Quasigroups and Related Systems **7** (2000), 1 – 5

Generation of NAFIL loops of small order

Raoul E. Cawagas

Abstract

NAFILs are non-associative finite invertible loops. These loops form a class that includes IP, Moufang, and Bol loops. This paper deals with the computer generation and determination of all non-isomorphic NAFILs of order $n = 5, 6, 7$. Results show that the number I_n of non-isomorphic NAFILs of order n are: $I_5 = 1$, $I_6 = 33$, and $I_7 = 2\ 333$.

1. Introduction

In 1998, I completed a two-year research project at the Polytechnic University of the Philippines to study **Non-Associative Finite Invertible Loops (NAFIL)**. This is a class of non-associative loops in which every element has a unique inverse and it includes the familiar IP, Moufang, and Bol loops. However, there are many interesting loops in this class that have not yet been studied as much as other loops.

Part of this project involved the determination of all distinct (non-isomorphic) NAFILs of orders $n = 5$, 6, and 7. For this, we developed a Pascal program called ICONSTRUCT to generate and determine the number I_n of distinct (non-isomorphic) finite invertible loops of

2000 Mathematics Subject Classification: 20N05
Keywords: invertible loop, non-associative finite loop
Research supported by the National Research Council of the Philippines (NRCP).

given orders. For NAFILs, the result of this study are summarized as follows:

Order n	5	6	7
Number I_n	1 (A)	33 (7A + 26NA)	16 (A only)

For orders 5 and 6, the numbers given are complete. For order 7, however, only the abelian (A) ones were determined. The non-abelian (NA) ones were not completely determined because of the enormous number of possible loops to be checked for isomorphism. For the same reason, we also did not consider loops of order $n \geqslant 8$.

2. Computer Generation of NAFILs

The problem of determining the number of non-isomorphic loops of a given type and order is a difficult one. It is known, for instance, that the number l_n of loops (both associative and non-associative) of orders $n = 5$, 6, and 7, are:

Order n	5	6	7
Number l_n	56	9 408	16 942 080
Number k_n	6	109	23 750

where k_n is the number of *isomorphy classes* (or non-isomorphic loops) of order n. Except for $n = 5$, which were completely determined by construction by A. A. Albert [1] in 1944, the figures given above were determined by calculation [2] using combinatorial formulas. What these loops actually are were not known because their Cayley tables were not generated.

Because of the importance of studying NAFILs of small order, we decided in 1996 to actually generate these loops, in the form of Cayley tables, by developing the program ICONSTRUCT. This program enabled us to generate all non-isomorphic loops of order $n \leqslant 6$ but it was not efficient enough to generate and determine all distinct NAFILs of order $n = 7$ in a reasonable period of time; except for the abelian ones. Using a Personal Computer (PC) with a Pentium processor, it is estimated that it would take several months for ICONSTRUCT to process 16 942 080 loops to determine all distinct NAFILs of order 7. Clearly, this would not be practical to undertake.

In November 1999, it has come to my attention that there are two softwares, SATO [3] and SEM [4], developed by Prof. Hantao Zhang at the University of Iowa, USA. These are finite model generators that were successfully used to study quasigroups. So, I posed the problem to Zhang and told him about our studies [5] with the program ICONSTRUCT and how it works.

After studying the matter, Zhang used his two software systems, SEM and SATO, to determine the number of distinct NAFILs of order $n = 7$. Given a set of logic formulas and a number n, SEM will generate models of size n satisfying the formulas. SATO can do the same; however, its input must be a set of propositional formulas. A set of formulas with fixed size, however, can be easily transformed to a set of propositional formulas.

Both SEM and SATO can easily generate all the NAFILs of a given small order. In fact, SATO generated 195 924 NAFILs of order 7 and SEM generated 29 679 (approximately 30K), both in a couple of minutes. The reason SEM produced less NAFILs is because SEM has a better mechanism to eliminate some isomorphic loops. Unlike ICONSTRUCT, however, both SATO and SEM were not designed to eliminate all NAFILs isomorphic to a given NAFIL.

The bottleneck of the problem is therefore how to identify isomorphic NAFILs. Zhang modified SATO so that it can check two given loops for isomorphism. Such a checking takes almost no time (about 0.01 second). Zhang then developed a simple shell script that removes, from a list of NAFILs, all loops isomorphic to a given NAFIL. Taking the approximately 30K NAFILs of order 7 yielded by SEM, the script would make approximately $90\mathrm{K}^2$ checkings for two loops if they were all distinct. The script takes about 0.1 second for each checking/removal of isomorphs. So it would take about 90 million seconds or over 10 thousand days to finish the job. Fortunately, it turns out that only 2 333 of them are distinct. So, using a PC with a single processor, the total computing time would have been about $30K^2 \times 2\,333 \times 0.1$ seconds, or 81 days. Clearly, as with ICONSTRUCT, this again would still have been not very practical to undertake.

Using a supercomputer of 48 Pentium II 400 processors, however, Zhang finished all the checkings in about three days. The supercomputer also made the rerun feasible. This is what Zhang did to determine the distinct (non-isomorphic) NAFILs of order $n = 7$. The result is as follows: There are exactly $I_7 = 2\ 333$ distinct (non-isomorphic) NAFILs of order $n = 7$ (16 abelian and 2 317 non-abelian). Zhang also processed the NAFILs of orders 5 and 6 and obtained the same results as those obtained by ICONSTRUCT. Hence we now have the following complete figures for all loops and non-isomorphic NAFILs of orders $n = 5$, 6, and 7:

Order n	5	6	7
Number l_n	56	9 408	16 942 080
Number k_n	6	109	23 750
Number I_n	1 (NA)	33 (7A+26NA)	2 333 (16A+2 317NA)

My research team at the SciTech R&D Center of the Polytechnic University of the Philippines is now studying these NAFIL loops to determine their structural properties. This is being done by means of a software called FINITAS [6] which we developed for studying finite algebraic structures. Our results so far have revealed many interesting properties of these loops.

For order $n = 8$, we have not been able to generate all distinct NAFILs, using the SEM and SATO programs, because they require too much disk space. On the other hand, a C program could replace the shell script so that it can be run at least twice faster. We can, however, generate specific examples of NAFILs of order $n \geqslant 8$ for study. For this, we have developed a simple program that generates NAFILs with a given number of selfinverse elements.

3. Summary

The generation and determination of all non-isomorphic NAFILs of order $n = 5, 6, 7$ have been successfully accomplished by means of three computer programs: ICONSTRUCT, SATO, and SEM. For $n = 7$, the use of a supercomputer (with 48 Pentium II 400 processors) was required. These programs not only generated the loops but also determined the exact number I_n of non-isomorphic loops of order $n \leqslant 7$.

The generated loops are in the form of Cayley tables. These Cayley tables are now being studied to determine the structure and basic properties of these loops.

Note: The file of the 2 333 Cayley tables of NAFILs of order 7 is available to interested researchers by sending an e-mail request to: **raoulec@pacific.net.ph**. The codes of SEM and SATO are also available at:
http://www.cs.uiowa.edu/~hzhang.

References

[1] **A. A. Albert**: *Quasigroups II*, Trans. Amer. Math. Soc. **55** (1944), 401 – 419.

[2] **B. F. Bryant and H. Schneider**: *Principal loop-isotopes of quasigroups*, Canad. J. Math. **18** (1966), 120 – 125.

[3] **H. Zhang, M. P. Bonacina and H. Hsiang**: *PSATO: a distributed propositional prover and its application to quasigroup problems*. Journal of Symbolic Computation **21** (1996), 543 – 560.

[4] **J. Zhang and H. Zhang**: *Generating models by SEM*, Proc. of International Conference on Automated Deduction (CADE-96), pp. 308 – 312, Lecture Notes in Artificial Intelligence 1104, Springer-Verlag.

[5] **R. E. Cawagas**: *Terminal Report: Development of the Theory of Finite Pseudogroups* (1998). Research supported by the National Research Council of the Philippines (1996-1998) under Project B-88 and B-95.

[6] **R. E. Cawagas**: *FINITAS: A Software for the Construction and Analysis of Finite Algebraic Structures*, PUP Journal of Research & Exposition (1998), Vol. 1, No. 1, pp. 1-10. (Full text of short communication presented at the 2nd European Congress of Mathematics, Budapest, Hungary, July 21-27, 1996.)

Scitech R&D Center
Received January 30, 2000
Polytechnic University of the Philippines, Sta. Mesa, Manila.