Quasigroups and Related Systems **7** (2000), 37 – 44

Squares in quadratical quasigroups

Vladimir Volenec

Abstract

"Geometrical" concept of square is defined and investigated in any quadratical quasigroup.

A groupoid $(Q, \cdot)$ is said to be *quadratical* if the identity

$$ab \cdot a = ca \cdot bc \tag{1}$$

holds and the equation $ax = b$ has a unique solution $x \in Q$ for any $a, b \in Q$ (cf. [10] and [3]). Every quadratical groupoid $(Q, \cdot)$ is a quasigroup, i.e. the equation $xa = b$ has a unique solution $x \in Q$ for any $a, b \in Q$. In a quadratical quasigroup $(Q, \cdot)$ the identities

$$aa = a \qquad \text{(idempotency)}, \tag{2}$$

$$a \cdot ba = ab \cdot a \qquad \text{(elasticity)}, \tag{3}$$

$$ab \cdot a = ba \cdot b, \tag{4}$$

$$ab \cdot cd = ac \cdot bd \qquad \text{(mediality)} \tag{5}$$

and the equivalency

$$ab = c \iff bc = ca \tag{6}$$

hold (cf. [10]).

If C is the set of all points of an Euclidean plane and if a groupoid $(C, \cdot)$ is defined so that $aa = a$ for any $a \in C$ and for any two

2000 Mathematics Subject Classification: 20N05
Keywords: quadratical quasigroup

different points $a, b \in C$ the point ab is the centre of the positively oriented square with two adjacent vertices a and b (Fig. 1), then $(C, \cdot)$ is a quadratical quasigroup. The figures in this quasigroup $(C, \cdot)$ can be used for illustration of "geometrical" relations in any quadratical quasigroup $(Q, \cdot)$ and for motivation of the study of this quasigroup.

From now on let $(Q, \cdot)$ be any quadratical quasigroup. The elements of the set Q are said to be *points*.

If an operation $\bullet$ is defined on the set Q by

$$a \bullet b = ab \cdot a = ca \cdot bc, \tag{7}$$

then $(Q, \bullet)$ is an idempotent medial commutative quasigroup (cf. [2]), i.e. the identities

$$\begin{aligned} a \bullet a &= a, && (8)\\ (a \bullet b) \bullet (c \bullet d) &= (a \bullet c) \bullet (b \bullet d), && (9)\\ a \bullet b &= b \bullet a \end{aligned}$$

hold, and the operations $\cdot$ and $\bullet$ are mutually medial, i.e. the identity

$$ab \bullet cd = (a \bullet c)(b \bullet d) \tag{10}$$

holds. For any two points a and b the point $a \bullet b$ is said to be the *midpoint* of a and b (cf. Fig. 1).

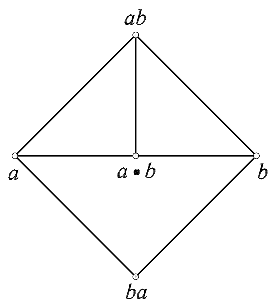

Fig. 1.

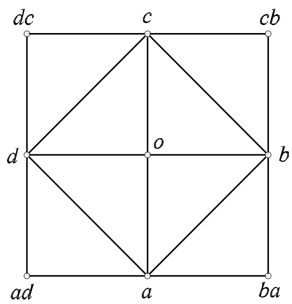

Fig. 2.

Theorem 1. *If any three of four products ab, bc, cd, da are equal, then all four products are equal (cf. Fig. 2).*

Proof. Let $ab = bc = cd$. The equality $bc = cd$ implies by (6) $db = c$. Therefore, by (4), we obtain

$$bd \cdot b = db \cdot d = cd = ab,$$

where from it follows $bd = a$ and then by (6) finally $da = ab$. □

Corollary 1. *Any three of four equalities*

$$ab = o, \quad bc = o, \quad cd = o, \quad da = o \tag{11}$$

imply the remaining equality. □

A quadrangle (a, b, c, d) is said to be a *square* and is denoted by $S(a, b, c, d)$ if any three of four products ab, bc, cd, da (and then all four products) are equal. More exactly, a quadrangle (a, b, c, d) is said to be a square with the *centre* o and is denoted by $S_o(a, b, c, d)$ if any three of four equalities (11) (and then all four equalities) hold.

If (e, f, g, h) is a cyclic permutation of (a, b, c, d), then $S(a, b, c, d)$ implies $S(e, f, g, h)$ and $S_o(a, b, c, d)$ implies $S_o(e, f, g, h)$.

The point o is said to be the *centre of a square on a segment* (a, b) if $S_o(a, b, c, d)$ holds for some points c and d.

Let us prove some simple results about squares.

Theorem 2. $S(a, b, c, d)$ *implies* $S_o(a, b, c, d)$, *where* $o = a \bullet c = b \bullet d$. (*cf. Fig.* 2)

Proof. Let $S_o(a, b, c, d)$ holds. From (11) we obtain

$$o \overset{(2)}{=} oo = da \cdot cd \overset{(7)}{=} a \bullet c,$$

and analogously $o = b \bullet d$. □

Theorem 3. *The statement* $S(a, b, c, d)$ *is equivalent with any of four* (*and then all four*) *equalities*

$$ac = d, \quad bd = a, \quad ca = b, \quad db = c. \tag{12}$$

Proof. According to the proof of Theorem 1 $S(a,b,c,d)$ implies $bd = a$, $db = c$ and analogously $ac = d$, $ca = b$. Conversely, because of cyclical permutations of (a,b,c,d), it suffices to prove the implications

$$ac = d,\ bd = a \quad \Longrightarrow \quad S(a,b,c,d),$$
$$ac = d,\ ca = b \quad \Longrightarrow \quad S(a,b,c,d).$$

From $ac = d$ and $bd = a$ by (6) it follows $cd = da$ and $da = ab$ and then Theorem 1 implies $S(a,b,c,d)$.

If $ac = d$ and $ca = b$, then we obtain

$$ab = a \cdot ca \overset{(3)}{=} ac \cdot a = da = ac \cdot a \overset{(4)}{=} ca \cdot c = bc$$

and Theorem 1 implies $S(a,b,c,d)$ again. □

Corollary 2. *For any two points a and b it holds $S_{a \bullet b}(a, ba, b, ab)$ and $ba \bullet ab = a \bullet b$ (cf. Fig. 1).* □

Theorem 4. *Let $S_{o'}(a',b',c',d')$ holds. The statements $S_o(a,b,c,d)$, $S_{oo'}(aa',bb',cc',dd')$, $S_{o'o}(a'a,b'b,c'c,d'd)$ are equivalent.*

Proof. It is sufficient to prove that the equalities $ab = o$ and $aa' \cdot bb' = oo'$ are equivalent if $a'b' = o'$ holds. But, this is obvious, because of

$$ab \cdot o' = ab \cdot a'b' \overset{(5)}{=} aa' \cdot bb'.$$ □

For any point p we obviously have $S_p(p,p,p,p)$. Therefore:

Corollary 3. *The following three statements:*

$$S_o(a,b,c,d), \quad S_{po}(pa,pb,pc,pd), \quad S_{op}(ap,bp,cp,dp)$$

are mutually equivalent. □

Theorem 5. *$S_o(a,b,c,d)$ implies $S_o(ba,cb,dc,ad)$ and $ad \bullet ba = a$, $ba \bullet cb = b$, $cb \bullet dc = c$, $dc \bullet ad = d$ (cf. Fig. 2).*

Proof. $S_o(a,b,c,d)$ obviously implies $S_o(b,c,d,a)$ and according to Theorem 4 it follows $S_o(ba,cb,dc,ad)$ because of $oo \overset{(2)}{=} o$. Further we obtain

$$ad \bullet ba \overset{(10)}{=} (a \bullet b)(d \bullet a) \overset{(9)}{=} (a \bullet b)(a \bullet d) =$$
$$\overset{(10)}{=} aa \bullet bd \overset{(2)}{=} a \bullet bd \overset{(12)}{=} a \bullet a \overset{(8)}{=} a. \qquad \square$$

Theorem 6. *Let $S_{o'}(a',b',c',d')$ holds. The statements $S_o(a,b,c,d)$ and $S_{o\bullet o'}(a \bullet a', b \bullet b', c \bullet c', d \bullet d')$ are equivalent.*

Proof. It suffices to prove the equivalency of the equalities $ab = o$ and $(a \bullet a')(b \bullet b') = o \bullet o'$ if the equality $a'b' = o'$ holds. This is obvious because of

$$ab \bullet o' = ab \bullet a'b' \overset{(10)}{=} (a \bullet a')(b \bullet b'). \qquad \square$$

Corollary 4. $S_o(a,b,c,d) \iff S_{p\bullet o}(p \bullet a,\ p \bullet b,\ p \bullet c,\ p \bullet d)$. $\square$

Corollary 5. $S_o(a,b,c,d) \implies S_o(a \bullet b,\ b \bullet c,\ c \bullet d,\ d \bullet a)$. $\square$

Theorem 7. *If $ab = c$, $b \bullet c = d$, $c \bullet a = e$, $a \bullet b = f$, then $bc = ca = f$, $af = e$, $fb = d$ and $S_{c\bullet f}(e,f,d,c)$ (cf. Fig. 3).*

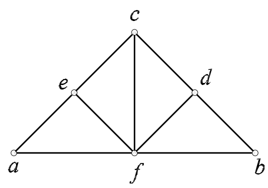

Fig. 3.

Proof. By Corollary 2 we have $S_f(a,ba,b,c)$ and $ba \bullet c = f$. Therefore, Corollary 4 implies $S_{c\bullet f}(e,f,d,c)$ because of $c \bullet a = e$, $c \bullet ba = f$, $c \bullet b = d$, $c \bullet c = c$. Further, we obtain

$$bc = b \cdot ab \overset{(3)}{=} ba \cdot b \overset{(7)}{=} b \bullet a = f,$$

$$ca = ab \cdot a \overset{(7)}{=} a \bullet b = f,$$

$$af = a(a \bullet b) \overset{(8)}{=} (a \bullet a)(a \bullet b) \overset{(10)}{=} aa \bullet ab \overset{(2)}{=} a \bullet c = e,$$

$$fb = (a \bullet b)b \overset{(8)}{=} (a \bullet b)(b \bullet b) \overset{(10)}{=} ab \bullet bb \overset{(2)}{=} c \bullet b = d. \qquad \square$$

Theorem 8. *If b' and c' are the centres of squares on the segments (c, a) and (a, b), then $b \bullet c$ is the centre of a square on the segment (c', b') (cf. Fig. 4).*

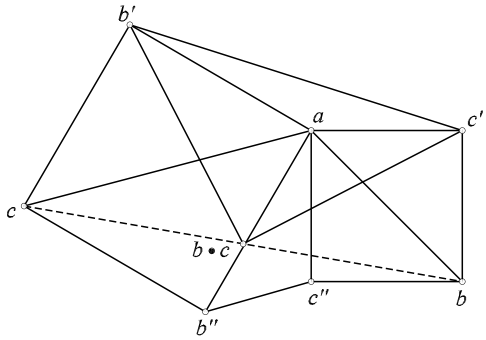

Fig. 4.

Proof. As $ca = b'$ and $ab = c'$, so we have $c'b' = ab \cdot ca \overset{(7)}{=} b \bullet c$. $\square$

In the case of the quasigroup $(C, \cdot)$ Theorem 8 proves a statement from [1], [7], [8], [9] and [11] which can be stated as a very famous problem of Captain Kidd burried treasure (cf. [6] and [4]).

The *rotation* about a point *a through a* (positively oriented) *right angle* is a transformation $x \longmapsto y$ of points such that $xy = a$.

Theorem 9. *If b', b'', c', c'' are the centres of squares on the segments (c, a), (a, c), (a, b), (b, a), then the rotation about the point $b \bullet c$ through a right angle maps the segment (c', b'') onto the segment (b', c'') (cf. Fig. 4).*

Proof. We have the equality from the above proof and analogously

$$b''c'' = ac \cdot ba \overset{(5)}{=} ab \cdot ca \overset{(7)}{=} b \bullet c. \qquad \square$$

Theorem 10. *Let* $S_o(p, a, u, b)$ *be fixed. If* (p, a', u', b') *is a square with the center* o*, then* $(o, b \bullet a', o', a \bullet b')$ *is a square with the centre* $o \bullet o'$ *and* $a \bullet b' = oo'$*,* $b \bullet a' = o'o$*,* $ba' = b'a = u \bullet u'$ *(cf. Fig.* 5*).*

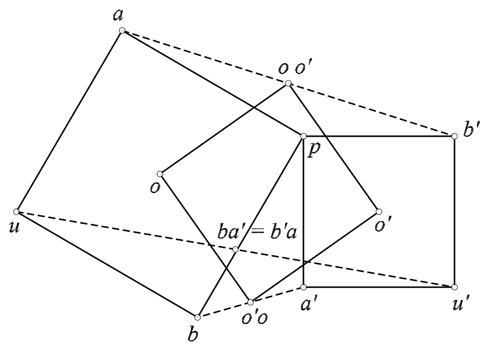

Fig. 5.

Proof. By Theorem 6 from $S_o(u, b, p, a)$ and $S_{o'}(p, a', u', b')$ it follows $S_{o \bullet o'}(u \bullet p, b \bullet a', p \bullet u', a \bullet b')$. But, $u \bullet p = o$ and $p \bullet u' = o'$ and we obtain $S_{o \bullet o'}(o, b \bullet a', o', a \bullet b')$, where from $oo' = a \bullet b'$, $o'o = b \bullet a'$ follows by Theorem 3. □

In the case of the quasigroup $(C, \cdot)$ Theorem 10 proves a result from [2] and [5].

References

[1] **L. Bankoff**: *Problem 540*, Crux Math. **6** (1980), 114.

[2] **A. I. Chegodaev**: *Application of geometric transformation in problem solving*, (in Russian), Mat. v škole 1962, 88 – 89.

[3] **W. A. Dudek**: *Quadratical quasigroups*, Quasigroups and Related Systems **4** (1997), 9 – 13.

[4] **A. Dunkels**: *Problem 400*, Crux Math. **4** (l978), 284.

[5] **V. M. Fishman**: *Solving of problems by geometric transformations*, (in Russian), Kvant 1975, No. 7, 30 – 35.

[6] **G. Gamow**: *One, Two, Three ... Infinity*, Viking Press, 1947.

[7] **Hoang Chung**: *Teaching students creative activity*, (in Russian), Mat. v škole 1966, No. 2, 77 – 81.

[8] **M. S. Klamkin and A. Liu**: *Problem 1605*, Crux Math. **17** (1991), 14.

[9] **E. A. Lihota**: *Variation of problem conditions in out of class activities*, (in Russian), Mat. v škole 1983, No. 6, cover pages 3 – 4.

[10] **V. Volenec**: *Quadratical groupoids*, Note di Mat. **13** (1993), 107 – 115.

[11] *Problem 3*, Math. Inform. Quart. **6** (1996), 213 – 214.

Department of Mathematics
University of Zagreb
10000 Zagreb
Bijenička c. 30
Croatia
e-mail: volenec@math.hr

Received June 20, 2000