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Fuzzy subquasigroups

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Abstract

Our work in this paper is concerned with the fuzzification of subasigroups in quasigroups. We investigate the connection between normal, maximal and completely normal fuzzy subquasigroups in unipotent quasigroups.

1. Introduction

Following the introduction of fuzzy sets by Zadeh [10], the fuzzy set theory developed by Zadeh himself and others have found many applications in the domain of mathematics and elsewhere. For example, in [7] are studied fuzzy subrings as well as fuzzy ideals in rings. Properties of some fuzzy ideals in semirings are investigated in [6]. Very similar results for some fuzzy ideals in BCI-algebras are proved in [5]. Connections between fuzzy groups and so-called level subgroups are find in [2], [3] and [9]. These algebras (i.e., rings, groups, BCI-algebras) are not similar, but used methods are very similar.

In this note some modifications of these methods will be applied to quasigroups.

2. Preliminaries

As it is well known a groupoid $(G, \cdot)$ is called a *quasigroup* if each of the equations $ax = b,\ xa = b$ has a unique solution for any $a, b \in G$.

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A quasigroup $(G,\cdot)$ may be also defined as an algebra $(G,\cdot,\backslash,/)$ with the three binary operations $\cdot,\backslash,/$ satisfying the identities

$$(xy)/y=x,\quad x\backslash(xy)=y,\quad (x/y)y=x,\quad x(x\backslash y)=y$$

(cf. [1] or [8]). We say also that $(G,\cdot,\backslash,/)$ is an *equasigroup* (i.e. equationally definable quasigroup) [8] or a *primitive quasigroup* [1].

The equasigroup $(G,\cdot,\backslash,/)$ corresponds to quasigroup $(G,\cdot)$ where

$$x\backslash y=z\Longleftrightarrow xz=y,\quad x/y=z\Longleftrightarrow zy=x.$$

In the theory of quasigroups, so-called *unipotent quasigroups*, i.e., quasigroups with the identity $xx=yy$, play an important role. These quasigroups are connected with Latin squares which have one fixed element in the diagonal (cf. [4]). Such quasigroups may be defined as quasigroups G with the special element θ satisfying the identity $xx=\theta$. Obviously, θ is uniquely determined and it is an idempotent, but, in general, it is not the (left, right) neutral element.

A nonempty subset S of a quasigroup $\mathcal{G}=(G,\cdot,\backslash,/)$ is called a *subquasigroup* it is closed with respect to these three operations, i.e., if $x*y\in G$ for all $*\in\{\cdot,\backslash,/\}$ and $x,y\in G$.

A function $\mu:G\to[0,1]$ is called *a fuzzy set* in a quasigroup G. The set $\mu_t=\{x\in G:\mu(x)\geqslant t\}$, where $t\in[0,1]$ is fixed, is called *a level subset of* μ. The set $\{x\in G:\mu(x)=\mu(\theta)\}$, where $\mathcal{G}$ is a unipotent quasigroup, is denoted by G_μ. $Im(\mu)$ denotes the image set of μ.

Let μ and ρ be two fuzzy set defined on G. According to [10] we say that μ is contained in ρ, and denote this fact by $\mu\subseteq\rho$, iff $\mu(x)\leqslant\rho(x)$ for all $x\in G$. Obviously $\mu=\rho$ iff $\mu(x)=\rho(x)$ for all $x\in G$.

3. Fuzzy subquasigroups

Definition 3.1. A fuzzy set μ in a quasigroup $\mathcal{G}=(G,\cdot,\backslash,/)$ is called a *fuzzy subquasigroup* of $\mathcal{G}$ if

$$\min\{\mu(xy),\,\mu(x\backslash y),\,\mu(x/y)\}\geqslant\min\{\mu(x),\,\mu(y)\}\quad\forall x,y\in G\,.$$

It is clear, that this definition is equivalent to the following

Definition 3.2. A fuzzy set μ in a quasigroup $\mathcal{G} = (G, \cdot, \backslash, /)$ is a *fuzzy subquasigroup* of $\mathcal{G}$ if

$$\mu(x * y) \geqslant \min\{\mu(x),\, \mu(y)\}$$

for all $* \in \{\cdot, \backslash, /\}$ and $x, y \in G$.

Proposition 3.3. *If μ is a fuzzy subquasigroup of a unipotent quasigroup $(G, \cdot, \backslash, /, \theta)$, then $\mu(\theta) \geqslant \mu(x)$ for any $x \in G$.*

Proof. Since $xx = \theta$ for any $x \in G$, then

$$\mu(\theta) = \mu(xx) \geqslant \min\{\mu(x), \mu(x)\} = \mu(x),$$

which completes the proof. □

Proposition 3.4. *If μ is a fuzzy subquasigroup of a quasigroup $\mathcal{G}$, then for all $x, y \in G$ we have*

(*a*) $\min\{\mu(x * y),\, \mu(x)\} = \min\{\mu(x * y),\, \mu(y)\} = \min\{\mu(x),\, \mu(y)\}$,

(*b*) $\mu(x) < \mu(y)$ *implies* $\mu(x * y) = \mu(x)$,

(*c*) $\mu(x) > \mu(y)$ *implies* $\mu(x * y) = \mu(y)$,

(*d*) $\mu(x) \neq \mu(y)$ *implies* $\mu(x * y) = \min\{\mu(x),\, \mu(y)\}$.

Proof. (a) At first we consider the case $x * y = xy$. Since $(xy)/y = x$ for all $x, y \in G$, then

$$\begin{aligned}\min\{\mu(xy), \mu(y)\} &\geqslant \min\{\min\{\mu(x),\, \mu(y)\},\, \mu(y)\}\\ &= \min\{\mu(x),\, \mu(y)\} = \min\{\mu((xy)/y),\, \mu(y)\}\\ &\geqslant \min\{\min\{\mu(xy), \mu(y)\}, \mu(y)\} = \min\{\mu(xy), \mu(y)\},\end{aligned}$$

which proves that

$$\min\{\mu(xy),\, \mu(y)\} = \min\{\mu(x),\, \mu(y)\}.$$

In the similar way, using $x \backslash (xy) = y$, we prove the second identity. Thus (a) holds for $x * y = xy$.

Now let $x * y = x \backslash y$. As in the previous case it is not difficult to see that

$$\min\{\mu(x \backslash y),\, \mu(x)\} \geqslant \min\{\mu(x),\, \mu(y)\}.$$

Since $y = x(x \backslash y)$ for all $x, y \in G$, then

$$\begin{aligned}\min\{\mu(x),\,\mu(y)\} &= \min\{\mu(x),\,\mu(x(x\setminus y))\}\\ &\geqslant \min\{\mu(x),\,\min\{\mu(x),\,\mu(x\setminus y)\}\}\\ &= \min\{\mu(x\setminus y),\,\mu(x)\}\,,\end{aligned}$$

which gives

$$\min\{\mu(x\setminus y),\,\mu(x)\} = \min\{\mu(x),\,\mu(y)\}.$$

At is well known $x\setminus y = z \Longleftrightarrow xz = y$. Thus, applying this fact to (a), where $x*y = xy$, we obtain

$$\begin{aligned}\min\{\mu(x\setminus y),\,\mu(y)\} &= \min\{\mu(z),\,\mu(xz)\} = \min\{\mu(z),\,\mu(x)\}\\ &= \min\{\mu(x\setminus y),\,\mu(x)\} = \min\{\mu(x),\,\mu(y)\}.\end{aligned}$$

Hence (a) holds also in the case $x*y = x\setminus y$.

If $x*y = x/y$ then $\min\{\mu(x/y),\,\mu(y)\} \geqslant \min\{\mu(x),\,\mu(y)\}$ by the assumption on μ. Thus, using the identity $x = (x/y)y$, we get

$$\begin{aligned}\min\{\mu(x),\,\mu(y)\} &= \min\{\mu((x/y)y),\,\mu(y)\}\\ &\geqslant \min\{\min\{\mu(x/y),\,\mu(y)\},\,\mu(y)\}.\\ &= \min\{\mu(x/y),\,\mu(y)\}\end{aligned}$$

Hence

$$\min\{\mu(x/y),\,\mu(y)\} = \min\{\mu(x),\,\mu(y)\}.$$

Since, by the definition, $xy = u \Longleftrightarrow uy = x$, then, as in the previous case, we obtain

$$\begin{aligned}\min\{\mu(x/y),\,\mu(x)\} &= \min\{\mu(u)\,\mu(uy)\} = \min\{\mu(u),\,\mu(y)\}\\ &= \min\{\mu(x/y),\,\mu(y)\} = \min\{\mu(x),\,\mu(y)\},\end{aligned}$$

which completes the proof of (a).

(b) Let $\mu(x) < \mu(y)$. Then from (a) we get

$$\min\{\mu(x*y),\,\mu(y)\} = \min\{\mu(x),\,\mu(y)\} = \mu(x).$$

This implies $\mu(x*y) = \mu(x)$.

(c) similarly as (b).

(d) is an immediate consequence of (b) and (c). □

Proposition 3.5. *A fuzzy set μ of a quasigroup $\mathcal{G} = (G,\cdot,\setminus,/)$ is a fuzzy subquasigroup iff for every $t\in[0,1]$, μ_t is either empty or a subquasigroup of G.*

Proof. If μ is a fuzzy subquasigroup of $\mathcal{G}$ and $\mu_t \neq \emptyset$, then for any $x, y \in \mu_t$ we have $\mu(x) \geqslant t$, $\mu(y) \geqslant t$. Thus

$$\mu(x * y) \geqslant \min\{\mu(x),\, \mu(y)\} \geqslant t$$

for any $* \in \{\cdot, \backslash, /\}$, which implies $x * y \in \mu_t$. This proves that μ_t is a subquasigroup of $\mathcal{G}$.

Conversely, let $x, y \in G$ and $t = \min\{\mu(x), \mu(y)\}$. Then, by the assumption, μ_t is a subquasigroup of $\mathcal{G}$, which gives $x * y \in \mu_t$. Hence $\mu(x * y) \geqslant t = \min\{\mu(x), \mu(y)\}$. Thus μ is a fuzzy subquasigroup of a quasigroup $\mathcal{G}$. □

Proposition 3.6. *Any subquasigroup of a quasigroup $\mathcal{G}$ can be realized as a level subquasigroup of some fuzzy subquasigroup of $\mathcal{G}$.*

Proof. Let S be a subquasigroup of a given quasigroup $\mathcal{G}$ and let μ be a fuzzy set in G defined by

$$\mu(x) = \begin{cases} t & if \;\; x \in S, \\ s & if \;\; x \notin S, \end{cases}$$

where $0 \leqslant s < t \leqslant 1$ are fixed. It is clear that $\mu_t = S$.

We prove that such defined μ is a fuzzy subquasigroup of G. Let $x, y \in G$. If $x, y \in S$, then also $x * y \in S$. Hence $\mu(x) = \mu(y) = \mu(x*y) = t$ and $\mu(x*y) \geqslant \min\{\mu(x),\, \mu(y)\}$. If $x, y \notin S$, then $\mu(x) = \mu(y) = s$, and, in the consequence, $\mu(x*y) \geqslant \min\{\mu(x),\, \mu(y)\} = s$. If at most one of x, y belongs to S, then at least one of $\mu(x)$ and $\mu(y)$ is equal to t. Therefore $\min\{\mu(x),\, \mu(y)\} = s$ and $\mu(x*y) \geqslant s$, which completes the proof. □

Proposition 3.7. *Two level subquasigroups μ_s, μ_t $(s < t)$ of a fuzzy subquasigroup are equal iff there is no $x \in G$ such that $s \leqslant \mu(x) < t$.*

Proof. Let $\mu_s = \mu_t$ for some $s < t$. If there exists $x \in G$ such that $s \leqslant \mu(x) < t$, then μ_t is a proper subset of μ_s, which is a contradiction. Conversely assume that there is no $x \in G$ such that $s \leqslant \mu(x) < t$. If $x \in \mu_s$, then $\mu(x) \geqslant s$, and so $\mu(x) \geqslant t$, because $\mu(x)$ does not lie between s and t. Thus $x \in \mu_t$, which gives $\mu_s \subseteq \mu_t$. The converse inclusion is obvious since $s < t$. Therefore $\mu_s = \mu_t$. □

From the above theorem it follows that the set of all level subquasigroups of a given fuzzy quasigroup μ of $\mathcal{G}$ is linearly ordered. If $\mathcal{G}$ is a unipotent quasigroup, then μ_{t_0}, where $t_0 = \mu(\theta)$, is the smallest level subquasigroup (because $\mu(x) \leqslant \mu(\theta)$ for all $x \in G$). In this case we have the chain

$$\mu_{t_0} \subset \mu_{t_1} \subset ... \subset \mu_{t_p} = G,$$

where $t_0 > t_1 > ... > t_p$.

Corollary 3.8. *Let μ be a fuzzy subquasigroup of G. If $Im(\mu) = \{t_1, t_2, ..., t_n\}$, where $t_1 < t_2 < ... < t_n$, then the family of levels μ_{t_i}, $1 \leqslant i \leqslant n$, constitutes all the level subquasigroups of μ.*

Proof. Let μ_s, where $s \in [0,1]$ and $s \notin Im(\mu)$, be a some level subquasigroups. If $s < t_1$, then $\mu_{t_1} \subseteq \mu_s$. Since $\mu_{t_1} = G$, it follows that $\mu_s = G$ and $\mu_s = \mu_{t_1}$. If $t_i < s < t_{i+1}$, then there is no $x \in G$ such that $s \leqslant \mu(x) < \mu_{t_{i+1}}$. Thus $\mu_s = \mu_{t_{i+1}}$, by Proposition 3.7. Obviously $\mu_s = \emptyset$ for $s > t_n$. This proves that for any $s \in [0,1]$ μ_s is either empty or belongs to $\{\mu_{t_i} : 1 \leqslant i \leqslant n\}$. □

The construction used in the proof of Proposition 3.6 shows that two different fuzzy subquasigroups may have an identical family of level subquasigroups. For example, only S and $G \setminus S$.

Proposition 3.9. *Let μ be a fuzzy subquasigroup with finite image. If $\mu_s = \mu_t$ for some $s, t \in Im(\mu)$, then $s = t$.*

Proof. Without loss of generality, let $s < t$. Since $s \in Im(\mu)$, then there exists $x \in G$ such that $\mu(x) = s < t$, and so $x \in \mu_s$ and $x \notin \mu_t$, which is a contradiction. □

Proposition 3.10. *Let μ and ρ be two fuzzy subquasigroups of a quasigroup $\mathcal{G}$ with identical family of level subquasigroups. If $Im(\mu) = \{t_1, ..., t_n\}$ and $Im(\rho) = \{s_1, ..., s_m\}$, where $t_1 > t_2 > ... > t_n$ and $s_1 > s_2 > ... > s_m$, then*

a) *$m = n$,*

b) *$\mu_{t_i} = \rho_{s_i}$ for $i = 1, ..., n$,*

c) *if $\mu(x) = t_i$, then $\rho(x) = s_i$ for $x \in G$ and $i = 1, ..., n$.*

Proof. (a) By Corollary 3.8 fuzzy subquasigroups μ and ρ have (respectively) the only $\{\mu_{t_i}\}$ and $\{\rho_{s_i}\}$ as the families of level subquasigroups. Since, by the assumption, these families are identical, then $m = n$.
(b) Follows from Corollary 3.8 and Proposition 3.7.
(c) Let $x \in G$ be such that $\mu(x) = t_i$ and $\rho(x) = s_j$. From (b) and $\mu(x) = t_i$ follows $x \in \rho_{s_i}$. Thus $\rho(x) \geqslant s_i$ and $s_j \geqslant s_i$, i.e. $\rho_{s_j} \subseteq \rho_{s_i}$. Since $x \in \rho_{s_j} = \mu_{t_j}$, we obtain $t_i = \mu(x) \geqslant t_j$. This gives $\mu_{t_i} \subseteq \mu_{t_j}$, and, in the consequence (by (b)) $\rho_{s_i} = \mu_{t_i} \subseteq \mu_{t_j} = \rho_{s_j}$. Thus $\rho_{s_i} = \rho_{s_j}$. But, by Proposition 3.9, $s_i = s_j$. Hence $\rho(x) = s_i$. □

Proposition 3.11. *Let μ and ρ be two fuzzy subquasigroups of $\mathcal{G}$ with identical family of levels. Then $\mu = \rho$ iff $Im(\mu) = Im(\rho)$.*

Proof. Let $Im(\mu) = Im(\rho) = \{s_1, ..., s_n\}$ and $s_1 > ....s_n$. By Proposition 3.10 for any $x \in G$ there exists s_i such that $\mu(x) = s_i = \rho(x)$. Thus $\mu(x) = \rho(x)$ for all $x \in G$, which gives $\mu = \rho$. □

Proposition 3.12. *Let $\{S_t : t \in T\}$, where $\emptyset \neq T \subseteq [0,1]$, be a collection of subquasigroups of a quasigroup $\mathcal{G}$ such that*

(i) $\quad G = \bigcup\limits_{t\in T} S_t,$

(ii) $\quad s > t \iff S_s \subset S_t \quad$ *for all* $s, t \in T$.

Then μ defined by

$$\mu(x) = \sup\{t \in T : x \in S_t\}$$

is a fuzzy subquasigroup of $\mathcal{G}$.

Proof. By Proposition 3.5, it is sufficient to show that every nonempty level μ_s is a subquasigroup of $\mathcal{G}$. Assume $\mu_s \neq \emptyset$ for some fixed $s \in [0,1]$. Then

$$s = \sup\{t \in T : t < s\} = \sup\{t \in T : S_s \subset S_t\}$$

or

$$s \neq \sup\{t \in T : t < s\} = \sup\{t \in T : S_s \subset S_t\}.$$

In the first case we have $\mu_s = \bigcap\limits_{t<s} S_t$, because

$$x \in \mu_s \iff x \in S_t \text{ for all } t < s \iff x \in \bigcap_{t<s} S_t\,.$$

In the second, there exists $\varepsilon > 0$ such that $(s-\varepsilon,\, s) \cap T = \emptyset$. We prove that in this case $\mu_s = \bigcup_{t\geqslant s} S_t$. Indeed, if $x \in \bigcup_{t\geqslant s} S_t$, then $x \in S_t$ for some $t \geqslant s$, which gives $\mu(x) \geqslant t \geqslant s$. Thus $x \in \mu_s$, i.e. $\bigcup_{t\geqslant s} S_t \subseteq \mu_s$.

Conversely, if $x \notin \bigcup_{t\geqslant s} S_t$, then $x \notin S_t$ for all $t \geqslant s$, which implies that $x \notin S_t$ for all $t > s-\varepsilon$, i.e. if $x \in S_t$ then $t \leqslant s-\varepsilon$. Thus $\mu(x) \leqslant s-\varepsilon$. Therefore $x \notin \mu_s$. Hence $\mu_s \subseteq \bigcup_{t\geqslant s} S_t$, and in the consequence $\mu_s = \bigcup_{t\geqslant s} S_t$. This completes our proof because (as it is not difficult to see) $\bigcup_{t\geqslant s} S_t$ and $\bigcap_{t<s} S_t$ are subquasigroups. □

Proposition 3.13. *Let μ be a fuzzy set in $\mathcal{G}$ and let $Im(\mu) = \{t_0, t_1, ..., t_n\}$, where $t_0 > t_1 > ... > t_n$. If $S_0 \subset S_1 \subset ... \subset S_n = G$ are subquasigroups of $\mathcal{G}$ such that $\mu(S_k \setminus S_{k-1}) = t_k$ for $k = 0, 1, ..., n$, where $S_{-1} = \emptyset$, then μ is a fuzzy subquasigroup.*

Proof. Let x, y be an arbitrary elements of G. For any fixed operation $* \in \{\cdot, \backslash, /\}$ there exists only one $k = 0, 1, ..., n$ such that $x * y \in S_k \setminus S_{k-1}$ (k depends on x, y and $*$).

We consider the following four cases:

1^o $\quad x*y \in S_k \setminus S_{k-1}, \;\; x, y \in S_k \setminus S_{k-1}$,

2^o $\quad x*y \in S_k \setminus S_{k-1}, \;\; x, y \notin S_k \setminus S_{k-1}$,

3^o $\quad x*y,\, x \in S_k \setminus S_{k-1}, \;\; y \notin S_k \setminus S_{k-1}$,

4^o $\quad x*y,\, y \in S_k \setminus S_{k-1}, \;\; x \notin S_k \setminus S_{k-1}$.

In the first case we have $\mu(x*y) = \mu(x) = \mu(y) = t_k$. Hence

$$\mu(x*y) = t_k = \min\{\mu(x),\, \mu(y)\}\,.$$

In the second case there exist $m \neq k$ and $n \neq k$ such that $x \in S_m \setminus S_{m-1}$ and $y \in S_n \setminus S_{n-1}$. Without loss of generality, we can assume $m \leqslant n$.

If $m \leqslant n < k$, then $S_m \subseteq S_n \subseteq S_{k-1} \subset S_k$ and $x, y \in S_n$. Thus $x*y \in S_n \subseteq S_{k-1}$, which is impossible.

If $m < k < n$, then $x,\, x*y \in S_k \subseteq S_{n-1} \subset S_n$, which for $x*y = xy$ gives $y = x\backslash(xy) \in S_k$. This is a contradiction. The case when $x*y = x\backslash y = u$ also is impossible because, by the assumption and the definition of $\backslash$ we have $y = xu \in S_k$. If $x * y = x/y = v$. Then $vy = x$ implies $y \in S_k$. A contradiction.

Thus must be $k < m \leqslant n$. Hence $\mu(x * y) = t_k$, $\mu(x) = t_m$, $\mu(y) = t_n$ and, in the consequence,

$$\mu(x * y) = t_k > t_n = \min\{\mu(x), \mu(y)\}.$$

The last two cases are obvious. □

Corollary 3.14. *Let μ be a fuzzy set in G with $Im(\mu) = \{t_0, t_1, ..., t_n\}$, where $t_0 > t_1 > ... > t_n$. If $S_0 \subset S_1 \subset ... \subset S_n = G$ are subquasigroups of $\mathcal{G}$ such that $\mu(S_k) \geqslant t_k$ for $k = 0, 1, ..., n$, then μ is a fuzzy subquasigroup in $\mathcal{G}$.* □

Corollary 3.15. *If $Im(\mu) = \{t_0, t_1, ..., t_n\}$, where $t_0 > t_1 > ... > t_n$, is the image of a fuzzy subquasigroup μ in $\mathcal{G}$, then all levels μ_{t_k} are subquasigroups of $\mathcal{G}$ such that $\mu(\mu_{t_0}) = t_0$ and $\mu(\mu_{t_k} \setminus \mu_{t_{k-1}}) = t_k$ for $k = 1, 2, ..., n$.*

Proof. All μ_{t_k} are subquasigroups by Proposition 3.5. Obviously $\mu(\mu_{t_0}) = t_0$. Since $\mu(\mu_{t_1}) \geqslant t_1$, then $\mu(x) = t_0$ for $x \in \mu_{t_0}$ and $\mu(x) = t_1$ for $x \in \mu_{t_0} \setminus \mu_{t_1}$. Repeating this procedure, we conclude that $\mu(\mu_{t_k} \setminus \mu_{t_{k-1}}) = t_k$ for $k = 1, 2, ..., n$. □

Proposition 3.16. *Let $\mathcal{G}$ be a unipotent quasigroup. If μ is a fuzzy subquasigroup in $\mathcal{G}$ with the image $Im(\mu) = \{t_i : i \in I\}$ and $\Omega = \{\mu_t : t \in Im(\mu)\}$, then*

- (a) *there exists a unique $t_0 \in Im(\mu)$ such that $t_0 \geqslant t$ for all $t \in Im(\mu)$,*
- (b) *G is the set-theoretic union of all $\mu_t \in \Omega$,*
- (c) *the members of Ω form a chain,*
- (d) *Ω contains all level subquasigroups of μ iff μ attains its infimum on all subquasigroups of $\mathcal{G}$.*

Proof. (a) Follows from the fact that in a unipotent quasigroup $t_0 = \mu(\theta) \geqslant \mu(x)$ for all $x \in G$ (see Proposition 3.3).

(b) If $x \in G$, then $\mu(x) = t(x) \in Im(\mu)$. This implies $x \in \mu_{t(x)} \subseteq \bigcup \mu_t \subseteq G$, where $t \in Im(\mu)$, which proves (b).

(c) Since $\mu_{t_i} \subseteq \mu_{t_j} \iff t_i \geqslant t_j$ for $i, j \in I$, then the set Ω is totally ordered by inclusion.

(d) Suppose that Ω contains all level subquasigroups of μ. Let S be a subquasigroup of $\mathcal{G}$. If μ is constant on S, then we are done. Assume that μ is not constant on AS. We consider two cases: (1) $S = G$ and (2) $S \subset G$. For $S = G$ let $\beta = \inf Im(\mu)$. Then $\beta \leqslant t \in Im(\mu)$, i.e. $\mu_\beta \supseteq \mu_t$ for all $t \in Im(\mu)$. But $\mu_0 = G \in \Omega$ because Ω contains all level subquasigroups of μ. Hence there exists $t' \in Im(\mu)$ such that $\mu_{t'} = G$. It follows that $\mu_\beta \supset \mu_{t'} = G$ so that $\mu_\beta = \mu_{t'} = G$ because every level subquasigroup of μ is a subquasigroup of $\mathcal{G}$.

Now it sufficient to show that $\beta = t'$. If $\beta < t'$, then there exists $t'' \in Im(\mu)$ such that $\beta \leq t'' < t'$. This implies $\mu_{t''} \supset \mu_{t'} = G$, which is a contradiction. Therefore $\beta = t' \in Im(\mu)$.

In the case $S \subset G$ we consider the fuzzy set μ_S defined by

$$\mu_S(x) = \begin{cases} \alpha & for \quad x \in S, \\ 0 & for \quad x \in G \setminus S. \end{cases}$$

From the proof of our Proposition 3.6 it follows that μ_A is a fuzzy subquasigroup of $\mathcal{G}$.

Let

$$J = \{i \in I : \mu(y) = t_i \;\; for\ some\ y \in S\}$$

and $\Omega_S = \{(\mu_S)_{t_i} : i \in J\}$. Noticing that Ω_S contains all level subquasigroups of μ_S, then there exists $x_0 \in S$ such that $\mu(x_0) = \inf\{\mu_S(x) : x \in S\}$, which implies that $\mu(x_0) = \{\mu(x) : x \in S\}$. This proves that μ attains its infimum on all subquasigroups of $\mathcal{G}$.

To prove the converse let μ_α be a level subquasigroup of μ. If $\alpha = t$ for some $t \in Im(\mu)$, then clearly $\mu_\alpha \in \Omega$. If $\alpha \neq t$ for all $t \in Im(\mu)$, then there does not exist $x \in G$ such that $\mu(x) = \alpha$. Let $S = \{x \in G : \mu(x) > \alpha\}$. Obviously $\theta \in S$. Let now $x, y \in S$. Then $\mu(x) > \alpha$ and $\mu(y) > \alpha$. From the fact that μ is a fuzzy subquasigroup we obtain

$$\mu(x * y) \geqslant \min\{\mu(x),\, \mu(y)\} > \alpha\,,$$

which proves $x{*}y \in S$ for all $* \in \{\cdot, \backslash, /\}$. Hence S is a subquasigroup. By hypothesis, there exists $y \in S$ such that $\mu(y) = \inf\{\mu(x) : x \in S\}$. But $\mu(y) \in Im(\mu)$ implies $\mu(y) = t'$ for some $t' \in Im(\mu)$. Hence $\inf\{\mu(x) : x \in S\} = t' > \alpha$. Note that there does not exist $z \in G$ such that $\alpha \leq \mu(z) < t'$. This gives $\mu_\alpha = \mu_{t'}$. Hence $\mu_\alpha \in \Omega$. Thus Ω contains all level subquasigroups of μ. □

Proposition 3.17. *Let $\mathcal{G}$ be a quasigroup such that every descending chain $S_1 \supset S_2 \supset ...$ of subquasigroups of $\mathcal{G}$ terminates at finite step. If μ is a fuzzy subquasigroup in $\mathcal{G}$ such that a sequence of elements of $Im(\mu)$ is strictly increasing, then μ has finite number of values.*

Proof. Assume that $Im(\mu)$ is not finite. Let $0 \leqslant t_1 < t_2 < ... \leqslant 1$ be a strictly increasing sequence of elements of $Im(\mu)$. Then every $\mu_i = \{x \in G : \mu(x) \geqslant t_i\}$ is a subquasigroup of $\mathcal{G}$. For $x \in \mu_i$ we have $\mu(x) \geqslant t_i > t_{i-1}$, which implies $x \in \mu_{i-1}$. Thus $\mu_i \subseteq \mu_{i-1}$. But for $t_{i-1} \in Im(\mu)$ there exists $x_{i-1} \in G$ such that $\mu(x_{i-1}) = t_{i-1}$. This gives $x_{i-1} \in \mu_{i-1}$ and $x_{i-1} \notin \mu_i$. Hence $\mu_i \subset \mu_{i-1}$, and so we obtain a strictly descending chain $\mu_1 \supset \mu_2 \supset \mu_3 \supset ...$ of subquasigroups, which is not terminating. This contradiction completes the proof. □

Proposition 3.18. *If every fuzzy subquasigroup μ in $\mathcal{G}$ has the finite image, then every descending chain of subquasigroup of $\mathcal{G}$ terminates at finite step.*

Proof. Suppose there exists a strictly descending chain

$$S_0 \supset S_1 \supset S_2 \supset ...$$

of subquasigroups of G which does not terminate at finite step. We prove that μ defined by

$$\mu(x) = \begin{cases} \frac{n}{n+1} & for \quad x \in S_n \setminus S_{n+1}, \\ 1 & for \quad x \in \bigcap S_n, \end{cases}$$

where $n = 0, 1, 2, ...$ and $S_0 = G$, is a fuzzy subquasigroup with an infinite number of values.

If $x * y \in \bigcap S_n$, then obviously $\mu(x * y) = 1 \geqslant \min\{\mu(x), \mu(y)\}$.

If $x * y \notin \bigcap S_n$, then $x * y \in S_p \setminus S_{p+1}$ for some $p \geqslant 0$. Since $x, y \in \bigcap S_n$ implies $x * y \in \bigcap S_n$ then at least one of x, y belongs to some $S_t \setminus S_{t+1}$. Let $x \in \bigcap S_n$ and $y \in S_t \setminus S_{t+1}$, where $t \leqslant p$. The case $t > p$ is impossible because gives $x, y \in S_t$ and, in the consequence, $x * y \in S_t \subseteq S_{p+1}$, which is a contradiction.

For $t \leqslant p$ we have

$$\mu(x * y) = \frac{p}{p+1} \geqslant \min\{\mu(x), \mu(y)\} = \frac{t}{t+1}.$$

If $x*y \in S_p \setminus S_{p+1}$, $x \in S_s \setminus S_{s+1}$ and $y \in S_t \setminus S_{t+1}$, then $s \leqslant p$ or $t \leqslant p$. Indeed, $s > p$ and $t > p$ give $x, y \in S_m$ for $m = \min\{s, t\}$. Thus $m > p$ and $x*y \in S_m \subset S_{p+1}$, which is impossible. Hence $s \leqslant p$ or $t \leqslant p$ and

$$\mu(x*y) = \frac{p}{p+1} \geqslant \min\{\mu(x), \mu(y)\} = \min\left\{\frac{s}{s+1}, \frac{t}{t+1}\right\}.$$

This proves that μ is a fuzzy subquasigroup with an infinite number of different values. Obtained contradiction completes our proof. □

Proposition 3.19. *Every ascending chain of subquasigroups of a quasigroup $\mathcal{G}$ terminates at finite step iff the set of values of any fuzzy subquasigroup in G is a well-ordered subset of* $[0,1]$.

Proof. If the set of values of a fuzzy subquasigroup μ is not well-ordered, then there exists a strictly decreasing sequence $\{t_n\}$ such that $t_n = \mu(x_n)$ for some $x_n \in G$. But in this case subquasigroups $B_n = \{x \in G : \mu(x) \geqslant t_n\}$ form a strictly ascending chain, which is a contradiction.

To prove the converse suppose that there exist a strictly ascending chain $A_1 \subset A_2 \subset A_3 \subset \ldots$ of subquasigroups. Then $S = \bigcup\limits_{n \in N} A_n$ is a subquasigroup of $\mathcal{G}$ and μ defined by

$$\mu(x) = \begin{cases} 0 & for\ x \notin S, \\ \frac{1}{k} & where\ k = \min\{n \in N : x \in A_n\} \end{cases}$$

is a fuzzy set on G.

We prove that μ is a fuzzy subquasigroup. If $x \notin S$ or $y \notin S$, then

$$\mu(x*y) \geqslant \min\{\mu(x), \mu(y)\} = 0.$$

If $x, y \in S$, then also $x*y \in S$. Let m, n, p be minimal such that $x \in A_m$, $y \in A_n$ and $x*y \in A_p$. Obviously $x, y, x*y \in A_k$, where $k = \max\{m, n\} \geqslant p$. Thus

$$\mu(x*y) = \frac{1}{p} \geqslant \frac{1}{k} = \min\{\mu(x), \mu(y)\}.$$

This proves that μ is a fuzzy subquasigroup. Since the chain of subquasigroups $A_1 \subset A_2 \subset A_3 \subset \ldots$ is not terminating, μ has a strictly descending sequence of values. This contradicts that the value set of any fuzzy subquasigroup is well-ordered. The proof is complete. □

4. Normal fuzzy subquasigroups

Definition 4.1. A fuzzy set μ of G is said to be *normal* if there exists $x \in G$ such that $\mu(x) = 1$.

A simple example of a normal fuzzy set is a characteristic function χ_A, where A is a fixed subset of G.

Note that if μ is a normal fuzzy subquasigroup of a unipotent quasigroup $\mathcal{G}$, then $\mu(\theta) = 1$, and hence in a unipotent quasigroup μ is normal iff $\mu(\theta) = 1$.

Proposition 4.2. *Given a fuzzy subquasigroup μ of a unipotent quasigroup $\mathcal{G}$ let μ^+ be a fuzzy set in $\mathcal{G}$ defined by $\mu^+(x) = \mu(x) + 1 - \mu(\theta)$ for all $x \in G$. Then μ^+ is a normal fuzzy subquasigroup of $\mathcal{G}$ which contains μ.*

Proof. We have $\mu^+(\theta) = \mu(\theta) + 1 - \mu(\theta) = 1 \geqslant \mu^+(x)$ for all $x \in G$. Let $x, y, z \in G$. Then

$$\begin{aligned}\min\{\mu^+(x*y),\, \mu^+(y)\} &= \min\{\mu(x*y) + 1 - \mu(\theta), \mu(y) + 1 - \mu(\theta)\} \\ &= \min\{\mu(x*y), \mu(y)\} + 1 - \mu(\theta) \\ &\leqslant \mu(x*y) + 1 - \mu(\theta) \\ &= \mu^+(x*y).\end{aligned}$$

This shows that μ^+ is a fuzzy subquasigroup of a unipotent quasigroup $\mathcal{G}$. Clearly $\mu \subseteq \mu^+$, completing the proof. □

Corollary 4.3. *Let μ and μ^+ be as in the above Proposition. If there is $x \in G$ such that $\mu^+(x) = 0$, then $\mu(x) = 0$.*

Proof. Since $\mu \subseteq \mu^+$, it is straightforward. □

It is clear that in a unipotent quasigroup a fuzzy set μ is normal iff $\mu^+ = \mu$.

Proposition 4.4. *If μ is a fuzzy subquasigroup of a unipotent quasigroup $\mathcal{G}$, then $(\mu^+)^+ = \mu^+$. Moreover if μ is normal, then $(\mu^+)^+ = \mu$.*

Proof. Straightforward. □

Proposition 4.5. *If μ and ν are fuzzy subquasigroups of a unipotent quasigroup $\mathcal{G}$ such that $\mu \subseteq \nu$ and $\mu(\theta) = \nu(theta)$, then $G_\mu \subseteq G_\nu$.*

Proof. Let $x \in G_\mu$. Then $\nu(x) \geqslant \mu(x) = \mu(\theta) = \nu(\theta)$ and so $\nu(x) = \nu(\theta)$, i.e., $x \in G_\nu$, proving $G_\mu \subseteq G_\nu$. □

Corollary 4.6. *If μ and ν are normal fuzzy subquasigroups of a unipotent quasigroups $\mathcal{G}$ such that $\mu \subseteq \nu$, then $G_\mu \subseteq G_\nu$.* □

Proposition 4.7. *Let μ be a fuzzy subquasigroup of a unipotent quasigroup $\mathcal{G}$. If there exists a fuzzy subquasigroup ν of G such that $\nu^+ \subseteq \mu$, then μ is normal.*

Proof. Assume that there exists a fuzzy subquasigroup ν such that $\nu^+ \subseteq \mu$. Then $1 = \nu^+(\theta) \leqslant \mu(theta)$, and so $\mu(\theta) = 1$ and we are done. □

Denote by $\mathcal{N}(G)$ the set of all normal fuzzy subquasigroups of G. Note that $\mathcal{N}(G)$ is a poset under the set inclusion.

Proposition 4.8. *Let μ be a non-constant fuzzy subquasigroup of a unipotent quasigroup $\mathcal{G}$. If μ is a maximal element of $(\mathcal{N}(G), \subseteq)$, then μ takes only the values 0 and 1.*

Proof. Observe that $\mu(\theta) = 1$ since μ is normal. Let $x \in G$ be such that $\mu(x) \neq 1$. We claim that $\mu(x) = 0$. If not, then there exists $a \in G$ such that $0 < \mu(a) < 1$. Let ν be a fuzzy set in G defined by $\nu(x) := \frac{1}{2}(\mu(x) + \mu(a))$ for all $x \in G$. Then clearly ν is well-defined, and we have that for all $x \in G$,

$$\nu(\theta) = \frac{1}{2}(\mu(\theta) + \mu(a)) = \frac{1}{2}(1 + \mu(a)) \geqslant \frac{1}{2}(\mu(x) + \mu(a)) = \nu(x).$$

Moreover, for any $x, y \in G$ we obtain

$$\begin{aligned} \nu(x * y) &= \tfrac{1}{2}(\mu(x * y) + \mu(a)) \geqslant \tfrac{1}{2}(\min\{\mu(x), \mu(y)\} + \mu(a)) \\ &= \min\{\tfrac{1}{2}(\mu(x) + \mu(a)), \tfrac{1}{2}(\mu(y) + \mu(a))\} \\ &= \min\{\nu(x), \nu(y)\}. \end{aligned}$$

Hence ν is a fuzzy subquasigroup of $\mathcal{G}$. It follows from Proposition 4.2 that $\nu^+ \in \mathcal{N}(G)$ where ν^+ is defined by $\nu^+(x) = \nu(x) + 1 - \nu(\theta)$ for all $x \in G$. Clearly $\nu^+(x) \geqslant \mu(x)$ for all $x \in G$. Note that

$$\begin{aligned}\nu^+(a) &= \nu(a) + 1 - \nu(\theta)\\ &= \tfrac{1}{2}(\mu(a) + \mu(a)) + 1 - \tfrac{1}{2}(\mu(\theta) + \mu(a))\\ &= \tfrac{1}{2}(\mu(a) + 1) > \mu(a)\end{aligned}$$

and $\nu^+(a) < 1 = \nu^+(\theta)$. Hence ν^+ is non-constant, and μ is not a maximal element of $\mathcal{N}(G)$. This is a contradiction. □

We construct a new fuzzy subquasigroup from old. Let $t > 0$ be a real number. If $\alpha \in [0,1]$, α^t shall mean the positive root in case $t < 1$. We define $\mu^t : G \to [0,1]$ by $\mu^t(x) := (\mu(x))^t$ for all $x \in G$.

Proposition 4.9. *If μ is a fuzzy subquasigroup of a unipotent quasigroup $\mathcal{G}$, then so is μ^t and $G_{\mu^t} = G_\mu$.*

Proof. For any $x, y \in G$, we have $\mu^t(\theta) = (\mu(\theta))^t \geqslant (\mu(x))^t = \mu^t(x)$ and

$$\begin{aligned}\mu^t(x * y) &= (\mu(x * y))^t \geqslant (\min\{\mu(x),\, \mu(y)\})^t\\ &= \min\{(\mu(x))^t, (\mu(y))^t\} = \min\{\mu^t(x),\, \mu^t(y)\}.\end{aligned}$$

Hence μ^t is a fuzzy subquasigroup. Moreover

$$\begin{aligned}G_{\mu^t} &= \{x \in G : \mu^t(x) = \mu^t(\theta)\} = \{x \in G : (\mu(x))^t = (\mu(\theta))^t\}\\ &= \{x \in G : \mu(x) = \mu(theta)\} = G_\mu\,.\end{aligned}$$

This completes the proof. □

Corollary 4.10. *If $\mu \in \mathcal{N}(G)$, then so is μ^t.*

Proof. Straightforward. □

Definition 4.11. A fuzzy set μ defined on G is called *maximal* if it is non-constant and μ^+ is a maximal element of the poset $(\mathcal{N}(G), \subseteq)$.

Proposition 4.12. *If μ is a maximal fuzzy subquasigroup of a unipotent quasigroup $\mathcal{G}$, then*

(i) *μ is normal,*
(ii) *μ takes only the values 0 and 1,*
(iii) *$\mu_{G_\mu} = \mu$,*
(iv) *G_μ is a maximal subquasigroup.*

Proof. Let μ be a maximal fuzzy subquasigroup. Then μ^+ is a non-constant maximal element of the poset $(\mathcal{N}(G), \subseteq)$. It follows from Proposition 4.8 that μ^+ takes only the values 0 and 1. Note that $\mu^+(x) = 1$ iff $\mu(x) = \mu(\theta)$, and $\mu^+(x) = 0$ iff $\mu(x) = \mu(\theta) - 1$. By Corollary 4.3, we have $\mu(x) = 0$, that is, $\mu(\theta) = 1$. Hence μ is normal, and clearly $\mu^+ = \mu$. This proves (i) and (ii).

(iii) Clearly $\mu_{G_\mu} \subseteq \mu$ and μ_{G_μ} takes only the values 0 and 1. Let $x \in G$. If $\mu(x) = 0$, then obviously $\mu \subseteq \mu_{G_\mu}$. If $\mu(x) = 1$, then $x \in G_\mu$, and so $\mu_{G_\mu}(x) = 1$. This shows that $\mu \subseteq \mu_{G_\mu}$.

(iv) G_μ is a proper subquasigroup because μ is non-constant. Let S be a subquasigroup of $\mathcal{G}$ such that $G_\mu \subseteq S$. Noticing that, for any subquasigroups A and B of $\mathcal{G}$, $A \subseteq B$ iff $\mu_A \subseteq \mu_B$, then we obtain $\mu = \mu_{G_\mu} \subseteq \mu_S$. Since μ and μ_S are normal and since $\mu = \mu^+$ is a maximal element of $\mathcal{N}(G)$, we have that either $\mu = \mu_S$ or $\mu_S = \mathbf{1}$ where $\mathbf{1} : G \to [0, 1]$ is a fuzzy set defined by $\mathbf{1}(x) = 1$ for all $x \in G$. The later case implies that $S = G$. If $\mu = \mu_S$, then $G_\mu = G_{\mu_S} = S$, which follows from the prof of Proposition 4.6. This proves that G_μ is a maximal subquasigroup of $\mathcal{G}$, ending the proof. □

Definition 4.13. A normal fuzzy subquasigroup μ of G is called *completely normal* if there exists $x \in G$ such that $\mu(x) = 0$. The set of all completely normal fuzzy subquasigroups of G is denoted by $\mathcal{C}(G)$.

It is clear that $\mathcal{C}(G) \subseteq \mathcal{N}(G)$. The restriction of the partial ordering $\subseteq$ of $\mathcal{N}(G)$ gives a partial ordering of $\mathcal{C}(G)$.

Proposition 4.14. *If $\mathcal{G}$ is a unipotent quasigroup, then any non-constant maximal element of $(\mathcal{N}(G), \subseteq)$ is also a maximal element of $(\mathcal{C}(G), \subseteq)$.*

Proof. Let μ be a non-constant maximal element of $(\mathcal{N}(G), \subseteq)$. By Proposition 4.8, μ takes only the values 0 and 1, and so $\mu(\theta) = 1$ and $\mu(x) = 0$ for some $x \in G$. Hence $\mu \in \mathcal{C}(G)$. Assume that there exists $\nu \in \mathcal{C}(G)$ such that $\mu \subseteq \nu$. Obviously $\mu \subseteq \nu$ also in $\mathcal{N}(G)$. Since μ is maximal in $(\mathcal{N}(G), \subseteq)$ and since ν is non-constant, therefore $\mu = \nu$. Thus μ is maximal element of $(\mathcal{C}(G), \subseteq)$, ending the proof. □

Proposition 4.15. *In a unipotent quasigroup every maximal fuzzy subquasigroup is completely normal.*

Proof. Let μ be a maximal fuzzy subquasigroup. By Proposition 4.12 μ is normal and $\mu = \mu^+$ takes only the values 0 and 1. Since μ is non-constant, it follows that $\mu(\theta) = 1$ and $\mu(x) = 0$ for some $x \in G$, which completes the proof. □

Proposition 4.16. *Let μ be a fuzzy subquasigroup of a unipotent quasigroup $\mathcal{G}$. If $f : [0, \mu(\theta)] \to [0, 1]$ is an increasing function, then a fuzzy set μ_f defined on G by $\mu_f(x) = f(\mu(x))$ is a fuzzy quasigroup. Moreover,*

a) if $f(\mu(\theta)) = 1$, then μ_f is normal,

b) if $f(t) \geqslant t$ for all $t \leqslant \mu(\theta)$, then $\mu \subseteq \mu_f$.

Proof. Since f is increasing, then for all $x, y \in G$ we have

$$\begin{aligned} \mu_f(x * y) = f(\mu(x * y)) &\geqslant f(\min\{\mu(x),\, \mu(y)\}) \\ &= \min\{f(\mu(x)),\, f(\mu(y))\} \\ &= \min\{\mu_f(x),\, \mu_f(y)\}\,. \end{aligned}$$

This proves that μ_f is a fuzzy subquasigroup.

If $f(\mu(\theta)) = 1$, then clearly μ_f is normal.

If $f(t) \geqslant t$ for all $t \leqslant \mu(\theta)$, then $\mu(x) \leqslant f(\mu(x)) = \mu_f(x)$ for all $x \in G$, which implies $\mu \subseteq \mu_f$. □

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