves that μ^+ is a fuzzy subgroupoid of $\mathcal{G}$. Moreover $\mu^+(\theta) = \mu(\theta) + 1 - \mu(\theta) = 1 \geqslant \mu^+(x) \geqslant \mu(x)$ for all $x \in G$. □

Corollary 3.3. *Let μ and μ^+ be as in the above Proposition. Then $\mu^+(x_0) = 0$ (for some $x_0 \in G$) implies $\mu(x_0) = 0$. Moreover μ is normal iff μ^+ is normal.* □

Corollary 3.4. *For every fuzzy subgroupoid μ defined on a unipotent groupoid $\mathcal{G}$ we have $(\mu^+)^+ = \mu^+$. Moreover, if μ is normal, then $(\mu^+)^+ = \mu$.* □

Proposition 3.5. *If μ and ν are two fuzzy subgroupoids of a unipotent groupoid $\mathcal{G}$ such that $\mu \subseteq \nu$ and $\mu(\theta) = \nu(\theta)$, then $G_\mu \subseteq G_\nu$.*

Proof. Straightforward. □

Corollary 3.6. *If μ and ν are normal fuzzy subgroupoids of a unipotent groupoid $\mathcal{G}$ such that $\mu \subseteq \nu$, then $G_\mu \subseteq G_\nu$.* □

Proposition 3.7. *If for a fuzzy subgroupoid μ defined on a unipotent groupoid $\mathcal{G}$ there exists a fuzzy subgroupoid ν defined on $\mathcal{G}$ such that ν^+ is contained in μ, then μ is normal.*

Proof. Straightforward. □

Denote by $\mathcal{N}(G)$ the set of all normal fuzzy subgroupoids of G. Note that $\mathcal{N}(G)$ is partially ordered by inclusion.

Proposition 3.8. *Let μ be a non-constant fuzzy subgroupoid of a unipotent groupoid $\mathcal{G}$. If μ is a maximal element of $(\mathcal{N}(G), \subseteq)$, then μ takes only two values: 0 and 1.*

Proof. Observe that $\mu(\theta) = 1$ since μ is normal. Let $x \in G$ be such that $\mu(x) \neq 1$. We claim that $\mu(x) = 0$. If not, then there exists $a \in G$ such that $0 < \mu(a) < 1$. Let ν be a fuzzy set in G defined by $\nu(x) := \frac{1}{2}[\mu(x) + \mu(a)]$ for all $x \in G$. Then clearly ν is well-defined, and we have that for all $x \in G$,

$$\nu(\theta) = \frac{1}{2}[\mu(\theta) + \mu(a)] = \frac{1}{2}[1 + \mu(a)] \geqslant \frac{1}{2}[\mu(x) + \mu(a)] = \nu(x).$$

Moreover, for any $x_1, x_2, \ldots, x_n \in G$ we obtain

$$\begin{aligned} \nu(f(x_1^n)) &= \tfrac{1}{2}\left[\mu(f(x_1^n)) + \mu(a)\right] \geqslant \tfrac{1}{2}\left[\min\{\mu(x_1), \ldots, \mu(x_n)\} + \mu(a)\right] \\ &= \min\{\tfrac{1}{2}\left[\mu(x_1) + \mu(a)\right], \ldots, \tfrac{1}{2}\left[\mu(x_n) + \mu(a)\right]\} \\ &= \min\{\nu(x_1), \nu(x_2), \ldots, \nu(x_n)\}. \end{aligned}$$

Hence ν is a fuzzy subgroupoid of $\mathcal{G}$ and ν^+ is defined by $\nu^+(x) = \nu(x) + 1 - \nu(\theta)$ is normal (Proposition 3.2). Thus $\nu^+ \in \mathcal{N}(G)$ and $\nu^+(x) \geqslant \mu(x)$ for all $x \in G$.

Note that

$$\begin{aligned} \nu^+(a) &= \nu(a) + 1 - \nu(\theta) \\ &= \tfrac{1}{2}\left[\mu(a) + \mu(a)\right] + 1 - \tfrac{1}{2}\left[\mu(\theta) + \mu(a)\right] \\ &= \tfrac{1}{2}\left[\mu(a) + 1\right] > \mu(a) \end{aligned}$$

and $\nu^+(a) < 1 = \nu^+(\theta)$. Hence ν^+ is non-constant, and μ is not a maximal element of $\mathcal{N}(G)$. This is a contradiction. □

Definition 3.9. A fuzzy set μ defined on $\mathcal{G}$ is called *maximal* if it is non-constant and μ^+ is a maximal element of the poset $(\mathcal{N}(G), \subseteq)$.

Theorem 3.10. *Let μ be a maximal fuzzy subgroupoid of a unipotent groupoid $\mathcal{G}$. Then*

(i) *μ is normal,*
(ii) *μ takes only the values 0 and 1,*
(iii) *$G_\mu = \{x \in G : \mu(x) = \mu(\theta)\}$ is a maximal subgroupoid of $\mathcal{G}$,*
(iv) *$\mu_{G_\mu} = \mu$.*

Proof. Let μ be a maximal fuzzy subgroupoid. Then μ^+ is a non-constant maximal element of the poset $(\mathcal{N}(G), \subseteq)$ and (by Proposition 3.8) takes only the values 0 and 1. But $\mu^+(x) = 1$ iff $\mu(x) = \mu(\theta)$, and $\mu^+(x) = 0$ iff $\mu(x) = \mu(\theta) - 1$, which by Corollary 3.3 gives $\mu(x) = 0$. Hence $\mu(\theta) = 1$, i.e., μ is normal and $\mu^+ = \mu$. This proves (i) and (ii).

(iii) G_μ is a proper subgroupoid because μ is non-constant. Let S be a subgroupoid of $\mathcal{G}$ such that $G_\mu \subseteq S$. Noticing that, for any subgroupoids A and B of $\mathcal{G}$, $A \subseteq B$ iff $\mu_A \subseteq \mu_B$, then we obtain

$\mu = \mu_{G_\mu} \subseteq \mu_S$. Since μ and μ_S are normal and $\mu = \mu^+$ is a maximal element of $\mathcal{N}(G)$, we have that either $\mu = \mu_S$ or $\mu_S = \mathbf{1}$, where $\mathbf{1}$ is a fuzzy set defined by $\mathbf{1}(x) = 1$ for all $x \in G$. The later case implies that $S = G$. If $\mu = \mu_S$, then obviously $G_\mu = G_{\mu_S} = S$. This proves that G_μ is a maximal subgroupoid of $\mathcal{G}$.

(iv) Clearly $\mu_{G_\mu} \subseteq \mu$ and μ_{G_μ} takes only the values 0 and 1. Let $x \in G$. If $\mu(x) = 0$, then obviously $\mu \subseteq \mu_{G_\mu}$. If $\mu(x) = 1$, then $x \in G_\mu$, and so $\mu_{G_\mu}(x) = 1$. This shows that $\mu \subseteq \mu_{G_\mu}$. □

Definition 3.11. A normal fuzzy subgroupoid μ of $\mathcal{G}\mathcal{G}$ is called *completely normal* if there exists $x \in G$ such that $\mu(x) = 0$.

The set of all completely normal fuzzy subgroupoids of $\mathcal{G}$ is denoted by $\mathcal{C}(G)$. It is clear that $\mathcal{C}(G) \subseteq \mathcal{N}(G)$.

Proposition 3.12. *If $\mathcal{G}$ is a unipotent groupoid, then any non-constant maximal element of $(\mathcal{N}(G), \subseteq)$ is also maximal in $(\mathcal{C}(G), \subseteq)$.*

Proof. Let μ be a non-constant maximal element of $(\mathcal{N}(G), \subseteq)$. By Proposition 3.8, μ takes only the values 0 and 1, and so $\mu(\theta) = 1$ and $\mu(x) = 0$ for some $x \in G$. Hence $\mu \in \mathcal{C}(G)$.

Assume that there exists $\nu \in \mathcal{C}(G)$ such that $\mu \subseteq \nu$. Obviously $\mu \subseteq \nu$ also in $\mathcal{N}(G)$. Since μ is maximal in $(\mathcal{N}(G), \subseteq)$ and ν is non-constant, therefore $\mu = \nu$. Thus μ is maximal element of $(\mathcal{C}(G), \subseteq)$. □

Proposition 3.13. *Any maximal fuzzy subgroupoid of a unipotent groupoid is completely normal.*

Proof. Let μ be a maximal fuzzy subgroupoid. By Theorem 3.10 μ is normal and $\mu = \mu^+$ takes only the values 0 and 1. Since μ is non-constant, it follows that $\mu(\theta) = 1$ and $\mu(x) = 0$ for some $x \in G$, which completes the proof. □

Proposition 3.14. *Let μ be a fuzzy subgroupoid of a unipotent groupoid $\mathcal{G}$. If $h : [0, \mu(\theta)] \to [0, 1]$ is an increasing function, then a fuzzy set μ_h defined on $\mathcal{G}$ by $\mu_h(x) = h(\mu(x))$ is a fuzzy groupoid.*

Moreover, μ_h *is normal iff* $h(\mu(\theta)) = 1$.

Proof. Since h is increasing, then for all $x_1, x_2, \ldots, x_n \in G$ we have

$$\begin{aligned} \mu_h(f(x_1^n)) = h(\mu(f(x_1^n)) &\geqslant h(\min\{\mu(x_1), \mu(x_2), \ldots, \mu(x_n)\}) \\ &= \min\{h(\mu(x_1)), h(\mu(x_2)), \ldots, h(\mu(x_n))\} \\ &= \min\{\mu_h(x), \mu_h(x_2), \ldots, \mu_h(y)\}\,. \end{aligned}$$

This proves that μ_h is a fuzzy subgroupoid. The rest is obvious. □

4. Cartesian products of fuzzy subgroupoids

According to [3], the *Cartesian product* of two fuzzy sets μ and ν in G is defined by

$$(\mu \times \nu)(x, y) = \min\{\mu(x),\, \nu(y)\}$$

for all $x, y \in G$.

It is clear that $\mu \times \nu = \nu \times \mu$ and $L(\mu \times \nu, t) = L(\mu, t) \times L(\nu, t)$ for all $t \in [0, 1]$. If μ and ν are normal, then also $\mu \times \nu$ is normal. Moreover, if μ and ν are fuzzy subgroupoids in $\mathcal{G}$, then $\mu \times \nu$ is a fuzzy subgroupoid in $\mathcal{G} \times \mathcal{G}$, but not converse. Indeed, as an example we can consider two fuzzy sets μ and ν such that $\mu(x) \leqslant \nu(x)$ for all $x \in G$. The Cartesian product of μ and ν depends only on μ. It is a fuzzy subgroupoid in $\mathcal{G} \times \mathcal{G}$ iff μ is a fuzzy subgroupoid in $\mathcal{G}$.

Theorem 4.1. *Let* μ *and* ν *be two fuzzy sets in a unipotent* n*-ary groupoid* $\mathcal{G}$ *such that* $\mu \times \nu$ *is a fuzzy subgroupoid of* $\mathcal{G} \times \mathcal{G}$*. Then*

- (i) *either* $\mu(x) \leqslant \mu(\theta)$ *or* $\nu(x) \leqslant \nu(\theta)$ *for all* $x \in G$,
- (ii) *if* $\mu(x) \leqslant \mu(\theta)$ *for all* $x \in G$, *then either* $\mu(x) \leqslant \nu(\theta)$ *or* $\nu(x) \leqslant \nu(\theta)$,
- (iii) *if* $\nu(x) \leqslant \nu(\theta)$ *for all* $x \in G$, *then either* $\mu(x) \leqslant \mu(\theta)$ *or* $\nu(x) \leqslant \mu(\theta)$,
- (iv) *either* μ *or* ν *is a fuzzy subgroupoid of* $\mathcal{G}$.

Proof. (i) Suppose that $\mu(x) > \mu(\theta)$ and $\nu(y) > \nu(\theta)$ for some $x, y \in G$. Then

$$(\mu \times \nu)(x, y) = \min\{\mu(x),\, \nu(y)\} > \min\{\mu(\theta),\, \nu(\theta)\} = (\mu \times \nu)(\theta, \theta),$$

which is a contradiction. Thus either $\mu(x) \leqslant \mu(\theta)$ or $\nu(x) \leqslant \nu(\theta)$ for all $x \in G$.

(ii) Assume that $\mu(x) > \nu(\theta)$ and $\nu(y) > \nu(\theta)$ for some $x, y \in G$. Then $(\mu \times \nu)(\theta, \theta) = \min\{\mu(\theta),\, \nu(\theta)\} = \nu(\theta)$ and hence

$$(\mu \times \nu)(x, y) = \min\{\mu(x),\, \nu(y)\} > \nu(\theta) = (\mu \times \nu)(\theta, \theta).$$

This is a contradiction. Hence (ii) holds.

(iii) Similarly as (ii).

(iv) Let F be the n-ary operation in $\mathcal{G} \times \mathcal{G}$ induced by f. Since, by (i), either $\mu(x) \leqslant \mu(\theta)$ or $\nu(x) \leqslant \nu(\theta)$ for all $x \in G$, without loss of generality we may assume that $\nu(x) \leqslant \nu(\theta)$. It follows from (iii) that either $\mu(x) \leqslant \mu(\theta)$ or $\nu(x) \leqslant \mu(\theta)$.

If $\nu(x) \leqslant \mu(\theta)$ for all $x \in G$, then

$$\begin{aligned}
\nu(f(x_1^n)) &= \min\{\mu(\theta), \nu(f(x_1^n))\} = (\mu \times \nu)(\theta, f(x_1^n)) \\
&= (\mu \times \nu)(\, f(\theta, \theta, \ldots, \theta), f(x_1^n)\,) \\
&= (\mu \times \nu)(\, F((\theta, x_1),\, (\theta, x_2),\, \ldots,\, (\theta, x_n))\,) \\
&\geqslant \min\{\, (\mu \times \nu)(\theta, x_1),\, (\mu \times \nu)(\theta, x_2),\, \ldots,\, (\mu \times \nu)(\theta, x_n)\,\} \\
&= \min\{\, \mu(\theta),\, \nu(x_1),\, \nu(x_2),\, \ldots,\, \nu(x_n)\,\} \\
&= \min\{\, \nu(x_1),\, \nu(x_2),\, \ldots,\, \nu(x_n)\,\},
\end{aligned}$$

which proves that ν is a fuzzy subgroupoid in $\mathcal{G}$.

If $\nu(x) \leqslant \mu(\theta)$ is not satisfied, then $\nu(y) > \mu(\theta)$ for some $y \in G$ and, by the assumption, $\mu(x) \leqslant \mu(\theta)$ for all $x \in G$, which gives $\nu(\theta) \geqslant \nu(y) > \mu(\theta) \geqslant \mu(x)$, i.e. $\nu(\theta) \geqslant \mu(x)$ for all $x \in G$. Hence $(\mu \times \nu)(x, \theta) = \min\{\mu(x),\, \nu(\theta)\} = \mu(x)$ and, in the consequence,

$$\begin{aligned}
\mu(f(x_1^n)) &= (\mu \times \nu)(f(x_1^n),\, \theta) = (\mu \times \nu)(f(x_1^n),\, f(\theta, \theta, \ldots, \theta)\,) \\
&= (\mu \times \nu)(\, F((x_1, \theta),\, (x_2, \theta),\, \ldots,\, (x_n, \theta))\,) \\
&\geqslant \min\{\, (\mu \times \nu)(x_1, \theta),\, (\mu \times \nu)(x_2, \theta),\, \ldots, (\mu \times \nu)(x_n, \theta)\,\} \\
&= \min\{\, \mu(x_1),\, \mu(x_2),\, \ldots, \mu(x_n)\}
\end{aligned}$$

which proves that μ is a fuzzy subgroupoid of $\mathcal{G}$.

This completes the proof. $\square$

The *composition* (*intersection*) of fuzzy sets μ and ν in G defined as

$$(\mu\cdot\nu)(x) = (\mu\cap\nu)(x) = \min\{\,\mu(x),\,\nu(x)\,\}$$

for all $x\in G$ (cf. [13]) is a special product connected with $\mu\times\nu$.

The relationship between fuzzy subgroupoids $\mu\times\nu$ and $\mu\cdot\nu$ can be viewed by the following diagram:

$$\begin{tikzcd}
G \arrow[r, "d"] \arrow[d, "\mu\cdot\nu"'] & G\times G \arrow[dl, "\mu\times\nu"'] \arrow[d, shift right=2, "\mu"'] \arrow[d, shift left=2, "\nu"] \\
I & I\times I \arrow[l, "T"']
\end{tikzcd}$$

where $I=[0,1]$, $T(\alpha,\beta)=\min\{\alpha,\beta\}$ and $d: G\to G\times G$ is defined by $d(x)=(x,x)$ (cf. [7]).

Obviously $\mu\cdot\nu=\mu\cdot\nu$ and $\mu\cdot\mu=\mu$. If μ and ν are normal, then so is $\mu\cdot\nu$. Similarly, if μ and ν are fuzzy subgroupoids in $\mathcal{G}$, then so is $\mu\cdot\nu$. The converse is not true, because for $\mu\subseteq\nu$ the product $\mu\cdot\nu$ depends only on μ.

Definition 4.2. Let ν be a fuzzy set in G. The *strongest fuzzy relation* on G is a fuzzy set $\rho_\nu : G\times G\to[0,1]$ defined by

$$\rho_\nu(x,y) = \min\{\,\nu(x),\,\nu(y)\,\}$$

for all $x,y\in G$.

ρ_ν is an extension of a fuzzy set ν defined in G to the Cartesian product of G. Obviously $\rho_\nu(x,x)=\nu(x)$ for all $(x,x)\in G\times G$.

It is clear that a fuzzy set ρ_ν is normal in $G\times G$ iff a fuzzy set ν is normal in G.

The following result is proved in [3].

Proposition 4.3. *Let ν be a fuzzy set in G. Then all levels of ρ_ν have the form $L(\rho_\nu, t) = L(\nu, t) \times L(\nu, t)$, where $t \in [0,1]$.* $\square$

Applying this result to our Theorem 2.3, we obtain

Corollary 4.4. *Let ν be a fuzzy set in an n-ary groupoid $\mathcal{G}$. Then ρ_ν is a fuzzy subgroupoid of $\mathcal{G} \times \mathcal{G}$ iff all nonempty sets of the form $L(\nu, t) \times L(\nu, t)$ are subgroupoids of $\mathcal{G} \times \mathcal{G}$.* $\square$

Proposition 4.5. *If ν is a fuzzy subgroupoid of $\mathcal{G}$, then ρ_ν is a fuzzy subgroupoid of $\mathcal{G} \times \mathcal{G}$.*

Proof. Indeed, for all $x_i, y_i \in G$, $i = 1, 2, \dots, n$, we have

$$\begin{aligned}
&\rho_\nu(F((x_1, y_1), \, \dots \, , (x_n, y_n))) \\
&\qquad = \rho_\nu(f(x_1^n), \, f(y_1^n)) = \min\{ \nu(f(x_1^n)), \, \nu(f(y_1^n)) \} \\
&\qquad \geqslant \min\{ \min\{\nu(x_1), \dots, \nu(x_n)\}, \, \min\{\nu(y_1), \dots, \nu(y_n)\}\} \\
&\qquad = \min\{ \min\{\nu(x_1), \nu(y_1)\}, \, \dots, \, \min\{\nu(x_n), \nu(y_n)\} \} \\
&\qquad = \min\{ \rho_\nu(x_1, y_1), \, \rho_\nu(x_2, y_2), \dots, \, \rho_\nu(x_n, y_n) \} ,
\end{aligned}$$

which completes the proof. $\square$

Theorem 4.6. *Let ν be a fuzzy set of a unipotent n-ary groupoid $\mathcal{G}$. Then ρ_ν is a fuzzy subgroupoid of $\mathcal{G} \times \mathcal{G}$ iff ν is a fuzzy subgroupoid.*

Proof. Assume that ρ_ν is a fuzzy subgroupoid of $\mathcal{G} \times \mathcal{G}$. Since $\mathcal{G}$ is a unipotent groupoid, we have

$$F((x, y), \dots, (x, y)) = (\, f(x, \dots, x), \, f(y, \dots, y) \,) = (\theta, \theta)$$

for all $(x, y) \in G \times G$. Thus $\rho_\nu(\theta, \theta) \geqslant \rho_\nu(x, y)$ (by Lemma 2.2) and, in the consequence,

$$\nu(\theta) = \min\{\nu(\theta), \nu(\theta)\} = \rho_\nu(\theta, \theta) \geqslant \rho_\nu(x, x) = \nu(x)$$

for all $x \in G$.

Therefore

$$\begin{aligned}
\nu(f(x_1^n)) &= \min\{\nu(f(x_1^n)),\, \nu(\theta)\} \\
&= \rho_\nu(f(x_1^n),\, f(\theta,\dots,\theta)) = \rho_\nu(\, F(\,(x_1,\theta),\dots,(x_n,\theta)\,)\,) \\
&\geqslant \min\{\rho_\nu(x_1,\theta),\, \dots,\, \rho_\nu(x_n,\theta)\} \\
&= \min\{\,\min\{\nu(x_1),\nu(\theta)\},\, \dots,\, \min\{\nu(x_n),\nu(\theta)\}\,\} \\
&= \min\{\nu(x_1),\, \nu(x_2),\, \dots,\, \nu(x_n)\}.
\end{aligned}$$

This proves that ν is a fuzzy subgroupoid of $\mathcal{G}$.

The converse statement follows from the previous Proposition. $\square$

5. Fuzzy ideals

Definition 5.1. A fuzzy set μ on an n-ary groupoid (G,f) is called a *fuzzy k-ideal* if

$$\mu(f(x_1^n)) \geqslant \mu(x_k)$$

holds for all $x_1,x_2,\dots,x_n \in G$. If μ is a fuzzy k-ideal for every $k=1,2,\dots,n$, then it is called a *fuzzy ideal.*

A simple example of a (normal) fuzzy ideal is a characteristic function χ_S, where S is an ideal of (G,f).

It is clear that every fuzzy k-ideal is a fuzzy subgroupoid, but there are fuzzy subgroupoids which are not fuzzy ideals.

Example 5.2. Let f be an n-ary operation ($n \geqslant 2$) defined on the set $G=\{m\in N : m\geqslant 1\}$ by the formula

$$f(x_1^n) = x_1\cdot x_2\cdot\,\dots\,\cdot x_n + 2\,.$$

It is not difficult to see that a fuzzy set μ such that $\mu(x)=0.5$ for $x=2m$, and $\mu(x)=0.7$ for $x=2m+1$, is a fuzzy subgroupoid of (G,f). It is not a k-ideal because for $x_k=1$ and $x_i=2$, where $i\neq k$, we have $\mu(f(x_1^n))=0.5<\mu(x_k)$. Similarly, by routine calculations we know that $\nu(x)=1-\frac{1}{x}$ is a fuzzy k-ideal for every $k=1,2,\dots,n$. $\square$

Theorem 5.3. *A fuzzy set μ of an n-ary groupoid $\mathcal{G}$ is a fuzzy k-ideal iff for every $t \in [0,1]$, $L(\mu,t)$ is either empty or a k-ideal of $\mathcal{G}$.*

Proof. Assume that every nonempty level is a k-ideal (for fixed k). If $\mu(f(x_1^n)) \geqslant \mu(x_k)$ is not true, then there are $y_1, y_2, \ldots, y_n \in G$ such that $\mu(f(y_1^n)) < \mu(y_k)$. But in this case $\mu(f(y_1^n)) < t < \mu(y_k)$, for

$$t = \frac{1}{2}(\,\mu(f(y_1^n)) + \mu(y_k)\,),$$

which gives $y_k \in L(\mu,t)$. Thus $f(y_1^n) \in L(\mu,t)$, because $L(\mu,t)$ is a k-ideal. Hence $\mu(f(y_1^n)) \geqslant t$.

Obtained contradiction proves that μ must be a fuzzy k-ideal.

The converse statement is obvious. □

Theorem 5.4. *Any subgroupoid of $\mathcal{G}$ can be realized as a level k-ideal of some fuzzy k-ideal of $\mathcal{G}$.*

Proof. It is straightforward by Theorem 2.4. □

It is not difficult to see that all results from the previous parts will be true if we replace the word "subgroupoid" by "k-ideal" or by "ideal". $\mathcal{N}(G)$ denotes in this case the set of all normal fuzzy k-ideals (ideals, respectively).

6. Fuzzification of quasigroups

An *n-ary quasigroup* (briefly; *quasigroup*) is defined as a groupoid $\mathcal{G}$ in which for all $x_0, x_1, \ldots, x_n \in G$ and for all $i = 1, 2, \ldots, n$ there exists a uniquely determined $z_i \in G$ such that

$$f(x_1^{i-1}, z_i, x_{i+1}^n) = x_0\,.$$

In any n-ary quasigroup $\mathcal{G}$ for every $s = 1, 2, \ldots, n$ one can define the *s-th inverse n-ary operation* $f^{(s)}$ putting

$$f^{(s)}(x_1^n) = y \quad\Longleftrightarrow\quad f(x_1^{s-1}, y, x_{s+1}^n) = x_s\,.$$

Obviously, the operation $f^{(s)}$ is the s-inverse operation for the operation f iff

$$f^{(s)}(x_1^{s-1}, f(x_1^n), x_{s+1}^n) = x_s$$

for all $x_1, \ldots, x_n \in G$ (cf. [2]). In this case we have also

$$f(x_1^{s-1}, f^{(s)}(x_1^n), x_{s+1}^n) = x_s.$$

Therefore (as in the binary case) the class of all n-ary quasigroups may be treated as the variety of equationally definable algebras with $n+1$ fundamental operations $f, f^{(1)}, \ldots, f^{(n)}$. (For $n = 2$, $f^{(1)}$ and $f^{(2)}$ are the left and right inverse operation in the sense of [1] and [10]).

A nonempty subset S of G is called a *subquasigroup* of $\mathcal{G}$ if it is an n-ary quasigroup with respect to f.

According to [2], a nonempty subset S of an n-ary quasigroup (G, f) is an *n-ary subquasigroup* iff it is closed with respect to $n+1$ operations $f, f^{(1)}, \ldots, f^{(n)}$, i.e., iff $g(x_1^n) \in G$ for all $x_1, \ldots, x_n \in G$ and all $g \in \mathcal{F} = \{f, f^{(1)}, f^{(2)}, \ldots, f^{(n)}\}$.

Definition 6.1. A fuzzy set μ defined on G is called a *fuzzy subquasigroup* of an n-ary quasigroup $\mathcal{G} = (G, f)$ if

$$\mu(f(x_1^n)) \geqslant \min\{\mu(x_1), \ldots, \mu(x_n)\}$$

for all $x_1, \ldots, x_n \in G$ and all $g \in \mathcal{F}$.

Proposition 6.2. *If μ is a fuzzy subquasigroup of $\mathcal{G}$, then*

$$\min\{\mu(x_1), \ldots, \mu(x_{i-1}), \mu(g(x_1^n)), \mu(x_{i+1}, \ldots, \mu(x_n)\}$$
$$\min\{\mu(x_1), \ldots, \mu(x_i), \ldots, \mu(x_n)\}$$

for all $x_1, \ldots, x_n \in G$ and all $g \in \mathcal{F}$.

Proof. Indeed, for $g = f$ we have

$$\begin{aligned} &\min\{\mu(x_1), \ldots, \mu(x_{i-1}), \mu(f(x_1^n)), \mu(x_{i+1}, \ldots, \mu(x_n)\} \\ &\geqslant \min\{\mu(x_1), \ldots, \mu(x_{i-1}), \min\{\mu(x_1), \ldots \\ &\qquad \ldots, \mu(x_i), \ldots, \mu(x_n)\}, \mu(x_{i+1}, \ldots, \mu(x_n)\} \\ &= \min\{\mu(x_1), \ldots, \mu(x_{i-1}), \mu(x_i), \mu(x_{i+1}, \ldots, \mu(x_n)\} \end{aligned}$$

$$\begin{aligned}
&= \min\{\mu(x_1),\dots,\mu(x_{i-1}),\mu(f^{(i)}(x_1^{i-1},f(x_1^n),x_{i+1}^n),\mu(x_{i+1},\dots,\mu(x_n)\}\\
&\geqslant \min\{\mu(x_1),\dots,\mu(x_{i-1}),\min\{\mu(x_1),\dots\\
&\qquad \dots,\mu(x_{i-1}),\mu(f(x_1^n)),\mu(x_{i+1}),\dots,\mu(x_n)\},\mu(x_{i+1},\dots,\mu(x_n)\}\\
&= \min\{\mu(x_1),\dots,\mu(x_{i-1}),\mu(f(x_1^n)),\mu(x_{i+1},\dots,\mu(x_n)\}
\end{aligned}$$

for all $x_1,\dots,x_n \in G$.

In the case $g = f^{(i)}$ the proof is analogous, but we must use the identity $f(x_1^{i-1}, f^{(i)}(x_1^n), x_{i+1}^n) = x_i$. □

Theorem 6.3. *A fuzzy set μ of an n-ary groupoid $\mathcal{G}$ is a fuzzy subgroupoid iff for every $t \in [0,1]$, $L(\mu,t)$ is either empty or a subgroupoid of $\mathcal{G}$.*

Proof. Straightforward. □

Theorem 6.4. *Any subgroupoid of $\mathcal{G}$ can be realized as a level subgroupoid of some fuzzy subgroupoid of $\mathcal{G}$.*

It is not difficult to see that similar results as results presented in parts 2, 3 and 4 can be proved for n-ary quasigroups.

7. Anti fuzzy subgroupoids

The concept of anti fuzzy subgroups (in the binary case) was introduced by R. Biswas in [4] and was studied by many authors. we generalize this concept to the n-ary case.

Definition 7.1. A fuzzy set ρ defined on an n-ary groupoid $\mathcal{G}$ will be called an *anti fuzzy subgroupoid* if

$$\rho(f(x_1^n)) \leqslant \max\{\rho(x_1),\,\dots,\,\rho(x_n)\}$$

holds for all $x_1,\dots,x_n \in G$.

The set

$$U(\rho, t0 = \{x \in G : \rho(x) \leqslant t\},$$

where $t \in [0, 1$ is fixed, is called a *level cut of* ρ.

It is clear that (in general) anti fuzzy subgroupoids are not fuzzy subgroupoids and conversel, fuzzy subgroupoids are not anti fuzzy subgroupoids.

It is not difficult to see that the following two results are true.

Proposition 7.2 *If ρ is a fuzzy subgroupoid of a unipotent groupoid (G, f, θ), then $\rho(\theta) \leqslant \rho(x)$ for all $x \in G$.*

Proposition 7.2 *A fuzzy set ρ of an n-ary groupoid $\mathcal{G}$ is an anti fuzzy subgroupoid iff for every $t \in [0, 1]$, $U(\rho, t)$ is either empty or a subgroupoid of $\mathcal{G}$.*

Basing on this two simple propositions it is not difficult to observe that the most part of results presented in this article is valid also for anti fuzzy subgroupoids, ideals and subquasigroups.

Note also that fuzzy subgroupoids (ideals, subquasigroups) and anti fuzzy subgroupoids (ideals, subquasigroups) can be used to the construction of intuitionistic fuzzy groupoids (respectively: ideals and subquasigroups).

References

[1] **V. D. Belousov**: *Foundations of the theory of quasigroups and loops*, Nauka, Moscow 1967.

[2] **V. D. Belousov**: *n-Ary Quasigroups*, (Russian), Ştiinţa, Chişinău 1972.

[3] **P. Bhattacharya, N. P. Mukherjee**: *Fuzzy relations and fuzzy groups*, Inform. Sci. **36** (1985), 267 – 282.

[4] **R. Biswas**: *Fuzzy subgroups and anti fuzzy subgroups*, Fuzzy Sets and Systems **35** (1990), 121 – 124.

[5] **W. A. Dudek**: *Unipotent n-ary groups*, Demonstratio Math. **24** (1991), 75 – 81.

[6] **W. A. Dudek**: *Varieties of polyadic groups*, Filomat **9** (1995), 657 – 674.

[7] **W. A. Dudek**: *Fuzzy subquasigroups*, Quasigroups and Related Systems **5** (1998), 81 – 98.

[8] **W. A. Dudek and Y. B. Jun**: *Fuzzy BCC-ideals in BCC-groupoids*, Math. Montisnigri **10** (1999), 21 – 30.

[9] **Y. B. Jun, J. Neggers and H. S. Kim**: *Normal L-fuzzy ideals in semirings*, Fuzzy Sets and Systems **82** (1996), 383 – 386.

[10] **H. O. Pflugfelder**: *Quasigroups and loops: introduction*, Sigma Series in Pure Math., vol. **7**, Heldermann Verlag, Berlin 1990.

[11] **A. Rosenfeld**: *Fuzzy groups*, J. Math. Anal. Appl. **35** (1971), 512 – 517.

[12] **F. M. Sioson**: *Ideals in* $(m+1)$*–semigroups*, Ann. Math. Pura Appl. **68** (1965), 161 – 200.

[13] **L. A. Zadeh**: *Fuzzy sets*, Inform. Control **8** (1965), 338 – 353.

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