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A parastrophic equivalence in quasigroups

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Abstract

In this paper there are found of "lowest" representants of classes of a parastrophic equivalence in quasigroups satisfying identities of the type

$$w_1\Box_1(w_2\Box_2\dots(w_n\Box_n x))) \doteq x, \quad 1 < n,$$

where $\Box_i$ is a parastrophe of $\Box_1$ for all $i \leqslant n$ and $w_1,\dots,w_n$ are terms in $Q(\cdot)$ and its parastrophes that not contain variable x. These representants are listed for $1 < n < 5$ by a personal computer.

1. Introduction

With any given quasigroup $(Q,\cdot)$ there are associated five operations $*$, $/$, $\backslash$, $\triangle$, $\bigtriangledown$ (see the following part 1) that we shall call *conjugates* of $(\cdot)$ (see [1], [4]) or *parastrophes* of $(\cdot)$ (see [3]). If a quasigroup $(Q,\cdot)$ satisfies a given identity, say I, then in general, for example $(Q,/)$ will satisfy a different conjugate identity, say II. Therefore it is in some sense true that the theory of quasigroups that satisfy the identity I is equivalent to the theory of quasigroups which satisfy the identity II, as has been remarked by Stein in [4].

In [3], Sade has given some general rules for determining the identities satisfied by the parastrophes of a quasigroup $(Q,\cdot)$ when $(Q,\cdot)$ satisfies a given identity involving some elements of the set $\sum(\cdot) = \{\cdot,*,/,\bigtriangledown,\backslash,\triangle\}$. In [4], Stein has listed the conjugate identities for a

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number well-known identities. More extensive list is given in Belousov [1]. With respect to a parastrophic equivalence, Belousov in [2] has given a classification of all quasigroups identities which are of the type $x\Box_1\big(x\Box_2(x\Box_3 y)\big) \doteq y$, where $\Box_i \in \sum(\cdot)$ for all $i = 1, 2, 3$.

In this paper we give a generalization and a simplification of methods used in [2].

2. Preliminaries

Let $(Q,\cdot)$ be a fixed quasigroup, $\mathcal{T} = \{L, R, T, L^{-1}, R^{-1}, T^{-1}\}$, and $\sum(\cdot) = \{\cdot, *, /, \bigtriangledown, \backslash, \bigtriangleup\}$, where $x \cdot y = z \Leftrightarrow y * x = z \Leftrightarrow z/y = x \Leftrightarrow y \bigtriangledown z = x \Leftrightarrow x \setminus z = y \Leftrightarrow z \bigtriangleup x = y$.

Further, let $L_a x = a \cdot x$, $R_a x = x \cdot a$, $L_a^{/} x = a/x$, $R_a^{\bigtriangledown} x = x \bigtriangledown a$, $T_a x = x \setminus a$, $L_a L_a^{-1} x = x$, ... Then it holds the relations given by Table 1. This table we read like this: $L^{\bigtriangledown} = R^{-1}$, $(R^{-1})^{\backslash} = T^{-1}$, ..., $\varphi_2(R^{-1}) = \varphi_2 R^{-1} = (R^{-1})^{/} = R$, $\varphi_5 R = L^{-1}$, $R_a^{-1} = (R_a)^{-1}$, $T_a^{-1} = (T_a)^{-1}$.

Table 1

	$\cdot$	$*$	$/$	$\bigtriangledown$	$\backslash$	$\bigtriangleup$
L	L	R	T^{-1}	R^{-1}	L^{-1}	T
R	R	L	R^{-1}	T^{-1}	T	L^{-1}
T	T	T^{-1}	L^{-1}	L	R	R^{-1}
L^{-1}	L^{-1}	R^{-1}	T	R	L	T^{-1}
R^{-1}	R^{-1}	L^{-1}	R	T	T^{-1}	L
T^{-1}	T^{-1}	T	L	L^{-1}	R^{-1}	R
	φ_0	φ_1	φ_2	φ_3	φ_4	φ_5

From this table directly follows that $(\varphi_i x)^{-1} = \varphi_i(x^{-1})$ for all $i \in \{0, 1, ..., 5\}$ and for all $x \in \mathcal{T}$. If $(Q, \Box)$ is a quasigroup, then the mappings $L_a^{\Box}$, $R_a^{\Box}$, ..., $(T_a^{-1})^{\Box}$ are called *translations* of $\Box$. Every operation in $\sum(\cdot)$ is named a *parastrophe* of $(\cdot)$.

If a quasigroup $(Q,\cdot)$ satisfies a given identity, for example

$$y \doteq (x \setminus yz)/zx\,, \tag{1}$$

then in general each of its parastrophes will satisfy a different conjugate identities. Thus, for example, (1) is equivalent to $y \cdot zx \doteq x \setminus yz$; if denote $zx = u$, $yz = v$ and $yu = t$ (i.e. $x = z \setminus u$, $z = y \setminus v$, $u = y \setminus t$), then

$$t \doteq \big((y \setminus v) \setminus (y \setminus t)\big) \setminus v\,. \tag{2}$$

Hence, $(Q,\cdot)$ satisfies (1) iff $(Q,\setminus)$ satisfies (2). If (2) is written with terms of $(Q,\cdot)$, then obtain

$$c \doteq (ab \cdot ac)b\,. \tag{3}$$

Thus (3) is a conjugate identity to (1). Further, from (3) we have $R_b L_{ab} L_a \doteq 1$, i.e. $L_a R_b L_{ab} \doteq 1$. Whence $c \doteq a \cdot (ab \cdot c)b$ and if denote $a = y$, $ab = z$, $c = z \setminus x$, then

$$y \doteq \big(x \cdot (y \setminus z)\big) \bigtriangledown (z \setminus x) \tag{4}$$

is a conjugate identity to (1). (4) we get from (1) if all operations in (1) are substituted by Table 2, i.e. $(\cdot)$ is substituted by $\setminus = \varphi_4(\cdot)$, $*$ by $\bigtriangleup = \varphi_4(*)$, . . . , $\bigtriangleup$ by $* = \varphi_4(\bigtriangleup)$ (see Sade [3]).

Table 2

	$\cdot$	$*$	$/$	$\bigtriangledown$	$\setminus$	$\bigtriangleup$
$\cdot$	$\cdot$	$*$	$/$	$\bigtriangledown$	$\setminus$	$\bigtriangleup$
$*$	$*$	$\cdot$	$\bigtriangledown$	$/$	$\bigtriangleup$	$\setminus$
$/$	$/$	$\bigtriangleup$	$\cdot$	$\setminus$	$\bigtriangledown$	$*$
$\bigtriangledown$	$\bigtriangledown$	$\setminus$	$*$	$\bigtriangleup$	$/$	$\cdot$
$\setminus$	$\setminus$	$\bigtriangledown$	$\bigtriangleup$	$*$	$\cdot$	$/$
$\bigtriangleup$	$\bigtriangleup$	$/$	$\setminus$	$\cdot$	$*$	$\bigtriangledown$
	φ_0	φ_1	φ_2	φ_3	φ_4	φ_5

Table 3

	φ_0	φ_1	φ_2	φ_3	φ_4	φ_5
φ_0	φ_0	φ_1	φ_2	φ_3	φ_4	φ_5
φ_1	φ_1	φ_0	φ_5	φ_4	φ_3	φ_2
φ_2	φ_2	φ_3	φ_0	φ_1	φ_5	φ_4
φ_3	φ_3	φ_2	φ_4	φ_5	φ_1	φ_0
φ_4	φ_4	φ_5	φ_3	φ_2	φ_0	φ_1
φ_5	φ_5	φ_4	φ_1	φ_0	φ_2	φ_3

The identities (1) and (4) may be written by the way as

$$R^{/}_{zx} L^{\setminus}_{x} R_z \doteq 1, \qquad R^{\bigtriangledown}_{z\setminus x} L_x R^{\setminus}_{z} \doteq 1$$

and with respect to Table 1 and Table 2

$$R^{-1}_{zx} L^{-1}_{x} R_{z} \simeq 1, \qquad T^{-1}_{z\backslash x} L_{x} T_{z} \simeq 1\,.$$

The ordered tripletes $R^{-1}L^{-1}R$, $T^{-1}LT$ may be assigned to the identities (2), (3). Therefore the triple $R^{-1}L^{-1}R$ will be called *conjugate* to the triple $T^{-1}LT$.

In what follows we shall denote:

$\mathbf{N} = \{0, 1, 2, 3, \ldots\}$,

$\mathcal{T} = \{L, R, T, L^{-1}, R^{-1}, T^{-1}\}$,

$[0, n) = \{0, 1, 2, 3, \ldots, n-1\}$, $n \in \mathbf{N}$, $n > 0$,

$\mathcal{T}^n = \{\alpha : \alpha$ is a map $[0, n) \to \mathcal{T}\}$ for all $n \in \mathbf{N}$, $n > 0$,

if $\alpha \in \mathcal{T}^n$ then $\alpha = A_{n-1} \ldots A_2 A_1 A_0$ and $\alpha(i) = A_i$ for all $i \in [0, n)$,

$\mathcal{T}^\infty = \mathcal{T} \cup (\mathcal{T} \times \mathcal{T}) \cup (\mathcal{T} \times \mathcal{T} \times \mathcal{T}) \cup \ldots$,

$l(\alpha) = n \iff \alpha \in \mathcal{T}^n$,

$\omega : \mathcal{T}^\infty \to \mathcal{T}^\infty$, $(\omega\alpha)(i) = \alpha((i+1)(mod\, n))$ for all $i \in [0, n)$,
it holds $\alpha \in \mathcal{T}^n \Rightarrow \omega\alpha \in \mathcal{T}^n$,

$\sigma : \mathcal{T}^\infty \to \mathcal{T}^\infty$, $(\sigma\alpha)(j) = \alpha(n-1-j)$ for $n = l(\alpha)$ and
for all $j \in [0, n)$,

$\rho : \mathcal{T} \to \mathcal{T}$, $L \mapsto L^{-1} \mapsto L$, $R \mapsto R^{-1} \mapsto R$, $T \mapsto T^{-1} \mapsto T$,
i.e. $\rho(A) = A^{-1}$ for all $A \in \mathcal{T}$,

$\rho : \mathcal{T}^\infty \to \mathcal{T}^\infty$, $(\rho\alpha)(i) = \rho\big(\alpha(i)\big)$ for all $j \in [0, n)$, $n = l(\alpha)$,

$\kappa : \mathcal{T} \to [0, 6)$, $L \mapsto 0$, $R \mapsto 1$, $T \mapsto 2$, $L^{-1} \mapsto 3$, $R^{-1} \mapsto 4$,
$T^{-1} \mapsto 5$,

$\kappa : \mathcal{T}^\infty \to \mathbf{N}$, $\kappa\alpha = \sum_{i=0}^{n-1} 10^i \kappa(\alpha(i))$, $n = l(\alpha)$,

$\alpha < \beta$ for $\alpha, \beta \in \mathcal{T}^\infty \iff \kappa\alpha < \kappa\beta$,

$\varphi_i : \mathcal{T}^\infty \to \mathcal{T}^\infty$, $(\varphi_i\alpha)(j) = \varphi_i(\alpha(j))$ for all $i \in [0, 6)$ and
$j \in [0, n)$, $n = l(\alpha)$, where $\varphi_i : \mathcal{T} \to \mathcal{T}$ is given in Table 1,

$\mathcal{P}_1 -$ be the group generated by $\{\varphi_i : \mathcal{T}^\infty \to \mathcal{T}^\infty,\ i \in [0, 6)\}$,

$\mathcal{P}-$ the group generated by the set $\mathcal{P}_1 \cup \{\rho\sigma, \omega\}$,
(these maps are defined upon $\mathcal{T}^\infty$),

$\varphi_{i+6} : \mathcal{T}^\infty \to \mathcal{T}^\infty$, $\varphi_{i+6} = \sigma\rho\varphi_i$ for all $i \in [0,6)$, where
$\varphi_i : \mathcal{T} \to \mathcal{T}$ is given in Table 1,

$C(i,j,k)(\alpha) = \kappa\big((\omega^k\varphi_i\alpha)(j)\big)$ for all $i \in [0,12)$, $j,k \in [0,n)$, $n = l(\alpha)$.

Lemma 1.1. *Let* $\alpha \in \mathcal{T}^\infty$, $n = l(\alpha)$ *and let* $j,k \in [0,n)$. *Then the following relations hold*

(*i*) $\sigma^2 = \rho^2 = 1$, $\omega\sigma\omega = \sigma$, $\omega^{-1}(\alpha) = \omega^{n-1}(\alpha)$, $\omega^k(\alpha) = \omega^t(\alpha)$ *if* $k \equiv t (mod\, n)$,

(*ii*) *every two elements of the set* $\{\omega, \sigma, \rho, \varphi_2, \varphi_4\}$ *commute, besides* ω, σ *and* φ_2, φ_4,

(*iii*) $\mathcal{P}_1 = \{\varphi_i : i \in [0,12)\}$; $\mathcal{P}_1$ *is generated by* $\{\varphi_2, \varphi_4\}$,

(*iv*) $\mathcal{P} = \{\omega^k\varphi_i : k \in \mathbf{N}, i \in [0,12)\}$,

(*v*) $C(i,j,0)(\alpha) = i + (-1)^i\kappa\alpha(j)(mod\, 6)$ *for* $i = 0,1$,

(*vi*) $C(i,j,0))(\alpha) = 1 - i + (-1)^{i+1}\kappa\alpha(j)(mod\, 6)$ *for* $i = 2,3,4,5$,

(*vii*) $C(i+6,j,0)(\alpha) = C(i, n-1-j, 0) + 3(mod\, 6)$ *for* $i \in [0,6)$,

(*viii*) $C(i,j,k)(\alpha) = C(i, (j-k)(mod\, 6), 0)(\kappa\alpha)$ *for* $i \in [0,12)$.

Proof. (*i*) $\omega\sigma\omega\alpha(j) = \omega\sigma\big(\alpha(j+1)\big) = \omega\alpha(n-1-j-1) = \alpha(n-1-j) = \sigma\alpha(j)$. The rest of the proof is straightforward when we use Table 1 – Table 3 . □

Definition 1.2. $\alpha, \beta \in \mathcal{T}^\infty$ are called *parastrophic equivalent* if there exists $\varphi \in \mathcal{P}$ such that $\varphi(\alpha) = \beta$.

Obviously the parastrophic equivalence is an equivalence relation; by $[\alpha]$ it will be denoted the class of the relation that comprises α. With respect to (*iv*) we have

$$[\alpha] = \{\, \omega^k\varphi_i(\alpha) : i \in [0,12),\, k \in [0,n),\, n = l(\alpha)\,\},$$

and by (*v*) – (*viii*)

$$[\alpha] = \{\, C(i,n-1,k)(\alpha)C(i,n-2,k)(\alpha)\ldots C(i,0,k)(\alpha) : i \in [0,12)\,\},$$

where $k \in [0, n)$, $n = l(\alpha)$.

In the following (by a personal computer) it will be found the lowest element of a class $[\alpha]$ for all $\alpha \in \mathcal{T}^n$ and $1 < n < 5$.

2. The parastrophic equivalence in $\mathcal{T}^2 - \mathcal{T}^4$

Theorem 2.1. *Let $n \in \{2, 3, 4\}$. Then every α in T^n is parastrophic equivalent to exactly one of the following elements*

LL	LR	LT	LL^{-1}	$(PE2)$
LLL	LLR	LLT	LLL^{-1}	$(PE3)$
LRT	LRL^{-1}	LRR^{-1}	LRT^{-1}	
LTR^{-1}	$LR^{-1}T$			
$LLLL$	$LLTR$	$LRLR$	$LTLT$	$(PE4)$
$LLLR$	$LLTT$	$LRLT$	$LTLL^{-1}$	
$LLLT$	$LLTL^{-1}$	$LRLL^{-1}$	$LTLR^{-1}$	
$LLLL^{-1}$	$LLTR^{-1}$	$LRLR^{-1}$	$LTT^{-1}L^{-1}$	
$LLRR$	$LLL^{-1}R$	$LRLT^{-1}$	$LL^{-1}LL^{-1}$	
$LLRT$	$LLL^{-1}T$	$LRTL^{-1}$	$LL^{-1}T^{-1}T$	
$LLRL^{-1}$	$LLL^{-1}L^{-1}$	$LRTR^{-1}$		
$LLRR^{-1}$	$LLR^{-1}R$	$LRL^{-1}T$		
$LLRT^{-1}$	$LLR^{-1}T$	$LRL^{-1}R^{-1}$		
	$LLT^{-1}R$	$LRL^{-1}T^{-1}$		
		$LRR^{-1}T$		
		$LRR^{-1}L^{-1}$		
		$LRT^{-1}T$		
		$LRT^{-1}L^{-1}$		
		$LRT^{-1}R^{-1}$		

In [1] V.D. Belousov defines: A primitive quasigroup $(Q, \cdot, \backslash, /)$ is a $\Pi-$*quasigroup of type* (α, β, γ) if α, β, $\gamma \in \sum(\cdot)$ and the quasigroup satisfies the identity

$$L_x^\alpha L_x^\beta L_x^\gamma \doteq 1.$$

This identity is equivalent to the identity $A_x B_x C_x \doteq 1$ for some $A, B, C \in \mathcal{T}$, $A^{-1} \neq B$, $C \neq B^{-1}$, $A \neq C^{-1}$. Therefore we can say that Q is a *quasigroup of type* ABC.

By Belousov [2], two $\Pi-$quasigroups of types ABC, DEF, respectively, are called *parastrophic equivalent* if $ABC = \varphi(DEF)$ for some $\varphi \in \mathcal{P}$; it is in the view of the definition of the parastrophic equivalence given in this paper. Thus if from 10 elements of the set $PE3$ delete LLL^{-1}, LRR^{-1}, LRL^{-1} then obtain 7 elements that determine 7 equivalence classes of the parastrophic equivalence relation listed in [2, Table 1].

If we want to determine the equivalence class of the parastrophic equivalency (for example) of the identity

$$(x/y) \setminus (y \setminus x) \doteq x \tag{5}$$

(see [2, p. 16]), then proceed like this: (5) is equivalent to

$$y \setminus x \doteq (x/y)x,$$

i.e.

$$R_x \doteq R_x L_x^{/}$$

whence by Table 1

$$T_z \doteq R_z T_z^{-1}$$

and also

$$R_z T_z^{-1} T_z^{-1} \doteq 1, \qquad \varphi_3(RT^{-1}T^{-1}) = R^{-1}LL.$$

Hence (5) is parastrophic equivalent to

$$L_x L_x R_x^{-1} \doteq 1,$$

i.e. $x \cdot xy \doteq yx$ in $(Q, /)$.

The lowest element of the set $[RT^{-1}T^{-1}]$ we can determine by a computer. Similarly we can proceed for arbitrary $ABC \in \mathcal{T}^3$; more generally, for arbitrary $x \in \mathcal{T}^n$, $n > 1$.

By a computer we can get $card(PE5) = 148$, $card(PE6) = 718$, $card(PE7) = 3441$.

References

[1] **V.D. Belousov**: *Systems of quasigroups with generalized identities*, (Russian), Usp. Mat. Nauk **20** (1965), 75 – 146.

[2] **V.D. Belousov**: *Parastrophic orthogonal quasigroups*, (Russian), Kishinev 1983.

[3] **A. Sade**: *Quasigroupes parastrophiques. Expression et identités*, Math. Nachr. **20** (1959), 73 – 106.

[4] **S.K. Stein**: *On the foundations of quasigroups*, Trans. Amer. Math. Soc. **85** (1957), 228 – 256.

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