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On intuitionistic fuzzy subquasigroups of quasigroups

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Abstract

In this paper, we introduce the notion of an intuitionistic fuzzy subquasigroup of a quasigroup G, and then some related properties are investigated. Characterizations of intuitionistic fuzzy subquasigroup of a quasigroup G are given.

1. Introduction

After the introduction of the concept of fuzzy sets by Zadeh [11], several researches were conducted on the generalizations of the notion of fuzzy set. The idea of "intuitionistic fuzzy set" was first published by Atanassov [1, 2], as a generalization of the notion of fuzzy set. Jun and Kim considered the intuitionistic fuzzification of near-rings [8]. In [6], Dudek introduced the notion of fuzzy subquasigroup of a quasigroup G. Fuzzy subquasigroups with respect to a norm are considered by Dudek and Jun in [7]. In this paper, we apply the concepts of intuitionistic fuzzy sets to subquasigroups of a quasigroup and introduce the notion of an intuitionistic fuzzy subquasigroup of a quasigroup, and then some related properties are investigated. Also, we discuss equivalence relations on the family of all intuitionistic fuzzy subquasigroups of a quasigroup.

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2. Preliminaries

A groupoid $(G,\cdot)$ is called a *quasigroup* if each of the equations $ax=b$, $xa=b$ has a unique solution for any $a,b\in G$. A quasigroup $(G,\cdot)$ may be also defined as an algebra $(G,\cdot,\backslash,/)$ with the three binary operations $\cdot$, $\backslash$, $/$ satisfying the identies

$$(xy)/y=x,\quad x\backslash(xy)=y,\quad (x/y)y=x \quad\text{and}\quad x(x\backslash y)=y.$$

We say also that $(G,\cdot,\backslash,/)$ is an *equasigroup* (i.e., *equationally definable quasigroup*) [9] or a *primitive quasigroup* [3]. The quasigroup $(G,\cdot,\backslash,/)$ corresponds to quasigroup $(G,\cdot)$, where

$$x\backslash y=z \Longleftrightarrow xz=y \quad\text{and}\quad x/y=z \Longleftrightarrow zy=x.$$

A quasigroup is called *unipotent* if $xx=yy$ for all $x,y\in G$. These quasigroups are connected with Latin squares which have one fixed element in the diagonal (cf. [5]). Such quasigroups may be defined as quasigroups $(G,\cdot)$ with the special element θ satisfying the identity $xx=\theta$. In this case also $x\backslash\theta=x$ and $\theta/x=x$ for all $x\in G$.

A nonempty subset S of a quasigroup $\mathcal{G}=(G,\cdot,\backslash,/)$ is called a *subquasigroup* of $\mathcal{G}$ if it is closed under these three operations $\cdot$, $\backslash$, $/$, i.e., if $x*y\in S$ for all $*\in\{\cdot,\backslash,/\}$ and $x,y\in S$.

By a *fuzzy set* μ in a set G we mean a function $\mu: G\to[0,1]$. The *complement* of μ, denoted by $\overline{\mu}$, is the fuzzy set in G given by $\overline{\mu}(x)=1-\mu(x)$ for all $x\in G$.

For a unipotent quasigroup $\mathcal{G}=(G,\cdot,\backslash,/)$ and a fuzzy set μ in G, let G_μ denote the set of all elements of G such that $\mu(x)=\mu(\theta)$, i.e.,

$$G_\mu=\{x\in G:\mu(x)=\mu(\theta)\}.$$

$\mathrm{Im}(\mu)$ denote the image set of μ, $a\wedge b=\min\{a,\,b\}$, $a\vee b=\max\{a,\,b\}$.

An *intuitionistic fuzzy set* (IFS for short) of a nonempty set X is defined by Atanassov (cf. [2]) in the following way.

Definition 2.1. An *intuitionistic fuzzy set* A of a nonempty set X is an object having the form

$$A=\{(x,\mu_A(x),\gamma_A(x)):x\in X\},$$

where the functions $\mu_A: X\to[0,1]$ and $\gamma_A: X\to[0,1]$ denote the degree of membership (namely $\mu_A(x)$) and the degree of nonmembership (namely $\gamma_A(x)$) of each element $x \in X$ to the set A, respectively, and $0 \leqslant \mu_A(x) + \gamma_A(x) \leqslant 1$ for all $x \in X$.

For the sake of simplicity, we shall use the symbol $A = (\mu_A, \gamma_A)$ for the intuitionistic fuzzy set $A = \{(x, \mu_A(x), \gamma_A(x)) : x \in X\}$.

The concept of fuzzy subquasigroups was introduced in [6].

Definition 2.2. A fuzzy set μ in a quasigroup $\mathcal{G} = (G, \cdot, \backslash, /)$ is called a *fuzzy subquasigroup* of $\mathcal{G}$ if

$$\mu(xy) \wedge \mu(x\backslash y) \wedge \mu(x/y) \geqslant \mu(x) \wedge \mu(y)$$

for all $x, y \in G$.

It is clear that this definition is equivalent to the following.

Definition 2.3. A fuzzy set μ in a quasigroup $\mathcal{G} = (G, \cdot, \backslash, /)$ is a *fuzzy subquasigroup* of $\mathcal{G}$ if

$$\mu(x * y) \geqslant \mu(x) \wedge \mu(y)$$

for all $* \in \{\cdot, \backslash, /\}$ and all $x, y \in G$.

3. Intuitionistic fuzzy subquasigroups

In what follows let $\mathcal{G} = (G, \cdot, \backslash, /)$ denote a quasigroup, and we start by defining the notion of intuitionistic fuzzy subquasigroups.

Definition 3.1. An intuitionistic fuzzy set $A = (\mu_A, \gamma_A)$ in $\mathcal{G}$ is called an *intuitionistic fuzzy subquasigroup* of $\mathcal{G}$ if

(IF1) $\mu_A(x * y) \geqslant \mu_A(x) \wedge \mu_A(y)$ and $\gamma_A(x * y) \leqslant \gamma_A(x) \vee \gamma_A(y)$

hold for all $x, y \in G$.

Proposition 3.2. *If $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy subquasigroup of a quasigroup $\mathcal{G}$, then*

(i) $\mu_A(x * y) \wedge \mu_A(x) = \mu_A(x * y) \wedge \mu_A(y) = \mu_A(x) \wedge \mu_A(y)$,

(ii) $\gamma_A(x * y) \vee \gamma_A(x) = \gamma_A(x * y) \vee \gamma_A(y) = \gamma_A(x) \vee \gamma_A(y)$

for all $x, y \in G$.

Proof. (i) is by Proposition 3.4 from [6].

(ii) We first consider the case $x * y = xy$. Since $(xy)/y = x$ for all $x, y \in G$, we have

$$\begin{aligned}\gamma_A(xy) \vee \gamma_A(y) &\leqslant \big[\, \gamma_A(x) \vee \gamma_A(y) \,\big] \vee \gamma_A(y) \\ &= \gamma_A(x) \vee \gamma_A(y) = \gamma_A((xy)/y) \vee \gamma_A(y) \\ &\leqslant \big[\, \gamma_A(xy) \vee \gamma_A(y) \,\big] \vee \gamma_A(y) \\ &= \gamma_A(xy) \vee \gamma_A(y),\end{aligned}$$

which proves that $\gamma_A(xy) \vee \gamma_A(y) = \gamma_A(x) \vee \gamma_A(y)$.

In the similar way, using the identity $x\backslash(xy) = y$, we can show that $\gamma_A(xy) \vee \gamma_A(x) = \gamma(x) \vee \gamma_A(y)$.

Next we prove that the result for the case $x * y = x\backslash y$. Since $x(x\backslash y) = y$ for all $x, y \in G$, we get

$$\begin{aligned}\gamma_A(x) \vee \gamma_A(y) &= \gamma_A(x) \vee \gamma_A\big(\, x(x\backslash y) \,\big) \\ &\leqslant \gamma_A(x) \vee \big[\, \gamma_A(x) \vee \gamma_A(x\backslash y) \,\big] = \gamma_A(x) \vee \gamma_A(x\backslash y) \\ &\leqslant \gamma_A(x) \vee \big[\, \gamma_A(x) \vee \gamma_A(y) \,\big] = \gamma_A(x) \vee \gamma_A(y).\end{aligned}$$

Thus $\gamma_A(x) \vee \gamma_A(x\backslash y) = \gamma_A(x) \vee \gamma_A(y)$.

Noticing that $x\backslash y = z \Longleftrightarrow xz = y$, we obtain

$$\begin{aligned}\gamma_A(x\backslash y) \vee \gamma_A(y) &= \gamma_A(z) \vee \gamma_A(xz) = \gamma_A(z) \vee \gamma_A(x) \\ &= \gamma_A(x\backslash y) \vee \gamma_A(x) = \gamma_A(x) \vee \gamma_A(y).\end{aligned}$$

Finally, we should prove the result for the case $x * y = x/y$. Using the equality $(x/y)y = x$, we have

$$\begin{aligned}\gamma_A(x) \vee \gamma_A(y) &= \gamma_A\big(\, (x/y)y \,\big) \vee \gamma_A(y) \\ &\geqslant \big[\, \gamma_A(x/y) \vee \gamma_A(y) \,\big] \vee \gamma_A(y) = \gamma_A(x/y) \vee \gamma_A(y) \\ &\geqslant \big[\, \gamma_A(x) \vee \gamma_A(y) \,\big] \vee \gamma_A(y) = \gamma_A(x) \vee \gamma_A(y).\end{aligned}$$

It follows that $\gamma_A(x/y) \vee \gamma_A(y) = \gamma_A(x) \vee \gamma_A(y)$.

Since $x/y = z \Longleftrightarrow zy = x$ for all $x, y, z \in G$, we get

$$\begin{aligned}\gamma_A(x/y) \vee \gamma_A(x) &= \gamma_A(z) \vee \gamma_A(zy) = \gamma_A(z) \vee \gamma_A(y) \\ &= \gamma_A(x/y) \vee \gamma_A(y) = \gamma_A(x) \vee \gamma_A(y).\end{aligned}$$

This completes the proof. □

Corollary 3.3. *Let $A = (\mu_A, \gamma_A)$ be an intuitionistic fuzzy subquasigroup of $\mathcal{G}$ and let $* \in \{\cdot, \backslash, /\}$. Then $\mu_A(x * y) = \mu_A(x) \wedge \mu_A(y)$ (resp. $\gamma_A(x * y) = \gamma_A(x) \vee \gamma_A(y)$) whenever $\mu_A(x) \neq \mu_A(y)$ (resp. $\gamma_A(x) \neq \gamma_A(y)$).*

Proof. Straightforward. □

Lemma 3.4. *If $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy subquasigroup of $\mathcal{G}$ and e is a left (right) neutral element of $(G, \cdot)$, then $\mu_A(e) \geqslant \mu_A(x)$ and $\gamma_A(e) \leqslant \gamma_A(x)$ for all $x \in G$.*

Proof. Indeed, if $ex = x$, then also $x/x = e$ and $\mu_A(e) = \mu_A(x/x) \geqslant \mu_A(x) \wedge \mu_A(x) = \mu_A(x)$. Similarly $\gamma_A(e) = \gamma_A(x/x) \leqslant \gamma_A(x)$. □

Lemma 3.5. *If $A = (\mu_A, \gamma_A)$ is an intuitionictic fuzzy subquasigroup of a unipotent quasigroup $\mathcal{G}$, then $\mu_A(\theta) \geqslant \mu_A(x)$ and $\gamma_A(\theta) \leqslant \gamma_A(x)$ for all $x \in G$.*

Proof. Since $xx = \theta$ for all $x \in G$, we have

$$\mu_A(\theta) = \mu_A(xx) \geqslant \mu_A(x) \wedge \mu_A(x) = \mu_A(x)$$

and

$$\gamma_A(\theta) = \gamma_A(xx) \leqslant \gamma_A(x) \vee \gamma_A(x) = \gamma_A(x)$$

for all $x \in G$. □

Theorem 3.6. *If $A = (\mu_A, \gamma)$ is an intuitionistic fuzzy subquasigroup of $\mathcal{G}$, then so is $\Box A$, where $\Box A = \{(x, \mu_A(x), 1 - \mu_A(x)) : x \in G\}$.*

Proof. It is sufficient to show that $\overline{\mu_A}$ satisfies the second condition of (IF1). For any $x, y \in G$, we have

$$\begin{aligned} \overline{\mu_A}(x * y) &= 1 - \mu_A(x * y) \leqslant 1 - \big[\, \mu_A(x) \wedge \mu_A(y) \,\big] \\ &= \big[\, 1 - \mu_A(x) \,\big] \vee \big[\, 1 - \mu_A(y) \,\big] = \overline{\mu_A}(x) \vee \overline{\mu_A}(y). \end{aligned}$$

Therefore $\Box A$ is an intuitionistic fuzzy subquasigroup of $\mathcal{G}$. □

Theorem 3.7. *Let $\mathcal{G} = (G, \cdot, \backslash, /)$ be a unipotent quasigroup. If $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy subquasigroup of $\mathcal{G}$, then*

$$G_\mu = \{x \in G : \mu_A(x) = \mu_A(\theta)\} \quad \text{and} \quad G_\gamma = \{x \in G : \gamma_A(x) = \gamma_A(\theta)\}$$

are subquasigroups of $\mathcal{G}$.

Proof. Obviously $G_\mu \neq \emptyset \neq G_\gamma$. Let $x, y \in G_\mu$ and $* \in \{\cdot, \backslash, /\}$. Then $\mu_A(x * y) \geqslant \mu_A(x) \wedge \mu_A(y) = \mu_A(\theta)$. Since $\mu_A(\theta) \geqslant \mu_A(z)$ for all $z \in G$, it follows that $\mu_A(x * y) = \mu_A(\theta)$, i.e., $x * y \in G_\mu$.

Similarly $x, y \in G_\gamma$ implies $\gamma_A(x*y) \leqslant \gamma_A(x) \vee \gamma_A(y) = \gamma_A(\theta)$ and so $\gamma_A(x * y) = \gamma_A(\theta)$, i.e., $x * y \in G_\gamma$. This completes the proof. □

Corollary 3.8. *If $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy subquasigroup of $\mathcal{G}$ and e is a left (right) neutral element of $(G, \cdot)$, then*

$$G_\mu = \{x \in G : \mu_A(x) = \mu_A(e)\} \quad \textit{and} \quad G_\gamma = \{x \in G : \gamma_A(x) = \gamma_A(e)\}$$

are subquasigroups of $\mathcal{G}$. □

For any $\alpha \in [0, 1]$ and fuzzy set μ of G, the set

$$U(\mu; \alpha) = \{x \in G : \mu(x) \geqslant \alpha\} \quad \text{(resp. } L(\mu; \alpha) = \{x \in G : \mu(x) \leqslant \alpha\})$$

is called an *upper* (resp. *lower*) *α-level cut* of μ.

Theorem 3.9. *If $A = (\mu_A, \gamma)$ is an intuitionistic fuzzy subquasigroup of $\mathcal{G}$, then the sets $U(\mu_A; \alpha)$ and $L(\gamma_A; \alpha)$ are subquasigroups of $\mathcal{G}$ for every $\alpha \in \mathrm{Im}(\mu_A) \cap \mathrm{Im}(\gamma_A)$.*

Proof. Let $\alpha \in \mathrm{Im}(\mu_A) \cap \mathrm{Im}(\gamma_A) \subseteq [0, 1]$ and $* \in \{\cdot, \backslash, /\}$ and let $x, y \in U(\mu_A; \alpha)$. Then $\mu_A(x) \geqslant \alpha$ and $\mu_A(y) \geqslant \alpha$. It follows from the first condition of (IF1) that

$$\mu_A(x * y) \geqslant \mu_A(x) \wedge \mu_A(y) \geqslant \alpha \quad \text{so that} \quad x * y \in U(\mu_A; \alpha).$$

If $x, y \in L(\gamma_A; \alpha)$, then $\gamma_A(x) \leqslant \alpha$ and $\gamma_A(y) \leqslant \alpha$, and so

$$\gamma_A(x * y) \leqslant \gamma_A(x) \vee \gamma_A(y) \leqslant \alpha.$$

Hence we have $x * y \in L(\gamma_A; \alpha)$. Therefore $U(\mu_A; \alpha)$ and $L(\gamma_A; \alpha)$ are subquasigroups of $\mathcal{G}$. □

Theorem 3.10. *Let $A = (\mu_A, \gamma_A)$ be an* IFS *in $\mathcal{G}$ such that the nonempty sets $U(\mu_A; \alpha)$ and $L(\gamma_A; \alpha)$ are subquasigroups of $\mathcal{G}$ for all $\alpha \in [0, 1]$. Then $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy subquasigroup of $\mathcal{G}$.*

Proof. Let $\alpha \in [0,1]$. Assume that $U(\mu_A;\alpha) \neq \emptyset$ and $L(\gamma_A;\alpha) \neq \emptyset$ are subquasigroups of $\mathcal{G}$. We must show that $A = (\mu_A, \gamma_A)$ satisfies the condition (IF1).

Let $* \in \{\cdot, \backslash, /\}$. If the first condition of (IF1) is false, then there exist $x_0, y_0 \in G$ such that $\mu_A(x_0 * y_0) < \mu_A(x_0) \wedge \mu_A(y_0)$. Taking

$$\alpha_0 = \frac{1}{2}\big[\,\mu_A(x_0 * y_0) + \big[\,\mu_A(x_0) \wedge \mu_A(y_0)\big]\,\big],$$

we have $\mu_A(x_0 * y_0) < \alpha_0 < \mu_A(x_0) \wedge \mu_A(y_0)$. It follows that x_0, y_0 are in $U(\mu_A;\alpha_0)$ but $x_0 * y_0 \notin U(\mu_A;\alpha_0)$, which is a contradiction.

Assume that the second condition of (IF1) does not hold. Then $\gamma_A(x_0 * y_0) > \gamma_A(x_0) \vee \gamma_A(y_0)$ for some $x_0, y_0 \in G$. Let

$$\beta_0 = \frac{1}{2}\big[\,\gamma_A(x_0 * y_0) + \big[\,\gamma_A(x_0) \vee \gamma_A(y_0)\,\big]\,\big].$$

Then $\gamma_A(x_0 * y_0) > \beta_0 > \gamma_A(x_0) \vee \gamma_A(y_0)$ and so $x_0, y_0 \in L(\gamma_A;\beta_0)$ but $x_0 * y_0 \notin L(\gamma_A;\beta_0)$. This is a contradiction.

Thus (IF1) must be satisfied. $\square$

Theorem 3.11. *Let $\mathcal{H}$ be a subquasigroup of $\mathcal{G}$ and let $A = (\mu_A, \gamma_A)$ be an* IFS *in $\mathcal{G}$ defined by*

$$\mu_A(x) = \begin{cases} \alpha_0 & \text{if } x \in H, \\ \alpha_1 & \text{otherwise}, \end{cases} \qquad \gamma_A(x) = \begin{cases} \beta_0 & \text{if } x \in H, \\ \beta_1 & \text{otherwise}, \end{cases}$$

for all $x \in G$ and $\alpha_i, \beta_i \in [0,1]$ such that $\alpha_0 > \alpha_1$, $\beta_0 < \beta_1$ and $\alpha_i + \beta_i \leqslant 1$ for $i = 0,1$. Then $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy subquasigroup of $\mathcal{G}$ and $U(\mu_A;\alpha_0) = H = L(\gamma_A;\beta_0)$.

Proof. Let $x, y \in G$ and let $* \in \{\cdot, \backslash, /\}$. If any one of x and y does not belong to H, then

$$\mu_A(x * y) \geqslant \alpha_1 = \mu_A(x) \wedge \mu_A(y)$$

and

$$\gamma_A(x * y) \leqslant \beta_1 = \gamma_A(x) \vee \gamma_A(y).$$

Therefore $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy subquasigroup of a quasigroup $\mathcal{G}$. Obviously $U(\mu_A;\alpha_0) = H = L(\gamma_A;\beta_0)$. $\square$

Corollary 3.12. *Let χ_H be the characteristic function of a subquasigroup $\mathcal{H}$ of $\mathcal{G}$. Then $H = (\chi_H, \overline{\chi_H})$ is an intuitionistic fuzzy subquasigroup of $\mathcal{G}$.* □

Theorem 3.13. *If $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy subquasigroup of $\mathcal{G}$, then*

$$\mu_A(x) = \sup\{\alpha \in [0,1] : x \in U(\mu_A; \alpha)\}$$

and

$$\gamma_A(x) = \inf\{\alpha \in [0,1] : x \in L(\gamma_A; \alpha)\}$$

for all $x \in G$.

Proof. Let $\delta = \sup\{\alpha \in [0,1] : x \in U(\mu_A; \alpha)\}$ and let $\varepsilon > 0$ be given. Then $\delta - \varepsilon < \alpha$ for some $\alpha \in [0,1]$ such that $x \in U(\mu_A; \alpha)$. This means that $\delta - \varepsilon < \mu_A(x)$ so that $\delta \leqslant \mu_A(x)$ since ε is arbitrary.

We now show that $\mu_A(x) \leqslant \delta$. If $\mu_A(x) = \beta$, then $x \in U(\mu_A; \beta)$ and so

$$\beta \in \{\alpha \in [0,1] : x \in U(\mu_A; \alpha)\}.$$

Hence

$$\mu_A(x) = \beta \leqslant \sup\{\alpha \in [0,1] : x \in U(\mu_A; \alpha)\} = \delta.$$

Therefore

$$\mu_A(x) = \delta = \sup\{\alpha \in [0,1] : x \in U(\mu_A; \alpha)\}.$$

Now let $\eta = \inf\{\alpha \in [0,1] : x \in L(\gamma_A; \alpha)\}$. Then

$$\inf\{\alpha \in [0,1] : x \in L(\gamma_A; \alpha)\} < \eta + \varepsilon$$

for any $\varepsilon > 0$, and so $\alpha < \eta + \varepsilon$ for some $\alpha \in [0,1]$ with $x \in L(\gamma_A; \alpha)$. Since $\gamma_A(x) \leqslant \alpha$ and ε is arbitrary, it follows that $\gamma_A(x) \leqslant \eta$.

To prove $\gamma_A(x) \geqslant \eta$, let $\gamma_A(x) = \zeta$. Then $x \in L(\gamma_A; \zeta)$ and thus $\zeta \in \{\alpha \in [0,1] : x \in L(\gamma_A; \alpha)\}$. Hence

$$\inf\{\alpha \in [0,1] : x \in L(\gamma_A; \alpha)\} \leqslant \zeta,$$

i.e., $\eta \leqslant \zeta = \gamma_A(x)$. Consequently

$$\gamma_A(x) = \eta = \inf\{\alpha \in [0,1] : x \in L(\gamma_A; \alpha)\},$$

which completes the proof. □

Theorem 3.14. *Let* $\{\mathcal{H}_\alpha : \alpha \in \Lambda\}$, *where* Λ *is a nonempty subset of* $[0,1]$, *be a collection of subquasigroups of* $\mathcal{G}$ *such that*

(i) $G = \bigcup_{\alpha\in\Lambda} H_\alpha$,

(ii) $\alpha > \beta \Longleftrightarrow H_\alpha \subset H_\beta$ *for all* $\alpha, \beta \in \Lambda$.

Then an intuitionistic fuzzy set $A = (\mu_A, \gamma_A)$ *defined by*

$$\mu_A(x) = \sup\{\alpha \in \Lambda : x \in H_\alpha\} \ \text{and} \ \gamma_A(x) = \inf\{\alpha \in \Lambda : x \in H_\alpha\}$$

for all $x \in G$ *is an intuitionistic fuzzy subguasigroup of* $\mathcal{G}$.

Proof. According to Theorem 3.10, it is sufficient to show that the nonempty sets $U(\mu_A;\alpha)$ and $L(\gamma_A;\beta)$ are subquasigroups of $\mathcal{G}$.

In order to prove that $U(\mu_A;\alpha) \neq \emptyset$ is a subquasigroup of $\mathcal{G}$, we consider the following two cases:

$$\text{(i)} \ \ \alpha = \sup\{\delta \in \Lambda : \delta < \alpha\} \quad \text{and} \quad \text{(ii)} \ \ \alpha \neq \sup\{\delta \in \Lambda : \delta < \alpha\}.$$

Case (i) implies that

$$x \in U(\mu_A;\alpha) \Longleftrightarrow \big(x \in H_\delta \ for \ all \ \delta < \alpha\big) \Longleftrightarrow x \in \bigcap_{\delta<\alpha} H_\delta\,,$$

so that $U(\mu_A;\alpha) = \bigcap_{\delta<\alpha} H_\delta$ which is a subquasigroup of $\mathcal{G}$.

For the case (ii), we claim that $U(\mu_A;\alpha) = \bigcup_{\delta\geqslant\alpha} H_\delta$. If $x \in \bigcup_{\delta\geqslant\alpha} H_\delta$ then $x \in H_\delta$ for some $\delta \geqslant \alpha$. It follows that $\mu_A(x) \geqslant \delta \geqslant \alpha$, so that $x \in U(\mu_A;\alpha)$. This shows that $\bigcup_{\delta\geqslant\alpha} H_\delta \subseteq U(\mu_A;\alpha)$.

Now assume that $x \notin \bigcup_{\delta\geqslant\alpha} H_\delta$. Then $x \notin H_\delta$ for all $\delta \geqslant \alpha$. Since $\alpha \neq \sup\{\delta \in \Lambda : \delta < \alpha\}$, there exists $\varepsilon > 0$ such that $(\alpha-\varepsilon, \alpha)\cap\Lambda = \emptyset$. Hence $x \notin H_\delta$ for all $\delta > \alpha - \varepsilon$, which means that if $x \in H_\delta$ then $\delta \leqslant \alpha - \varepsilon$. Thus $\mu_A(x) \leqslant \alpha - \varepsilon < \alpha$, and so $x \notin U(\mu_A;\alpha)$. Therefore $U(\mu_A;\alpha) \subseteq \bigcup_{\delta\geqslant\alpha} H_\delta$, and thus $U(\mu_A;\alpha) = \bigcup_{\delta\geqslant\alpha} H_\delta$, which is a subquasigroup of $\mathcal{G}$.

Now we prove that $L(\gamma_A;\beta)$ is a subquasigroup of $\mathcal{G}$. We consider the following two cases:

$$\text{(iii)} \ \ \beta = \inf\{\eta \in \Lambda : \beta < \eta\} \quad \text{and} \quad \text{(iv)} \ \ \beta \neq \inf\{\eta \in \Lambda : \beta < \eta\}.$$

For the case (iii) we have

$$x \in L(\gamma_A;\beta) \Longleftrightarrow \big(x \in H_\eta \ \ for\ all\ \eta > \beta\big) \Longleftrightarrow x \in \bigcap_{\eta>\beta} H_\eta$$

and hence $L(\gamma_A;\beta) = \bigcap\limits_{\eta>\beta} H_\eta$ which is a subquasigroup of $\mathcal{G}$.

For the case (iv), there exists $\varepsilon > 0$ such that $(\beta, \beta+\varepsilon) \cap \Lambda = \emptyset$. We will show that $L(\gamma_A;\beta) = \bigcup\limits_{\eta\leqslant\beta} H_\eta$. If $x \in \bigcup\limits_{\eta\leqslant\beta} H_\eta$ then $x \in H_\eta$ for some $\eta \leqslant \beta$. It follows that $\gamma_A(x) \leqslant \eta \leqslant \beta$ so that $x \in L(\gamma_A;\beta)$. Hence $\bigcup\limits_{\eta\leqslant\beta} H_\eta \subseteq L(\gamma_A;\beta)$.

Conversely, if $x \notin \bigcup\limits_{\eta\leqslant\beta} H_\eta$ then $x \notin H_\eta$ for all $\eta \leqslant \beta$, which implies that $x \notin H_\eta$ for all $\eta < \beta+\varepsilon$, i.e., if $x \in H_\eta$ then $\eta \geqslant \beta+\varepsilon$. Thus $\gamma_A(x) \geqslant \beta+\varepsilon > \beta$, i.e., $x \notin L(\gamma_A;\beta)$. Therefore $L(\gamma_A;\beta) \subseteq \bigcup\limits_{\eta\leqslant\beta} H_\eta$ and consequently $L(\gamma_A;\beta) = \bigcup\limits_{\eta\leqslant\beta} H_\eta$ which is a subquasigroup of $\mathcal{G}$. This completes the proof. □

Theorem 3.15. $A = (\mu_A, \gamma_A)$ *is an intuitionistic fuzzy subquasigroup of* $\mathcal{G}$ *iff* μ_A *and* $\overline{\gamma_A}$ *are fuzzy subquasigroups of* $\mathcal{G}$.

Proof. Let $A = (\mu_A, \gamma_A)$ be an intuitionistic fuzzy subquasigroup of $\mathcal{G}$. Then clearly μ_A is a fuzzy subquasigroup of $\mathcal{G}$. Let $x, y \in G$ and $* \in \{\cdot, \backslash, /\}$. Then

$$\begin{aligned}\overline{\gamma_A}(x*y) &= 1-\gamma_A(x*y) \geqslant 1-\big[\,\gamma_A(x)\vee\gamma_A(y)\,\big]\\ &= \big[\,1-\gamma_A(x)\,\big]\wedge\big[\,1-\gamma_A(y)\,\big] = \overline{\gamma_A}(x)\wedge\overline{\gamma_A}(y)\,.\end{aligned}$$

Hence $\overline{\gamma_A}$ is a fuzzy subquasigroup of $\mathcal{G}$.

Conversely suppose that μ_A and $\overline{\gamma_A}$ are fuzzy subquasigroups of $\mathcal{G}$. If $x, y \in G$ and $* \in \{\cdot, \backslash, /\}$, then

$$\begin{aligned}1-\gamma_A(x*y) = \overline{\gamma_A}(x*y) &\geqslant \overline{\gamma_A}(x)\wedge\overline{\gamma_A}(y)\\ &= \big[\,1-\gamma_A(x)\,\big]\wedge\big[\,1-\gamma_A(y)\,\big]\\ &= 1-\big[\,\gamma_A(x)\vee\gamma_A(y)\,\big]\,,\end{aligned}$$

which proves $\gamma_A(x*y) \leqslant \gamma_A(x)\vee\gamma_A(y)$. This completes the proof. □

If $\mathcal{H}$ is a subquasigroup of $\mathcal{G}$, then $H = (\chi_H, \overline{\chi_H})$ is an intuitionistic fuzzy subquasigroup of $\mathcal{G}$ from Corollary 3.12, where χ_H is the characteristic function of H.

Let $IFS(\mathcal{G})$ be the family of all intuitionistic fuzzy subquasigroups of $\mathcal{G}$ and $\alpha \in [0,1]$ be a fixed real number. For any $A = (\mu_A, \gamma_A)$ and $B = (\mu_B, \gamma_B)$ from $IFS(\mathcal{G})$ we define two binary relations $\mathfrak{U}^\alpha$ and $\mathfrak{L}^\alpha$ on $IFS(\mathcal{G})$ as follows:

$$(A, B) \in \mathfrak{U}^\alpha \Longleftrightarrow U(\mu_A; \alpha) = U(\mu_B; \alpha)$$

and

$$(A, B) \in \mathfrak{L}^\alpha \Longleftrightarrow L(\gamma_A; \alpha) = L(\gamma_B; \alpha) \, .$$

These two relations $\mathfrak{U}^\alpha$ and $\mathfrak{L}^\alpha$ are equivalence relations, give rise to partitions of $IFS(\mathcal{G})$ into the equivalence classes of $\mathfrak{U}^\alpha$ and $\mathfrak{L}^\alpha$, denoted by $[A]_{\mathfrak{U}^\alpha}$ and $[A]_{\mathfrak{L}^\alpha}$ for any $A = (\mu_A, \gamma_A) \in IFS(\mathcal{G})$, respectively. And we will denote the quotient sets of $IFS(\mathcal{G})$ by $\mathfrak{U}^\alpha$ and $\mathfrak{L}^\alpha$ as $IFS(\mathcal{G})/\mathfrak{U}^\alpha$ and $IFS(\mathcal{G})/\mathfrak{L}^\alpha$, respectively.

If $\mathcal{S}(\mathcal{G})$ is the family of all subquasigroups of $\mathcal{G}$ and $\alpha \in [0,1]$, then we define two maps U_α and L_α from $IFS(\mathcal{G})$ to $\mathcal{S}(\mathcal{G}) \cup \{\emptyset\}$ as follows:

$$U_\alpha(A) = U(\mu_A; \alpha) \quad \text{and} \quad L_\alpha(A) = L(\gamma_A; \alpha),$$

respectively, for each $A = (\mu_A, \gamma_A) \in IFS(\mathcal{G})$. Then the maps U_α and L_α are well-defined.

Theorem 3.16. *For any $\alpha \in (0,1)$, the maps U_α and L_α are surjective from $IFS(\mathcal{G})$ onto $\mathcal{S}(\mathcal{G}) \cup \{\emptyset\}$.*

Proof. Let $\alpha \in (0,1)$. Note that $\mathbf{0}_\sim = (\mathbf{0}, \mathbf{1})$ is in $IFS(\mathcal{G})$, where $\mathbf{0}$ and $\mathbf{1}$ are fuzzy sets in $\mathcal{G}$ defined by $\mathbf{0}(x) = 0$ and $\mathbf{1}(x) = 1$ for all $x \in G$. Obviously, $U_\alpha(\mathbf{0}_\sim) = L_\alpha(\mathbf{0}_\sim) = \emptyset$. If $\mathcal{H}$ is a subquasigroup of $\mathcal{G}$, then for the intuitionistic fuzzy subquasigroup $H = (\chi_H, \overline{\chi_H})$, $U_\alpha(H) = U(\chi_H; \alpha) = H$ and $L_\alpha(H) = L(\overline{\chi_H}; \alpha) = H$. Hence U_α and L_α are surjective. □

Theorem 3.17. *The quotient sets $IFS(\mathcal{G})/\mathfrak{U}^\alpha$ and $IFS(\mathcal{G})/\mathfrak{L}^\alpha$ are equipotent to $\mathcal{S}(\mathcal{G}) \cup \{\emptyset\}$ for any $\alpha \in (0,1)$.*

Proof. Let $\alpha \in (0,1)$ and let $\overline{U_\alpha} : IFS(\mathcal{G})/\mathfrak{U}^\alpha \longrightarrow \mathcal{S}(\mathcal{G}) \cup \{\emptyset\}$ and $\overline{L_\alpha} : IFS(\mathcal{G})/\mathfrak{L}^\alpha \longrightarrow \mathcal{S}(\mathcal{G}) \cup \{\emptyset\}$ be the maps defined by

$$\overline{U_\alpha}([A]_{\mathfrak{U}^\alpha}) = U_\alpha(A) \quad \text{and} \quad \overline{L_\alpha}([A]_{\mathfrak{L}^\alpha}) = L_\alpha(A),$$

respectively, for each $A = (\mu_A, \gamma_A) \in IFS(\mathcal{G})$.

If $U(\mu_A;\alpha) = U(\mu_B;\alpha)$ and $L(\gamma_A;\alpha) = L(\gamma_B;\alpha)$ for $A = (\mu_A, \gamma_A)$ and $B = (\mu_B, \gamma_B)$ from $IFS(\mathcal{G})$, then $(A,B) \in \mathfrak{U}^\alpha$ and $(A,B) \in \mathfrak{L}^\alpha$, whence $[A]_{\mathfrak{U}^\alpha} = [B]_{\mathfrak{U}^\alpha}$ and $[A]_{\mathfrak{L}^\alpha} = [B]_{\mathfrak{L}^\alpha}$. Hence the maps $\overline{U_\alpha}$ and $\overline{L_\alpha}$ are injective.

To show that the maps $\overline{U_\alpha}$ and $\overline{L_\alpha}$ are surjective, let $\mathcal{H}$ be a subquasigroup of $\mathcal{G}$. Then for $H = (\chi_H, \overline{\chi_H}) \in IFS(\mathcal{G})$ we have $\overline{U_\alpha}([H]_{\mathfrak{U}^\alpha}) = U(\chi_H;\alpha) = H$ and $\overline{L_\alpha}([H]_{\mathfrak{L}^\alpha}) = L(\overline{\chi_H};\alpha) = H$. Also $\mathbf{0}_\sim = (\mathbf{0}, \mathbf{1}) \in IFS(\mathcal{G})$. Moreover $\overline{U_\alpha}([\mathbf{0}_\sim]_{\mathfrak{U}^\alpha}) = U(\mathbf{0};\alpha) = \emptyset$ and $\overline{L_\alpha}([\mathbf{0}_\sim]_{\mathfrak{L}^\alpha}) = L(\mathbf{1};\alpha) = \emptyset$. Hence $\overline{U_\alpha}$ and $\overline{L_\alpha}$ are surjective. □

For any $\alpha \in [0,1]$, we define another relation $\mathfrak{R}^\alpha$ on $IFS(\mathcal{G})$ as following:

$$(A,B) \in \mathfrak{R}^\alpha \Longleftrightarrow U(\mu_A;\alpha) \cap L(\gamma_A;\alpha) = U(\mu_B;\alpha) \cap L(\gamma_B;\alpha)$$

for any $A = (\mu_A, \gamma_A)$ and $B = (\mu_B, \gamma_B)$ from $IFS(\mathcal{G})$. Then the relation $\mathfrak{R}^\alpha$ is also an equivalence relation on $IFS(\mathcal{G})$.

Theorem 3.18. *For any $\alpha \in (0,1)$ and any $A = (\mu_A, \gamma_A) \in IFS(\mathcal{G})$ the map $I_\alpha : IFS(\mathcal{G}) \longrightarrow \mathcal{S}(\mathcal{G}) \cup \{\emptyset\}$ defined by*

$$I_\alpha(A) = U_\alpha(A) \cap L_\alpha(A)$$

is surjective.

Proof. Indeed, if $\alpha \in (0,1)$ is fixed, then for $\mathbf{0}_\sim = (\mathbf{0}, \mathbf{1}) \in IFS(\mathcal{G})$ we have

$$I_\alpha(\mathbf{0}_\sim) = U_\alpha(\mathbf{0}_\sim) \cap L_\alpha(\mathbf{0}_\sim) = U(\mathbf{0};\alpha) \cap L(\mathbf{1};\alpha) = \emptyset\,,$$

and for any $\mathcal{H} \in \mathcal{S}(\mathcal{G})$, there exists $H = (\chi_H, \overline{\chi_H}) \in IFS(\mathcal{G})$ such that $I_\alpha(H) = U(\chi_H;\alpha) \cap L(\overline{\chi_H};\alpha) = H$. □

Theorem 3.19. *For any* $\alpha \in (0,1)$*, the quotient set* $IFS(\mathcal{G})/\mathfrak{R}^{\alpha}$ *is equipotent to* $\mathcal{S}(\mathcal{G}) \cup \{\emptyset\}$*.*

Proof. Let $\alpha \in (0,1)$ and let $\overline{I_\alpha} : IFS(\mathcal{G})/\mathfrak{R}^{\alpha} \longrightarrow \mathcal{S}(\mathcal{G}) \cup \{\emptyset\}$ be a map defined by

$$\overline{I_\alpha}([A]_{\mathfrak{R}^\alpha}) = I_\alpha(A) \quad \text{ for each } \quad [A]_{\mathfrak{R}^\alpha} \in IFS(\mathcal{G})/\mathfrak{R}^{\alpha}.$$

If $\overline{I_\alpha}([A]_{\mathfrak{R}^\alpha}) = \overline{I_\alpha}([B]_{\mathfrak{R}^\alpha})$ for any $[A]_{\mathfrak{R}^\alpha}$, $[B]_{\mathfrak{R}^\alpha} \in IFS(\mathcal{G})/\mathfrak{R}^{\alpha}$, then

$$U(\mu_A;\alpha) \cap L(\gamma_A;\alpha) = U(\mu_B;\alpha) \cap L(\gamma_B;\alpha),$$

hence $(A,B) \in \mathfrak{R}^{\alpha}$ and $[A]_{\mathfrak{R}^\alpha} = [B]_{\mathfrak{R}^\alpha}$. It follows that $\overline{I_\alpha}$ is injective.

For $\mathbf{0}_\sim = (\mathbf{0},\mathbf{1}) \in IFS(\mathcal{G})$ we have $\overline{I_\alpha}(\mathbf{0}_\sim) = I_\alpha(\mathbf{0}_\sim) = \emptyset$. If $H \in \mathcal{S}(\mathcal{G})$, then for $H = (\chi_H, \overline{\chi_H}) \in IFS(\mathcal{G})$, $\overline{I_\alpha}(H) = I_\alpha(H) = H$. Hence $\overline{I_\alpha}$ is a bijective map. □

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