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Invertible elements in associates and semigroups. 2

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Abstract

This work is a continuation of [12]. Some additional invertibility criteria for elements of associates and n-ary semigroups are found. The corresponding axiomatics for polyagroups and n-ary groups are established.

The study of (i,j)-associative $(n+1)$-ary groupoids is reduced in [8] to the study of so-called associate of the type (s,n), where $s|n$. A bracketting rule and a decomposition of the main operation was described in [10]. Some criteria of invertibility of elements are found in [12]. Here, we give some additional criteria of invertibility and find axiomatics for polyagroups and n-groups.

The following theorem is proved in [10]

Theorem 1. *Let (Q,f) be an associate of a type (r,s,n). If the words w_1 and w_2 differ from each other by the bracketting only and the coordinate of every f's occurrence in the words w_1 and w_2 is divisible by r and also there exists a one-to-one correspondence between f's occurrences in the word w_1 and those in the word w_2 such that the corresponding coordinates are congruent modulo s, then the formula $w_1 = w_2$ is an identity in (Q,f).*

By the *coordinate of the i-th occurrence of the symbol f in a word w* is mean a number of all individual variables and constants, appearing

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in the word w from the beginning of w to the i-th occurrence of the operation symbol f.

A transformation $\lambda_{i,a}$ of the set Q, which is determined by the equality

$$\lambda_{i,a}(x) = f(\overset{i}{a}, x, \overset{n-i}{a}), \tag{1}$$

is said to be an *i-th shift of the groupoid (Q, f) induced by an element a*. Hence, the i-th shift is a partial case of the translation (see [1]). If the i-th shift is a substitution of the set Q, then the element a is called *i-invertible*. If an element a is i-invertible for all $i = 0, 1, \ldots, n$, then it is called *invertible*. Invertible elements in n-semigroups are described by Gluskin in [6] and [7].

The following theorem is proved in [12]

Theorem 2. *An element $a \in Q$ is invertible in an associate (Q, f) of the type (s, n) iff there exists an element $\bar{a} \in Q$ such that*

$$f(\bar{a}, a, \ldots a, x) = x, \qquad f(x, a, \ldots a, \bar{a}) = x \tag{2}$$

for all $x \in Q$.

1. Criterion of invertibility

Corollary 1. *An element a is invertible in an associate (Q, f) of the type (s, n) iff there exist $\hat{a}$ and $\breve{a}$ such that*

$$f(\hat{a}, a, \ldots, a, x) = x, \qquad f(x, a, \ldots, a, \breve{a}) = x \tag{3}$$

hold for all $x \in Q$.

Proof. If an element a is r-multiple invertible, then (2) are true according to Theorem 2. Therefore (3) with $\hat{a} = \breve{a} = \bar{a}$ hold.

Conversely, assume that (3) hold. Putting $x = \breve{a}$ in the first equality, and $x = \hat{a}$ in the second, we obtain

$$f(\hat{a}, a, \ldots, a, \breve{a}) = \breve{a} \quad \text{and} \quad f(\hat{a}, a, \ldots, a, \breve{a}) = \hat{a}.$$

Hence $\hat{a} = \breve{a}$. Thus (2) hold.

The invertibility of a follows from Theorem 2. □

Lemma 1. *If an element a is i-invertible in an associate (Q,f) of the type (s,n), then every i-th skew element to a is also j-th skew for all $j \equiv i \pmod{s}$.*

Proof. Since the i-th shift induced by a is a substitution of the set Q, then

$$a = \lambda_{i,a}^{-1}\lambda_{i,a}(a) \overset{(1)}{=} \lambda_{i,a}^{-1} f(\overset{n+1}{a}) \overset{(1)}{=} \lambda_{i,a}^{-1} f(\overset{j}{a}, \lambda_{i,a}\lambda_{i,a}^{-1}(a), \overset{n-j}{a})$$

$$\overset{(4)}{=} \lambda_{i,a}^{-1} f(\overset{j}{a}, f(\overset{i}{a}, \bar{a}^i, \overset{n-i}{a}), \overset{n-j}{a}) \overset{T1}{=} \lambda_{i,a}^{-1} f(\overset{i}{a}, f(\overset{j}{a}, \bar{a}^i, \overset{n-j}{a}), \overset{n-i}{a})$$

$$\overset{(1)}{=} \lambda_{i,a}^{-1}\lambda_{i,a} f(\overset{j}{a}, \bar{a}^i, \overset{n-j}{a}) = f(\overset{j}{a}, \bar{a}^i, \overset{n-j}{a}).$$

Thus $f(\overset{j}{a}, \bar{a}^i, \overset{n-j}{a}) = a$. This means, that $\bar{a}^i$ is the j-th skew to a. □

If an element a of a multiary groupoid is i-invertible, then the element $\lambda_{i,a}^{-1}(a)$ coincides with the *i-th skew* of the element a, which is denoted by $\bar{a}^i$ $(\bar{a} := \bar{a}^0)$ and is determined by the equality

$$f(\overset{i}{a}, \bar{a}^i, \overset{n-i}{a}) = a. \tag{4}$$

The following theorem is valid.

Theorem 3. *In any associate (Q,f) of the type (s,n) for any element a and for any $i = 0,1,\ldots,n-1$; $k = 1,\ldots,\frac{n}{s}-1$ the following conditions are equivalent:*

1) *a is invertible;*
2) *a is i- and $(n-i)$-invertible;*
3) *there exist elements $\hat{a}$ and $\breve{a}$ from Q such that*

$$f(\overset{i}{a}, \hat{a}, \overset{n-i-1}{a}, x) = x \quad \textit{and} \quad f(x, \overset{n-i-1}{a}, \breve{a}, \overset{i}{a}) = x \tag{5}$$

hold for all $x \in Q$.

4) *a is ks-invertible.*

Proof. 1) ⇒ 2) by the definition of invertibility.

2) ⇒ 3). Since the element a is i- and $(n-i)$-invertible, the i-th and $(n-i)$-th shifts are substitutions of the set Q.

Let $i \leqslant n-s$. To prove the relation (5), we consider the following equalities:

$$\begin{aligned} x = \lambda_{i,a}^{-1}\lambda_{i,a}(x) &\overset{(1)}{=} \lambda_{i,a}^{-1} f(\overset{i}{a}, x, \overset{n-i}{a}) \\ &\overset{L1}{=} \lambda_{i,a}^{-1} f(\overset{i}{a}, x, \overset{s-1}{a}, f(\overset{n-s-i}{a}, \bar{a}^{(n-i)}, \overset{i+s}{a}), \overset{n-s-i}{a}) \\ &\overset{T1}{=} \lambda_{i,a}^{-1} f(\overset{i}{a}, f(x, \overset{n-i-1}{a}, \bar{a}^{(n-i)}, \overset{i}{a}), \overset{n-i}{a}) \\ &\overset{(1)}{=} \lambda_{i,a}^{-1}\lambda_{i,a} f(x, \overset{n-i-1}{a}, \bar{a}^{(n-i)}, \overset{i}{a}) = f(x, \overset{n-i-1}{a}, \bar{a}^{(n-i)}, \overset{i}{a}). \end{aligned}$$

Hence, the second equality from (5) holds.

To prove the first, observe that

$$\begin{aligned} x = \lambda_{n-i,a}^{-1}\lambda_{n-i,a}(x) &\overset{(1)}{=} \lambda_{n-i,a}^{-1} f(\overset{n-i}{a}, x, \overset{i}{a}) \\ &\overset{L1}{=} \lambda_{n-i,a}^{-1} f(\overset{n-s-i}{a}, f(\overset{i+s}{a}, \bar{a}^{i}, \overset{n-s-i}{a}), \overset{s-1}{a}, x, \overset{i}{a}) \\ &\overset{T1}{=} \lambda_{n-i,a}^{-1} f(\overset{n-i}{a}, f(\overset{i}{a}, \bar{a}^{i}, \overset{n-i-1}{a}, x), \overset{i}{a}) \\ &\overset{(1)}{=} \lambda_{n-i,a}^{-1}\lambda_{n-i,a} f(\overset{i}{a}, \bar{a}^{i}, \overset{n-i-1}{a}, x) = f(\overset{i}{a}, \bar{a}^{i}, \overset{n-i-1}{a}, x). \end{aligned}$$

This proves that for $i \leqslant n-s$ the relation (5) holds.

Let $i > s$. At first, we prove the validity of the relations

$$f(\overset{i-s}{a}, \bar{a}^{i}, \overset{n-i+s-1}{a}, x) = x, \tag{6}$$

$$f(x, \overset{n-i+s-1}{a}, \bar{a}^{(n-i)}, \overset{i-s}{a}) = x. \tag{7}$$

Make a chain of conclusions:

$$\begin{aligned} x = \lambda_{i,a}^{-1}\lambda_{i,a}(x) &\overset{(1)}{=} f(\overset{i}{a}, \lambda_{i,a}^{-1}(x), \overset{n-i}{a}) \overset{(4)}{=} \lambda_{i,a}^{-1} f(\overset{i-s}{a}, f(\overset{i}{a}, \bar{a}^{i}, \overset{n-i}{a}), \overset{s-1}{a}, x, \overset{n-i}{a}) \\ &\overset{T1}{=} \lambda_{i,a}^{-1} f(\overset{i}{a}, f(\overset{i-s}{a}, \bar{a}^{i}, \overset{n-i+s-1}{a}, x), \overset{n-i}{a}) \\ &\overset{(1)}{=} \lambda_{i,a}^{-1}\lambda_{i,a} f(\overset{i-s}{a}, \bar{a}^{i}, \overset{n-i+s-1}{a}, x) = f(\overset{i-s}{a}, \bar{a}^{i}, \overset{n-i+s-1}{a}, x). \end{aligned}$$

This proves (6). To prove (7) note that

$$\begin{aligned} x = \lambda_{n-i,a}^{-1}\lambda_{n-i,a}(x) &\overset{(1)}{=} \lambda_{n-i,a}^{-1} f(\overset{n-i}{a}, x, \overset{i}{a}) \\ &\overset{(4)}{=} \lambda_{n-i,a}^{-1} f(\overset{n-i}{a}, x, \overset{s-1}{a}, f(\overset{n-i}{a}, \bar{a}^{(n-i)}, \overset{i}{a}), \overset{i-s}{a}) \\ &\overset{T1}{=} \lambda_{n-i,a}^{-1} f(\overset{n-i}{a}, f(x, \overset{n-i+s-1}{a}, \bar{a}^{(n-i)}, \overset{i-s}{a}), \overset{i}{a}) \end{aligned}$$

$$\overset{(1)}{=} \lambda_{n-i,a}^{-1}\lambda_{n-i,a} f(x, \overset{n-i+s-1}{a}, \bar{a}^{(n-i)}, \overset{i-s}{a}) = f(x, \overset{n-i+s-1}{a}, \bar{a}^{(n-i)}, \overset{i-s}{a}).$$

Using the obtained relation, we get correctness of the first of equalities (5). Indeed,

$$x \overset{(6)}{=} f(\overset{i-s}{a}, \bar{a}^i, \overset{n-i+s-1}{a}, x) \overset{(4)}{=} f(f(\overset{i}{a}, \bar{a}^i, \overset{n-i}{a}), \overset{i-s-1}{a}, \bar{a}^i, \overset{n-i+s-1}{a}, x)$$

$$\overset{T1}{=} f(\overset{i}{a}, \bar{a}^i, \overset{n-i-1}{a}, f(\overset{i-s}{a}, \bar{a}^i, \overset{n-i+s-1}{a}, x)) \overset{(6)}{=} f(\overset{i}{a}, \bar{a}^i, \overset{n-i-1}{a}, x).$$

In the same way:

$$x \overset{(7)}{=} f(x, \overset{n-i+s-1}{a}, \bar{a}^{(n-i)}, \overset{i-s}{a})$$

$$\overset{(4)}{=} f(x, \overset{n-i+s-1}{a}, \bar{a}^{(n-i)}, \overset{i-s-1}{a}, f(\overset{n-i}{a}, \bar{a}^{(n-i)}, \overset{i}{a}))$$

$$\overset{T1}{=} f(f(x, \overset{n-i+s-1}{a}, \bar{a}^{(n-i)}, \overset{i-s}{a}), \overset{n-i-1}{a}, \bar{a}^{(n-i)}, \overset{i}{a}) \overset{(6)}{=} f(x, \overset{n-i-1}{a}, \bar{a}^{(n-i)}, \overset{i}{a}),$$

which proves the second equality from (5). Thus 2) implies 3).

3) $\Rightarrow$ 4). If $i = 0$, then (5) implies (3), which, by Corollary 1, proves that a is an invertible element. In particular, it is j-invertible for all j.

If $i > 0$, then for

$$\hat{a} := f(\overset{i}{a}, f(\bar{a}^i, \overset{n-1}{a}, \bar{a}^i), \overset{n-i-1}{a}, \bar{a}^{(n-i)}), \tag{8}$$

$$\breve{a} := f(\bar{a}^i, \overset{n-i-1}{a}, f(\bar{a}^{(n-i)}, \overset{n-1}{a}, \bar{a}^{(n-i)}), \overset{i}{a}) \tag{9}$$

we have

$$f(\hat{a}, \overset{n-1}{a}, x) \overset{(8)}{=} f(f(\overset{i}{a}, f(\bar{a}^i, \overset{n-1}{a}, \bar{a}^i), \overset{n-i-1}{a}, \bar{a}^{(n-i)}), \overset{n-1}{a}, x)$$

$$\overset{T1}{=} f(\overset{i}{a}, f(\bar{a}^i, \overset{n-i-1}{a}, f(\overset{i}{a}, \bar{a}^i, \overset{n-i-1}{a}, \bar{a}^{(n-i)}), \overset{i}{a}), \overset{n-i-1}{a}, x)$$

$$\overset{(5)}{=} f(\overset{i}{a}, f(\bar{a}^i, \overset{n-i-1}{a}, \bar{a}^{(n-i)}, \overset{i}{a}), \overset{n-i-1}{a}, x)$$

$$\overset{(5)}{=} f(\overset{i}{a}, \bar{a}^i, \overset{n-i-1}{a}, x) \overset{(5)}{=} x.$$

The second equality from (3) may be proved in the same way. Indeed,

$$f(x, \overset{n-1}{a}, \breve{a}) \overset{(9)}{=} f(x, \overset{n-1}{a}, f(\bar{a}^i, \overset{n-i-1}{a}, f(\bar{a}^{(n-i)}, \overset{n-1}{a}, \bar{a}^{(n-i)}), \overset{i}{a}))$$

$$\overset{T1}{=} f(x, \overset{n-i-1}{a}, f(\overset{i}{a}, f(\bar{a}^i, \overset{n-i-1}{a}, \bar{a}^{(n-i)}, \overset{i}{a}), \overset{n-i-1}{a}, \bar{a}^{(n-i)}), \overset{i}{a})$$

$$\overset{(5)}{=} f(x, \overset{n-i-1}{a}, f(\overset{i}{a}, \bar{a}^i, \overset{n-i-1}{a}, \bar{a}^{(n-i)}), \overset{i}{a}) \overset{(5)}{=} f(x, \overset{n-i-1}{a}, \bar{a}^{(n-i)}, \overset{i}{a}) \overset{(5)}{=} x.$$

Hence, the relations (3) are valid and therefore, by Corollary 1, the element a is invertible.

4) $\Rightarrow$ 1). Let $j \equiv 0 \pmod{s}$, $0 < j < n$, i.e. $j = ks$, where $k = 1, \ldots, n/s - 1$, and let an element a be j-invertible.

Since the element a is ks-invertible, the ks-th shift is a substitution of the set Q. Observe that for

$$y := \lambda_{ks,a}^{-1}(z), \qquad z := \lambda_{ks,a}(y). \tag{10}$$

the following two equalities hold

$$\lambda_{ks,a}^{-1} f(z, \overset{ks-1}{a}, x, \overset{n-ks}{a}) = f(\lambda_{ks,a}^{-1}(z), \overset{n-1}{a}, x), \tag{11}$$

$$\lambda_{ks,a}^{-1} f(\overset{ks}{a}, x, \overset{n-ks-1}{a}, z) = f(x, \overset{n-1}{a}, \lambda_{ks,a}^{-1}(z)). \tag{12}$$

Indeed,

$$\begin{aligned}
\lambda_{ks,a}^{-1} f(z, \overset{ks-1}{a}, x, \overset{n-ks}{a}) &\overset{(10)}{=} \lambda_{ks,a}^{-1} f(\lambda_{ks,a}(y), \overset{ks-1}{a}, x, \overset{n-ks}{a}) \\
&\overset{(1)}{=} \lambda_{ks,a}^{-1} f(f(\overset{ks}{a}, y, \overset{n-ks}{a}), \overset{ks-1}{a}, x, \overset{n-ks}{a}) \\
&\overset{T1}{=} \lambda_{ks,a}^{-1} f(\overset{ks}{a}, f(y, \overset{n-1}{a}, x), \overset{n-ks}{a}) \\
&\overset{(1)}{=} \lambda_{ks,a}^{-1} \lambda_{ks,a} f(y, \overset{n-1}{a}, x) \overset{(1)}{=} f(y, \overset{n-1}{a}, x) \\
&\overset{(10)}{=} f(\lambda_{ks,a}^{-1}(z), \overset{n-1}{a}, x).
\end{aligned}$$

Similarly

$$\begin{aligned}
\lambda_{ks,a}^{-1} f(\overset{ks}{a}, x, \overset{n-ks-1}{a}, z) &\overset{(1)}{=} \lambda_{ks,a}^{-1} f(\overset{ks}{a}, x, \overset{n-ks-1}{a}, f(\overset{ks}{a}, y, \overset{n-ks}{a})) \\
&\overset{T1}{=} \lambda_{ks,a}^{-1} f(\overset{ks}{a}, f(x, \overset{n-1}{a}, y), \overset{n-ks}{a}) \\
&\overset{(1)}{=} \lambda_{ks,a}^{-1} \lambda_{ks,a} f(x, \overset{n-1}{a}, y) \\
&\overset{(1)}{=} f(x, \overset{n-1}{a}, y) \overset{(10)}{=} f(x, \overset{n-1}{a}, \lambda_{ks,a}^{-1}(z)).
\end{aligned}$$

Now, putting $z := a$ in (11) we obtain

$$\lambda_{ks,a}^{-1} f(\overset{ks}{a}, x, \overset{n-ks}{a}) = f(\lambda_{ks,a}^{-1}(a), \overset{n-1}{a}, x),$$

$$\lambda_{ks,a}^{-1}\lambda_{ks,a}(x) = f(\bar{a}^{ks}, \overset{n-1}{a}, x),$$

which together with the definitions of a shift and the definition of a skew element gives

$$x = f(\bar{a}^{ks}, \overset{n-1}{a}, x) \tag{13}$$

for all $x \in Q$. This means, that the first equality from (3) holds. To verify the second one we put $z = a$ in (12). Then

$$\lambda_{ks,a}^{-1} f(\overset{ks}{a}, x, \overset{n-ks}{a}) = f(x, \overset{n-1}{a}, \lambda_{ks,a}^{-1}(a)),$$

which, as in the previous case, implies

$$\lambda_{ks,a}^{-1}\lambda_{ks,a}(x) = f(x, \overset{n-1}{a}, \bar{a}^{ks})$$

Thus

$$x = f(x, \overset{n-1}{a}, \bar{a}^{ks}) \tag{14}$$

for all $x \in Q$. Corollary 1 and (13), (14) imply the invertibility of a.

This completes the proof of Theorem 3. □

Note, that for binary semigroups the following assertion is valid.

Lemma 2. *Let $(Q, \cdot)$ be a binary semigroup and shift $\lambda_{0,a}$ ($\lambda_{1,a}$) be a substitution of Q, then the element $e_r := \lambda_{0,a}^{-1}(a)$ ($e_\ell := \lambda_{1,a}^{-1}(a)$) is a right (respectively left) unit, and $a_r^{-1} := \lambda_{0,a}^{-2}(a)$ ($a_\ell^{-1} := \lambda_{1,a}^{-2}(a)$) is a right (respectively left) inverse element of the element a in semigroup $(Q, \cdot)$.*

Proof. Indeed,

$$\lambda_{0,a}(x \cdot e_r) = x \cdot e_r \cdot a = x \cdot \lambda_{0,a}(e_r) = x \cdot \lambda_{0,a}\lambda_{0,a}^{-1}(a) = x \cdot a = \lambda_{0,a}(x).$$

Since $\lambda_{0,a}$ is a substitution of the set Q, then the proved equality

$$\lambda_{0,a}(x \cdot e_r) = \lambda_{0,a}(x)$$

gives $x \cdot e_r = x$ for all $x \in Q$, that is the element e_r is a right unit element in the semigroup $(Q, \cdot)$.

In the same way one can prove that e_ℓ is a left unit element in $(Q,\cdot)$.

To establish that the element a_r^{-1} is a right inverse of a, note that

$$\lambda_{0,a}(a\cdot a_r^{-1}) = a\cdot a_r^{-1}\cdot a = a\cdot \lambda_{0,a}\lambda_{0,a}^{-2}(a) = a\cdot \lambda_{0,a}^{-1}(a) = a\cdot e_r = a.$$

Applying $\lambda_{0,a}^{-1}$ to the equality $\lambda_{0,a}(a\cdot a_r^{-1}) = a$, we get

$$a\cdot a_r^{-1} = \lambda_{o,a}^{-1}(a) = e_r.$$

Hence, the element a is right invertible.

Similarly we can prove that the element a_ℓ^{-1} is a left inverse of a, when the shift $\lambda_{1,a}$ is a substitution of the set Q. □

Corollary 2. *An element a of a binary semigroup is invertible iff it is* 0*-invertible and* 1*-invertible simultaneously.*

An element a of an associate (Q,f) of the type (s,n) is said to be: *right* (*left*) *invertible*, if the shift $\lambda_{0,a}$ (respectively $\lambda_{1,a}$) is a substitution of the set Q.

An element a of an $(n+1)$-ary groupoid (Q,f) will be called *inner invertible*, if the shift $\lambda_{i,a}$ is a substitution of the set Q for some $i = 1,\dots,n-1$.

Corollary 3. *An element a is invertible in an associate (Q,f) of the type (s,n) iff it is right and left invertible simultaneously.*

The *Proof* follows from the point 2) of Theorem 3 when $i = 0$.

Corollary 4. *In any $(n+1)$-ary semigroup (Q,f) for any element a and for any numbers $i = 1,\dots,n-1$; $k = 1,\dots,\frac{n}{s}-1$ the following assertions are equivalent:*

1) *a is invertible,*
2) *a is inner invertible,*
3) *a is right and left invertible,*
4) *there exist elements $\hat{a}$ and $\breve{a}$ in Q such that for arbitrary $x\in Q$ the following equalities hold:*

$$f(\overset{i}{a},\hat{a},\overset{n-i-1}{a},x) = x, \qquad f(x,\overset{n-i-1}{a},\breve{a},\overset{i}{a}) = x. \tag{15}$$

2. Axiomatics of polyagroups

Definition 1. A groupoid (Q,f) is called a *polyagroup of a type* (s,n) iff it is a quasigroup and an associate of the type (s,n).

It is easy to see that for $s=1$ a polyagroup of a type (s,n) is an $(n+1)$-ary group.

Directly from Theorem 3 and the definition of a polyagroup we obtain:

Theorem 4. *In an associate (Q,f) of the type (s,n) for any $i=0,1,\dots,n-1$ the following conditions are equivalent:*

1) *the associate is a polyagroup,*
2) *every element of the associate is invertible,*
3) *every element of the associate is i- and $(n-i)$-invertible,*
4) *for every element y there exist elements $\hat{y}$ and $\breve{y}$ in Q such that for arbitrary $x\in Q$ the following two equalities hold*

$$f(\overset{i}{y},\hat{y},\overset{n-i-1}{y},x)=x, \qquad f(x,\overset{n-i-1}{y},\breve{y},\overset{i}{y})=x,$$

5) *every element is ks-invertible, for some $k=1,\dots,\frac{n}{s}-1$.*

Since for $s=1$ a polyagroup of a type (s,n) is an $(n+1)$-group (an associate of the type $(1,n)$ is an $(n+1)$-semigroup), then as a simple consequence of the above Theorem, we obtain the following characterizations of $(n+1)$-ary groups, which are proved in [3 – 5].

Corollary 5. *In an $(n+1)$-semigroup (Q,f) for any $i=0,1,\dots,n-1$ the following assertions are equivalent:*

1) *a semigroup is an $(n+1)$-group,*
2) *every element of the semigroup is invertible,*
3) *every element is a right and left invertible,*
4) *every element is inner invertible,*
5) *for every element y there exist elements $\hat{y}$ and $\breve{y}$ in Q such that for arbitrary $x\in Q$ the following two equalities hold*

$$f(\overset{i}{y},\hat{y},\overset{n-i-1}{y},x)=x, \qquad f(x,\overset{n-i-1}{y},\breve{y},\overset{i}{y})=x.$$

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