Quasigroups and Related Systems **6** (1999), 71 – 80

On congruences on n-ary T-quasigroups

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Abstract

We consider the class of n-ary quasigroups which are uniquely determined by some abelian group and their automorphisms. Connections between different groups corresponding to the same n-ary quasigroup are described.

According to Toyoda's theorem, if $Q(\cdot)$ is a medial (entropic) quasigroup, i.e., if it satisfies the identity $xy \cdot uv = xu \cdot yv$, then there exists an abelian group $Q(+)$, its authomorphisms φ, ψ and an element $g \in Q$ such that $\varphi\psi = \psi\varphi$ and $x \cdot y = \varphi x + \psi y + g$ for every $x, y \in Q$. Without the requirement $\varphi\psi = \psi\varphi$ such kind of abelian group isotopes are called T-quasigroups and was considered by Kepka and Nemec (cf. [2], [3]). Toyoda's theorem may be generalized to the case $n \geqslant 2$ (cf. [1]). So T-quasigroups of arity n can be defined analogously with the binary case. Most of the results proved for binary T-quasigroups can be generalized for n-T-quasigroups of any finite arity $n \geqslant 2$. At the same time the theory of n-quasigroups gives often new aspects of the proved for $n = 2$ facts. For example, for $n \geqslant 3$ there are n-groups $Q(A)$ and their congruences θ such that $C_a(A)$ is not an n-group for any class of congruence $C_a(A) \in Q/\theta$ (cf. [5]).

To avoid repetitions assume that $n \geqslant 2$ and $\overline{1,n} = 1, 2, \ldots, n$.

Definition 1. An n-quasigroup is called an *n-ary T-quasigroup* (or, shortly, an *n-T-quasigroup*) if there are a binary abelian group $Q(+)$, its automorphisms $\gamma_1, \gamma_2, \ldots, \gamma_n$ and an element $g \in Q$ such that

$$A(x_1^n) = \gamma_1 x_1 + \gamma_2 x_2 + \cdots + \gamma_n x_n + g$$

1991 Mathematics Subject Classification: 20N15
Keywords: n-ary quasigroup, T-quasigroup, isotope, congruence

for every $(x_1^n) \in Q^n$. The $(n+2)$-tuple $(Q(+), \gamma_1, \gamma_2, \ldots, \gamma_n, g)$ is called a *T-form* of $Q(A)$ and the group $Q(+)$ is called a *T-group* of $Q(A)$.

It follows from the definition of medial quasigroups that an n-T-quasigroup $Q(A)$ is medial iff $\gamma_i\gamma_j = \gamma_j\gamma_i$, for every $i, j \in \overline{\{1, n\}}$, where $(Q(+), \gamma_1, \gamma_2, \ldots, \gamma_n, g)$ is a T-form of $Q(A)$ (cf. [1]).

Proposition 1. *Any two T-groups corresponding to the same n-T-quasigroup are isomorphic.*

The proof follows from the Albert's theorem: isotopic groups are isomorphic.

An n-quasigroup $Q(A)$ is called an *n-ary isotope of a binary group* $Q(\circ)$ if there exist $n+1$ permutations $\alpha_1, \alpha_2, \ldots, \alpha_{n+1} \in S_Q$ such that

$$A(x_1^n) = \alpha_{n+1}^{-1}(\alpha_1 x_1 \circ \alpha_2 x_2 \circ \cdots \circ \alpha_n x_n)$$

for every $(x_1^n) \in Q^n$. If $\alpha_{n+1} = \varepsilon$ is the identical permutation, then $Q(A)$ is called a *principal n-ary isotope* of $Q(\circ)$ (cf. [4]). If, in addition, $\alpha_1, \alpha_2, \ldots, \alpha_{n+1}$ are linear mappings of the group $Q(\circ)$, i.e. if there exist some automorphisms $\theta_1, \theta_2, \ldots, \theta_{n+1}$ of $Q(\circ)$ and elements $a_1, a_2, \ldots, a_{n+1} \in Q$ such that $\alpha_i(x) = \theta_i(x) \circ a_i$ for $i = 1, 2, \ldots, n+1$, then the n-ary isotope $Q(A)$ is called *linear over* $Q(\circ)$.

Proposition 2. *If an n-T-quasigroup $Q(A)$ is an n-ary principal isotope of a binary group $Q(\circ)$ then $Q(A)$ is linear over $Q(\circ)$.*

Proof. Let $(Q(+), \gamma_1, \gamma_2, \ldots, \gamma_n, g)$ be a T-form of an n-T-quasigroup $Q(A)$ and let $Q(A)$ be an n-ary isotope of the binary group $Q(\circ)$. Then

$$\gamma_1 x_1 + \gamma_2 x_2 + \cdots + \gamma_n x_n + g = \alpha_1 x_1 \circ \alpha_2 x_2 \circ \cdots \circ \alpha_n x_n \qquad (1)$$

for every $(x_1^n) \in Q^n$ and for some $\alpha_1, \alpha_2, \ldots, \alpha_n \in S_Q$. Making the permutation $x_i \to \gamma_i^{-1} x_i$ for $i = \overline{1, n-1}$; $x_n \to R_g^{-1}\gamma_n^{-1} x_n$ in (1), where $R_g(x) = x + g$ for every $x \in Q$, we get:

$$x_1 + x_2 \cdots + x_n = \varphi_1 x_1 \circ \varphi_2 x_2 \circ \cdots \circ \varphi_n x_n, \qquad (2)$$

where $\varphi_i = \alpha_i \gamma_i^{-1}$, $i = \overline{1, n-1}$; $\varphi_n = \alpha_n R_g^{-1} \gamma_n^{-1}$. Now taking $x_1 = x$, $x_2 = y$, $x_3 = x_4 = \cdots = x_n = 0$ in (2) (0 is the neutral element of $Q(+)$), we obtain:

$$x + y = \varphi_1 x \circ \varphi_2 y \circ a, \qquad (3)$$

where $a = \varphi_3 0 \circ \cdots \circ \varphi_n 0$. Thus the groups $Q(+)$ and $Q(\circ)$ are isotopic and then, by Albert's theorem, they are isomorphic: $Q(+) \cong Q(\circ)$. In particular, we get that $Q(\circ)$ is abelian too. Taking $x_1 = \cdots = x_{i-1} = x_{i+1} = \cdots = x_n = 0$ in (2) we get $\varphi_i x_i \circ a_i = x_i$, or $\varphi_i x_i = x_i \circ a_i^{-1}$, where $a_i = \varphi_1 0 \circ \cdots \circ \varphi_{i-1} 0 \circ \varphi_{i+1} 0 \circ \cdots \circ \varphi_n 0$, $i = \overline{1,n}$. Therefore the equality (3) can be written in the form:

$$x + y = x \circ y \circ 0^{n-2} \circ b, \tag{4}$$

where $b = a_1^{-1} \circ \cdots \circ a_n^{-1}$.

Putting $\beta(x) = x \circ 0^{n-2} \circ b$ we obtain

$$\begin{aligned} \beta(x+y) &= (x+y) \circ 0^{n-2} \circ b = (x \circ y \circ 0^{n-2} \circ b) \circ 0^{n-2} \circ b \\ &= (x \circ 0^{n-2} \circ b) \circ (y \circ 0^{n-2} \circ b) = \beta(x) \circ \beta(y), \end{aligned}$$

which proves that the mapping β is an isomorphism from $Q(+)$ onto $Q(\circ)$. Moreover, $\beta\gamma_i\beta^{-1} \in Aut\, Q(\circ)$ for each $i \in \overline{1,n}$.

Denoting $\beta\gamma_i\beta^{-1}$ by θ_i, $i = \overline{1,n}$, using the equalities $\varphi_i = \alpha_i\gamma_i^{-1}$, $1 \leqslant i \leqslant n-1$, $\varphi_n = \alpha_n R_g^{-1}\gamma_n^{-1}$ and $\varphi_j(x_j) = x_j \circ a_j^{-1}$, $j = \overline{1,n}$, we have: $\alpha_i\gamma_i^{-1}(x) = x \circ a_i^{-1}$, or $\gamma_i(x) = \alpha_i(x) \circ a_i$, $1 \leqslant i \leqslant n-1$ and $\gamma_n(x) = \alpha_n(x) \circ d_n$, where $d_n = a_n \circ 0^{n-2} \circ (-g) \circ b$. Thus

$$\begin{aligned} \theta_i(x) = \beta\gamma_i\beta^{-1}(x) &= \beta\gamma_i(x \circ (0^{n-2} \circ b)^{-1}) \\ &= \beta[\alpha_i(x \circ (0^{n-2} \circ b)^{-1}) \circ d_i] \\ &= \alpha_i(x \circ (0^{n-2} \circ b)^{-1}) \circ d_i \circ 0^{n-2} \circ b \end{aligned}$$

involves

$$\alpha_i(x) = \theta_i(x \circ 0^{n-2} \circ b) \circ d_i^{-1} \circ (0^{n-2} \circ b)^{-1} = \theta_i(x) \circ c_i,$$

where

$$c_i = \theta_i(0^{n-2} \circ b) \circ d_i^{-1} \circ (0^{n-2} \circ b)^{-1},$$

i.e. the permutations α_i, $1 \leqslant i \leqslant n$ are linear over $Q(\circ)$. Moreover,

$$\begin{aligned} c_i &= \theta_i(0^{n-2} \circ b) \circ d_i^{-1} \circ (0^{n-2} \circ b)^{-1} \\ &= \alpha_i(e) \circ d_i \circ (0^{n-2} \circ b) \circ (0^{-1} \circ b)^{-1} \circ d_i^{-1} = \alpha_i(e), \end{aligned}$$

where e is the unit of $Q(\circ)$, i.e. $c_i = \alpha_i(e)$, $1 \leqslant i \leqslant n$. Hence $\alpha_i = \theta_i(x) \circ \alpha_i(e)$ fot all $1 \leqslant i \leqslant n$. □

Proposition 3. *Let $Q(+)$ be a T-group of an n-ary T-quasigroup $Q(A)$ and let $P(A)$ be an n-ary subquasigroup of $Q(A)$. If the neutral element 0 of $Q(+)$) belongs to P, then $P(A)$ is an n-T-quasigroup and $P(+)$ is a T-group of $P(A)$.*

Proof. Let $(Q(+), \gamma_1, \dots \gamma_n, g)$ be a T-form of $Q(A)$. If $0 \in P$, then $A(\overset{n}{0}) = g \in P$. More, if $A(\overset{i-1}{0}, x, \overset{n-i}{0}) = 0$, where $1 \leqslant i \leqslant n$, then $x \in P$, i.e. $\gamma_i(x) + g = 0$, so $x = \gamma_i^{-1}(-g) \in P$ for every $1 \leqslant i \leqslant n$.

If $x \in P$ then $\gamma_i(y) = A(\overset{i-1}{0}, y, \overset{n-i-1}{0}, \gamma_n^{-1}(-g)) = x$, implies $y \in P$, i.e. $x \in P$ gives $\gamma_i^{-1}(x) \in P$ for every $i = \overline{1, n}$. Thus, for every $x, y \in P$ we have $\gamma_1^{-1}(x),\ \gamma_2^{-1}(y) \in P$. Therefore

$$A(\gamma_1^{-1}(x), \gamma_2^{-1}(y), \overset{n-3}{0}, \gamma_n^{-1}(-g)) = x + y \in P.$$

Further, for $x \in P$ there exists an element $y \in P$ such that

$$x + y = A(\gamma_1^{-1}(x), \gamma_2^{-1}(y), \overset{n-3}{0}, \gamma_n^{-1}(-g)) = 0,$$

i.e. $y = -x \in P$. Thus $P(+)$ is a subgroup of $Q(+)$ and $P(A)$ is an n-T-quasigroup with T-form $(P(+), \gamma_1|_P, \dots, \gamma_n|_P, g)$. □

Proposition 4. *Let $Q(A)$ be an n-T-quasigroup and $P(A)$ be an n-ary subquasigroup of $Q(A)$. Then for every $a \in P$ there exists a binary group $Q(\circ)$ with the unit a such that $P(\circ)$ is a T-group of $P(A)$.*

Proof. Let $(Q(+), \gamma_1, \dots, \gamma_n, g)$ be a T-form of $Q(A)$ and $P(A)$ be an n-ary subquasigroup of $Q(A)$. If $a \in P$ then $Q(\circ) \simeq Q(+)$, where the binary operation $(\circ)$ is defined by $x \circ y = R_a^{-1}x + y$, and a is the unit of the group $Q(\circ)$. According to Proposition 2, from the equalities

$$\begin{aligned} A(x_1^n) &= \gamma_1 x_1 + \dots + \gamma_n x_n + g \\ &= R_a\gamma_1 x_1 \circ \dots \circ R_a\gamma_n x_n \circ g = \varphi_1 x_1 \circ \dots \circ \varphi_n x_n, \end{aligned}$$

where $\varphi_i = R_a\gamma_i$ for every $1 \leqslant i \leqslant n-1$ and $\varphi_n = R_g^{(\circ)} R_a\gamma_n$, follows that there exist $\theta_1, \theta_2, \dots, \theta_n \in Aut\ Q(\circ)$ such that

$$\theta_1 x_1 \circ \dots \circ \theta_n x_n \circ \varphi_1(a) \circ \dots \circ \varphi_n(a) = A(x_1^n).$$

According to Proposition 3, $P(A)$ is an n-T-quasigroup, $P(\circ)$ is one of its T-groups and $(P(\circ), \theta_1|_P, \dots, \theta_n|_P, d)$, where $d = \varphi_1(a) \circ \dots \circ \varphi_n(a)$, is a T-form of $P(A)$. □

Corollary 1. *If $Q(A)$ is an n-T-quasigroup, then for every $a \in Q$ there exists a T-group of $Q(A)$ with the neutral element a.* □

Corollary 2. *Every n-ary subquasigroup of an n-T-quasigroup is an n-T-quasigroup.* □

Let $Q(A)$ be an n-ary groupoid and let θ be an equivalence relation on Q. Then we say that θ is a *congruence relation on the n-groupoid* $Q(A)$ iff the following statement holds

$$a_i \theta b_i,\ i = \overline{1,n} \implies A(a_1^n)\theta A(b_1^n) \tag{4}$$

for every $(a_1^n), (b_1^n) \in Q^n$. The statement (4) is equivalent to

$$a\theta b \implies A(c_1^{i-1}, a, c_i^{n-1})\theta A(c_1^{i-1}, b, c_i^{n-1}) \tag{5}$$

for every $a, b \in Q$ and for every $(c_1^n) \in Q^n$.

Definition 2. The congruence θ defined on the n-groupoid $Q(A)$ is called *normal* if for every $i = \overline{1,n}$ and for every $(c_1^n) \in Q^n$

$$A(c_1^{i-1}, a, c_{i+1}^n)\theta A(c_1^{i-1}, b, c_{i+1}^n) \Longrightarrow a\theta b.$$

Proposition 5. *Let θ be a normal congruence of an n-T-quasigroup $Q(A)$. Then θ is a congruence of any its T-group.*

Proof. Let $(Q(+), \gamma_1, \ldots, \gamma_n, g)$ be a T-form of an n-T-quasigroup $Q(A)$ and let θ be a normal congruence of $Q(A)$. Then

$$\begin{aligned} a\theta b &\iff A(\gamma_1^{-1}(a), \overset{n-2}{0}, \gamma_n^{-1}(-g))\,\theta\, A(\gamma_1^{-1}(b), \overset{n-2}{0}, \gamma_n^{-1}(-g)) \\ &\iff \gamma_1^{-1}(a)\theta\gamma_1^{-1}(b), \end{aligned}$$

therefore

$$A(\gamma_1^{-1}(a), \gamma_2^{-1}(c), \overset{n-3}{0}, \gamma_n^{-1}(-g))\theta A(\gamma_1^{-1}(b), \gamma_2^{-1}(c), \overset{n-3}{0}, \gamma_n^{-1}(-g))$$

or $(a + c)\theta(b + c)$, i.e. θ is a congruence on $Q(+)$. □

Proposition 6. *Let $(Q(+), \gamma_1, \ldots, \gamma_n, g)$ be a T-form of an n-T-quasigroup $Q(A)$ and let θ be a congruence of $Q(+)$. Then*

1. *θ is a congruence on $Q(A)$ if and only if $\gamma_i|_{Ker\theta}$ are endomorphisms of the group $Ker\theta$ for every $i = \overline{1,n}$.*
2. *θ is a normal congruence on $Q(A)$ if and only if $\gamma_i|_{Ker\theta}$ are automorphisms of the group $Ker\theta$ for every $i = \overline{1,n}$.*

Proof. 1) Let θ be a congruence on $Q(A)$. If $a \in \operatorname{Ker}\theta$ then $a\theta 0$, i.e.

$$A(\overset{i-1}{0}, a, \overset{n-i-1}{0}, \gamma_n^{-1}(-g))\,\theta\, A(\overset{n-1}{0}, \gamma_n^{-1}(-g))$$

for every $1 \leqslant i \leqslant n-1$, therefore $\gamma_i(a)\theta 0$ for every $1 \leqslant i \leqslant n-1$. Analogously we get $\gamma_n(a) \in \operatorname{Ker}\theta$. Thus $\gamma_i(a) \in \operatorname{Ker}\theta$ for every $i = \overline{1,n}$ and $a \in \operatorname{Ker}\theta$. If $a, b \in \operatorname{Ker}\theta$, then $a\theta 0$ and $b\theta 0$, therefore $a\theta b$ and $(a+b)\theta 0$. Thus $a + b \in \operatorname{Ker}\theta$ and then $\gamma_i(a+b) \in \operatorname{Ker}\theta$ for every $i = \overline{1,n}$. But $\gamma_i(a), \gamma_i(b) \in \operatorname{Ker}\theta$ involves $\gamma_i(a) + \gamma_i(b) \in \operatorname{Ker}\theta$. Thus $\gamma_i|_{\operatorname{Ker}\theta}$ is an endomorphism on $\operatorname{Ker}\theta$ for every $i = \overline{1,n}$.

Conversely, let $\gamma_i|_{\operatorname{Ker}\theta}$, $i = \overline{1,n}$ be endomorphisms of $\operatorname{Ker}\theta$ and let $a\theta b$. Then $(a-b)\theta 0$, i.e. $\gamma_i(a-b) \in \operatorname{Ker}\theta$, therefore $\gamma_i(a)\theta\gamma_i(b)$ for every $i = \overline{1,n}$. So

$$(\gamma_1(c_1) + \dots + \gamma_{i-1}(c_{i-1}) + \gamma_i(a) + \gamma_{i+1}(c_{i+1}) + \dots \gamma_n(c_n))$$

and

$$\theta(\gamma_1(c_1) + \dots + \gamma_{i-1}(c_{i-1}) + \gamma_i(b) + \gamma_{i+1}(c_{i+1}) + \dots \gamma_n(c_n))$$

are equivalent. Thus

$$A(c_1^{i-1}, a, c_{i+1}^n)\theta A(c_1^{i-1}, b, c_{i+1}^n)$$

for every $(c_1^n) \in Q^n$, $i = \overline{1,n}$. Hence θ is a congruence on $Q(A)$.

2) Let θ be a normal congruence on $Q(A)$. Then $\gamma_i|_{\operatorname{Ker}\theta}$ is an endomorphism of $\operatorname{Ker}\theta$. Moreover,

$$\begin{aligned} a\theta 0 &\Longleftrightarrow A(\gamma_1^{-1}(a), \overset{n-2}{0}, \gamma_n^{-1}(-g))\theta A(\overset{n-1}{0}, \gamma_n^{-1}(-g)) \\ &\Longleftrightarrow \gamma_1^{-1}(a)\theta 0 \iff \gamma_1^{-1}(a) \in \operatorname{Ker}\theta. \end{aligned}$$

Analogously can be proved that $a\theta 0 \iff \gamma_i^{-1}(a) \in \operatorname{Ker}\theta$ for every $i = \overline{2,n}$. Therefore γ_i is invertible on $\operatorname{Ker}\theta$. So $\gamma_i|_{\operatorname{Ker}\theta}$ is an automorphism of $\operatorname{Ker}\theta$ for every $i = \overline{1,n}$.

On the other hand, if $\gamma_i|_{\operatorname{Ker}\theta} \in Aut\ \operatorname{Ker}\theta$, $i = \overline{1,n}$, then θ is a congruence on $Q(A)$. Moreover,

$$\begin{aligned} A(c_1^{i-1}, a, c_{i+1}^n)\theta A(c_1^{i-1}, b, c_{i+1}^n) &\Longleftrightarrow \gamma_i(a-b) \in \operatorname{Ker}\theta \\ &\Longleftrightarrow a - b \in \operatorname{Ker}\theta \Longleftrightarrow a\theta b \end{aligned}$$

for every $i = \overline{1,n}$ and $(c_1^n) \in Q^n$, i.e. θ is a normal congruence on $Q(A)$. □

Remark 1. Not every congruence of an n-T-quasigroup is normal as the following example shows. Let $Q(+)$ be the additive group of rational numbers. Then $Q(A)$, where $A(x_1^n) = 2x_1 + 2x_2 + \cdots + 2x_n$ for every $(x_1^n) \in Q^n$, is a medial n-quasigroup. The binary relation η defined by

$$x\eta y \iff x - y \in Z$$

is a congruence on $Q(A)$. Moreover, if $x_i \eta y_i$ for every $1 \leqslant i \leqslant n$, then also $x_i - y_i \in Z$, and $(x_1 + \cdots + x_n) - (y_1 + \cdots + y_n) \in Z$, which implies $A(x_1^n)\eta A(y_1^n)$. This proves that η is a congruence on $Q(A)$. But $\operatorname{Ker}\eta = Z$ and $\gamma_i|_Z$, $i = \overline{1,n}$ are not automorphisms of $Z(+)$, so η is not normal on $Q(A)$. Examples for $n = 2$ are given in [2]. □

Proposition 7. *Let $(Q(+), \gamma_1, \ldots, \gamma_n, g)$ be a T-form of an n-T-quasigroup $Q(A)$. If at least one of the automorphisms γ_i has a finite order, then every congruence on $Q(A)$ is a congruence on $Q(+)$.*

Proof. Let $\gamma_1^m = \varepsilon$ for some fixed m and let θ be a congruence on $Q(A)$. Then $a\theta b$ implies $A(a, \overset{n-2}{0}, \gamma_n^{-1}(-g))\theta A(b, \overset{n-2}{0}, \gamma_n^{-1}(-g))$, i.e. $\gamma_1(a)\,\theta\,\gamma_1(b)$. Thus, by induction, we have $\gamma_1^{m-1}(a)\,\theta\,\gamma_1^{m-1}(b)$, which gives $\gamma_1^{-1}(a)\,\theta\,\gamma_1^{-1}(b)$. Hence

$$A(\gamma_1^{-1}(a), \gamma_2^{-1}(c), \overset{n-3}{0}, \gamma_n^{-1}(-g))$$

and

$$A(\gamma_1^{-1}(b), \gamma_2^{-1}(c), \overset{n-3}{0}, \gamma_n^{-1}(-g))(a+c)\theta(b+c)$$

are equivalent, i.e. $(a+c)\theta(b+c)$ for every $c \in Q$. So θ is a congruence of $Q(+)$. □

Definition 3. An n-ary subquasigroup $P(A)$ of an n-T-quasigroup $Q(A)$ is called *normal in* $Q(A)$ if there exists a normal congruence θ of $Q(A)$ such that P is a class of equivalence of θ.

Proposition 8. *Every n-ary subquasigroup of an n-T-quasigroup $Q(A)$ is normal.*

Proof. Let $(Q(+), \gamma_1, \ldots, \gamma_n, g)$ be a T-form of $Q(A)$ and let $P(A)$ be an n-ary subquasigroup of $Q(A)$. If $0 \in P$ is the neutral element of $Q(+)$ then $P(A)$ is an n-T-quasigroup and $(P(+), \gamma_1|_P, \ldots, \gamma_n|_P, g)$ is a T-form of $P(A)$. So as $P(+)$ is an invariant subgroup of $Q(+)$ the factor-group Q/P defines a congruence θ on $Q(+)$ such that P is a class of equivalence of θ. Since $\operatorname{Ker}\theta = P$ we get that $\gamma_i|_P = \gamma_i|_{\operatorname{Ker}\theta}$ are automorphisms of $\operatorname{Ker}\theta$, i.e. θ is a normal congruence on $Q(A)$ and $P(A)$ is a normal subquasigroup of $Q(A)$. According to Proposition 4, for every element $a \in Q$ there is a T-group of $Q(A)$, having a as a neutral element. □

Remark 2. As it is well known, if θ is a congruence of a binary group $Q(\cdot)$ then there is exactly one $C_a \in Q/\theta$ such that $(C_a, \cdot)$ is a subgroup of $Q(\cdot)$. J.Ušan proved that for $n \geqslant 2$ there are n -groups $Q(A)$ and their congruences θ such that for any $C_a \in Q/\theta$, $C_a(A)$ is not an n-group.

Proposition 9. *Let* $(Q_1(+), \gamma_1, \dots, \gamma_n, g_1)$ *and* $(Q_2(\circ), \alpha_1, \dots, \alpha_n, g_2)$ *be two* T*-forms of* n*-*T*-quasigroups* $Q_1(A)$ *and* $Q_2(B)$*, respectively. If* $\eta : Q_1(A) \to Q_2(B)$ *is a morphism of* n*-quasigroups, then the mapping* $\varphi : Q_1(+) \to Q_2(\circ)$ *defined by* $\varphi(x) = \eta(x) \circ (\eta(0))^{-1}$ *is a group morphism and* $\varphi\gamma_i = \alpha_i\varphi$ *for every* $i = \overline{1,n}$*. Moreover,* φ *is an isomorphism if and only if* η *is an isomorphism (*0 *is the neutral element of* $Q(+)$*).*

Proof. From $\eta A(x_1^n) = B(\eta x_1, \eta x_2, \dots, \eta x_n)$ follows

$$\eta(\gamma_1 x_1 + \cdots + \gamma_n x_n + g_1) = \alpha_1\eta(x_1) \circ \cdots \circ \alpha_n\eta(x_n) \circ g_2. \qquad (7)$$

Putting in (7) $x_i = 0$ for $1 \leqslant i \leqslant n-1$ and $x_n = \gamma_n^{-1}(-g_1)$, we get:

$$\eta(0) = \alpha_1\eta(0) \circ \cdots \circ \alpha_{n-1}\eta(0) \circ \alpha_n\eta(\gamma_n^{-1}(-g_1)) \circ g_2.$$

Therefore

$$\alpha_1\eta(0) \circ \cdots \circ \alpha_{i-1}\eta(0) \circ \alpha_{i+1}\eta(0) \circ \cdots \circ \alpha_{n-1}\eta(0) \circ \alpha_n\eta\gamma_n^{-1}(-g_1) \circ g_2 = \eta(0) \circ (\alpha_i\eta(0))^{-1}.$$

Thus

$$\eta\gamma_i(x_i) = \alpha_i\eta(x_i) \circ \eta(0) \circ (\alpha_i\eta(0))^{-1},$$

and

$$\eta\gamma_i(x_i) \circ (\eta(0))^{-1} = \alpha_i\eta(x_i) \circ (\alpha_i\eta(0))^{-1} = \alpha_i\eta(x_i) \circ \alpha_i(\eta(0))^{-1} = \alpha_i(\eta(x_i) \circ (\eta(0))^{-1}).$$

From the last equalities we get $\varphi\gamma_i(x_i) = \alpha_i\varphi(x_i)$ for every $x_i \in Q$, i.e. $\varphi\gamma_i = \alpha_i\varphi$ for every $i = \overline{1,n}$.

Putting $x_i = 0$ for $1 \leqslant i \leqslant n-1$ and $x_n = \gamma_n^{-1}(-g_1)$ in (7), we see that

$$\eta(\gamma_1(x_1) + \gamma_2(x_2)) = \alpha_1\eta(x_1) \circ \alpha_2\eta(x_2) \circ \eta(0) \circ (\alpha_1\eta(0) \circ \alpha_2\eta(0))^{-1}$$

implies

$$\eta(\gamma_1(x_1) + \gamma_2(x_2)) \circ (\eta(0))^{-1} = \alpha_1(\eta(x_1) \circ \eta(0)^{-1}) \circ \alpha_2(\eta(x_2) \circ \eta(0)^{-1}),$$

which gives

$$\varphi(\gamma_1(x_1) + \gamma_2(x_2)) = \alpha_1\varphi(x_1) \circ \alpha_2\varphi(x_2) = \varphi\gamma_1(x_1) \circ \varphi\gamma_2(x_2)$$

for every $x_1, x_2 \in Q$. Thus φ is a group morphism from $Q_1(+)$ to $Q_2(\circ)$. □

Corollary 3. *Let $Q_1(A)$ and $Q_2(B)$ be two n-T-quasigroups and $Q_1(+)$, $Q_2(\circ)$ be their T-groups, respectively. Then a morphism η from $Q_1(A)$ to $Q_2(B)$ is a morphism from $Q_1(+)$ to $Q_2(\circ)$ if and only if $\eta(0) = e$, where 0 and e are the neutral elements of $Q_1(+)$ and $Q_2(\circ)$, respectively.*

Proof. The mapping $\varphi : Q_1(+) \to Q_2(\circ)$ such that

$$\varphi(x) = \eta(x) \circ \eta(0)^{-1}$$

is a morphism of groups. So $\varphi(x+y) = \varphi(x) \circ \varphi(y)$ is equivalent to

$$\eta(x+y) \circ \eta(0)^{-1} = \eta(x) \circ \eta(0)^{-1} \circ \eta(y) \circ \eta(0)^{-1}.$$

Thus

$$\eta(x+y) \circ \eta(0)^{-1} = \eta(x) \circ \eta(y) \circ \eta(0)^{-1}.$$

Hence $\eta(0) = e$ if and only if $\eta(x+y) = \eta(x) \circ \eta(y)$. □

Proposition 10. *Let $Q(A)$ be an n-T-quasigroup, K and H be two n-ary subquasigroups of $Q(A)$. If there is a congruence θ on $Q(A)$ such that $H, K \in Q/\theta$ then H and K are isomorphic.*

Proof. So as $K(A)$ and $H(A)$ are normal subquasigroups in $Q(A)$ there is a normal congruence ρ on $Q(A)$ such that $K(A)$ is one of its classes. If $(c_1^n) \in K^n$ and $a\rho b$ for some $a, b \in Q$ then there are $x, y \in Q$ such that $A(c_1^{i-1}, x, c_{i+1}^n) = a$ and $A(c_1^{i-1}, x, y, c_{i+2}^n) = b$. Hence from $a\rho b$ follows $c_{i+1}\rho y$, i.e. $y \in K$.

Let θ be a congruence on $Q(A)$ such that $K, H \in Q/\theta$. Then $y, c_{i+1} \in K$ implies $y\theta c_{i+1}$, thus $a\rho b$ implies $a\theta b$.

Let $K \in Q/\rho \cap Q/\theta$ and $a\theta b$. Then for every $(c_1^n) \in K^n$ there exists $x \in Q$ such that $A(c_1^{i-1}, x, a, c_{i+2}^n) \in K$. But $a\theta b$ implies $A(c_1^{i-1}, x, a, c_{i+2}^n)\theta A(c_1^{i-1}, x, b, c_{i+2}^n)$, hence $A(c_1^{i-1}, x, b, c_{i+2}^n) \in K$. So as $K \in Q/\rho$ we get: $A(c_1^{i-1}, x, a, c_{i+2}^n)\rho A(c_1^{i-1}, x, b, c_{i+2}^n)$ thus $a\rho b$, i.e. $a\theta b$ implies $a\rho b$. So we get that $\theta = \rho$ thus θ must be a normal congruence too.

Let $H(+)$ and $K(\circ)$ be T-groups of $H(A)$ and $K(A)$, respectively. Let 0 and e be the neutral elements of $H(+)$ and $K(\circ)$, respectively. The mapping $\sigma : Q(+) \to Q(\circ)$ defined by $\sigma(x) = x \circ 0^{-1}$, is a group isomorphism (by Proposition 9, for $\eta = \varepsilon$).

From $\sigma(0) = e$ it follows that $\sigma \in \operatorname{Aut} Q(A)$. So as θ is a normal congruence on $Q(A)$, θ is a congruence on $Q(\circ)$. Therefore $a\theta b \Leftrightarrow a \circ {}^{-1}\theta b \circ 0^{-1} \Leftrightarrow \sigma(a)\theta\sigma(b)$. Thus $\sigma(H) = K$. □

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Received October 15, 1999