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On topological n-semigroups

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Abstract

We study n–ary semigroups with a topology. We established the conditions under which topology from n–semigroup is continuing to enveloping semigroup up to topology coordinated with semigroup operation.

1. Introduction

The same set may be a carrier of an algebraic structure and a topological space structure at once. As this takes place, the algebraic operations are assumed to be continuous in the appropriate topology. This permits using both the algebraic and topological methods simultaneously for investigation of those objects.

The subjects of this article are n-semigroups with topology. In this work there been obtained the results concerning the continuity of n-ary operation, established the conditions under which topology from n-semigroup is continuing to enveloping semigroup up to topology coordinated with semigroup by operation. There has been introduced the notion of right (left) reversible n-semigroup and was shown that if on such semigroup with locally compact topology the translations are continuous, then n-ary operation is continuous and n-semigroup is topologically imbedded into local compact group.

We keep to the terminology from [8]. However, we write n-semigroup and n-group instead of n-ary semigroup, n-ary group. If $< X, () >$

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is n-semigroup and $c_1^{n-1} \in X^{n-1}$, then the mapping $x \mapsto (c_1^k x c_{k+1}^{n-1})$ from X to X is called *translation*. If each translation of $< X, (\,) >$ is injective then $< X, (\,) >$ is called a *cancellative n-semigroup.* Let $a_1, ..., a_k, c_{l+1}, ..., c_n$ be the elements of n-semigroup $< X, (\,) >$ and $B_{k+1}, ..., B_l$ be the subsets of this semigroup. Image of Cartesian product $\{a_1\} \times ... \times \{a_k\} \times B_{k+1} \times ... \times B_l \times \{c_{l+1}\} \times ... \times \{c_n\}$ on mapping $(\,)$ we will designate as $(a_1^k B_{k+1}^l c_{l+1}^n)$. By symbol X^k we designate Cartesian product of k specimens of the X set and set $\{x_1...x_k : x_1, x_2, \ldots, x_k \in X\}$ if X is the subset of binary semigroup.

2. Topological n–semigroups and groups

A triple $< X, (\,), \tau >$ is called a *topological n-semigroup* if X is a nonempty set, τ is a topology on X and $(\,) : X^n \to X$ is an associative operation, which is a continuous mapping.

Let $< X, (\,) >$ be an n-semigroup and $a_1^{n-2} \in X^{n-1}$. Then formula

$$x * y = (x a_1^{n-2} y) \quad (x, y \in X) \tag{1}$$

defines an associative binary operation $*$ on X, which is continuous if $< X, (\,), \tau >$ is a topological n-semigroup.

If τ is any topology on an n-semigroup $< X, (\,) >$ and the operation $*$ is continuous then the n-ary operation $(\,)$ may be disconnected. This illustrates the next example.

Example 1. Let $X = (1, +\infty)$, $n > 2$ and the operation $(\,)$ be an ordinary sum of n numbers. The topology τ on X we define with help of the next base $\{(a, b) : 1 < a < b < \infty\} \cup \{n + 1\}$ of open sets. Then the operation $(\,)$ is disconnected in the point $c_1^n \in X^n$, where $c_i = (n + 1)/n$, $i = 1, 2, .., n$. But the operation $*$ is continuous if $a_1^{n-2} \in X^{n-2}$ and $a_1 + ... + a_{n-2} > n - 1$. □

Theorem 1. *Let n–semigroup $< x, (\,) >$ and a topology τ on X be such that for a certain sequence $a_1^{n-2} \in X^{n-2}$ the binary operation $*$ defined by the formula (1) is continuous, for a certain $a \in X$ the identity $(x a_1^{n-2} a) = x$ is realized and the translation $\beta(x) = (a x a_1^{n-2})$*

is continuous. Then $< X, (\,), \tau >$ *is a topological* n*–semigroup.*

The proof of this theorem is given in [4].

Theorem 2. *Let* $< x, (\,) >$ *be an* n*–semigroup and let a locally compact topology* τ *on* X *be such that for certain sequence* $a_1^{n-2} \in X^{n-2}$ *and for each* $x \in X$ *translations* $x \mapsto (xa_1^{n-2}z)$, $x \mapsto (za_1^{n-2}x)$ *are continuous and open. Moreover, let for a certain* $a \in X$ *the identity* $(xa_1^{n-2}a) = x$ *is realized and the translation* $\beta(x) = (axa_1^{n-2})$ *is continuous. If a binary semigroup* $< X, * >$*, where* $*$ *is defined by* (1), *is algebraically embedded into a binary group, then* $< X, (\,), \tau >$ *is a topological* n*–semigroup.*

Proof. Let G be a binary group such that $< X, * >$ is its subsemigroup which generates G. By Corollary 2.3 from [6] (see also [7]) G admits a unique locally compact topology τ_G making it a topological group such that $\tau \subset \tau_G$. Then the binary operation $*$ defined by (1) is continuous. □

A triple $< X, (\,), \tau >$ is called a *topological n-group* if $< X, (\,) >$ is an n-group, the operation $(\,)$ is continuous and the solution x of each equations $(xb_1^{n-1}) = b$ and $(b_1^{n-1}x) = b$ is continuously depend on $b_1^{n-1}b \in X^n$. This definition is equivalent the definition topological n-group which gives G. Crombez and G. Six in [1] for $n > 2$ and it is equivalent the definition topological n-group which gives S. A. Rusakov for $n > 1$ (cf. [8]).

Note that if $< X, (>$ is an n–group the for every $a_1^{n-2} \in X^{n-2}$ there exists an uniquely determined element $a \in X$ such that $(xa_1^{n-2}a) = x$ for any $x \in X$. This element is called *right inverse* to the sequence a_1^{n-2}.

Theorem 3. *Let* $< X, (\,) >$ *be a cancellative* n*-semigroup and* τ *be a compact topology on* X *such that each translation in* $< X, (\,), \tau >$ *is continuous. Then* $< X, (\,), \tau >$ *is a topological* n*-group.*

Proof. Let $a_1^{n-2} \in X^{n-2}$ and $*$ be a binary operation on X which is defined by formula (1). Then $< X, * >$ is the binary cancellative semigroup, each a translation $< X, *, \tau >$ is continuous and τ is a compact topology. Hence, $< X, *, \tau >$ is the compact binary group. Let a be a neutral element of this group and $c_1^{n-1}c \in X^n$. The equation $x * (ac_1^{n-1}) = c$ has the solution in $< X, * >$. As $x * (ac_1^{n-1}) = (xa_1^{n-2}ac_1^{n-1}) = ([x * a]c_1^{n-1}) = (xc_1^{n-1})$, then the equation $(xc_1^{n-1}) = c$ is solved in $< X, () >$. It is similarly shown that the equation $(c_1^{n-1}x) = c$ is solved in $< X, () >$. Hence, by results obtained in [4], $< X, () >$ is a topological n-group. □

3. Enveloping semigroups

Let $< X, () >$ be an n-semigroup. A binary semigroup $< S, \bullet >$ is called an *enveloping semigroup* for $< X, () >$ if $X \subset S$, $(x_1^n) = x_1 ... x_n$ for each sequence $x_1^n \in X^n$ and $S = X \cup X^2 \cup ... \cup X^{n-1}$. An enveloping semigroup $< S, \bullet >$ for $< X, () >$ is called the *universal enveloping semigroup* if the sets $X, X^2, ..., X^{n-1}$ are disjoints. The smallest universal enveloping semigroup is called the *universal covering semigroup*. In [3] is mentioned that S. Markovski proved that for any cancellative n-semigroup exists a cancellative enveloping (universal) semigroup.

Theorem 4. *Let $< X, (), \tau >$ be a cancellative n-semigroup and let τ be a topology on X such that each translation of $< X, (), \tau >$ is continuous and open. Then there exists an universal enveloping semigroup $< S, \bullet >$ for n-semigroup $< X, () >$ and there exists a topology τ_S on S such that each translation of $< S, \bullet, \tau >$ is open and continuous mapping, $X, X^2, ..., X^{n-1} \in \tau_S$ and restriction of τ_S to X is the same as τ. If the topology τ is Hausdorff then the topology τ_S is Hausdorff. If the topology τ is locally compact then τ_S is locally compact topology.*

Proof. Let $< S, \bullet >$ be a cancellative universal enveloping semigroup for an n-semigroup $< X, () >$. Then $S = X \cup X^2 \cup ... \cup X^{n-1}$, where $X^i = \{x_1 ... x_1 \,:\, x_1, ..., x_i \in X\}$ and $X^i \cap X^j = \emptyset$ if $i \neq j$. Let

$$\mathcal{A} = \{a_1 ... a_k B b_1 ... b_l \,:\, 0 \geq k, l \geq n-1, \;\; B \in \tau, \;\; a_1^k \in X^k, \;\; b_1^l \in X^l\}.$$

We show that there exists a unique topology τ_S on S having $\mathcal{A}$ as a basis open sets.

Let $a_1 \ldots a_k A c_1 \ldots c_l \cap d_1 \ldots d_s B b_1 \ldots b_m \neq \emptyset$, where $0 \geq k, l, s, m \geq n-1$, $A, B \in \tau$ and $a_1^k, b_1^m, c_1^l, d_1^s$ are the sequences of elements from the n-semigroup X.

We must consider the following four cases:

a) $k+l < n-1, \quad s+m < n-1,$
b) $k+l < n-1, \quad s+m \geq n-1,$
c) $k+l \geq n-1, \quad s+m < n-1,$
d) $k+l \geq n-1, \quad s+m \geq n-1.$

a) Let $k+l < n-1$ and $s+m < n-1$. Then $k+l = s+m$ and there exists $a \in A$, $b \in B$ such that

$$a_1 \ldots a_k a c_1 \ldots c_l = d_1 \ldots d_s b b_1 \ldots b_m \,.$$

Let $x \in X$. Then

$$(\overset{(n-k-l-1)}{x} \; a_1^k a c_1^l) = (\overset{(n-k-l-1)}{x} \; d_1^s b b_1^m)\,.$$

The set $(\overset{(n-k-l-1)}{x} \; a_1^k A c_1^l)$ is nonempty and belongs to τ. As all translations of $< X, (\,), \tau >$ are continuous, then there exist $U \in \tau$ and $b \in U \subset B$ such that

$$(\overset{(n-k-l-1)}{x} \; d_1^s U b_1^m) \subset (\overset{(n-k-l-1)}{x} \; a_1^k A c_1^l)\,.$$

From here we have

$$x^{n-k-l-1} d_1 \ldots d_s U b_1 \ldots b_m \subset x^{n-k-l-1} a_1 \ldots a_k A c_1 \ldots c_l$$

in S. As S is a cancellative semigroup, then

$$d_1 \ldots d_s U b_1 \ldots b_m \subset a_1 \ldots a_k A c_1 \ldots c_l \,.$$

Hence

$$d_1 \ldots d_s U b_1 \ldots b_m \subset a_1 \ldots a_k A c_1 \ldots c_l \cap d_1 \ldots d_s B b_1 \ldots b_m \,.$$

b) Let $k+l < n-1$, $s+m \geq n-1$. Then $s+m = n-1+k+l$. Let $x \in X$. Then for any $a \in A$ and $b \in B$ we have

$$(\overset{(n-k-l-1)}{x} \; a_1^k a c_1^l) = (\overset{(n-k-l-1)}{x} \; d_1^s b b_1^m)\,.$$

Then, as above, there exists $U \in \tau$ and $b \in U \subset B$ such that

$$d_1 \ldots d_s U b_1 \ldots b_m \subset a_1 \ldots a_k A c_1 \ldots c_l \cap d_1 \ldots d_s B b_1 \ldots b_m \,.$$

c) Analogously as b).

d) Let $k + l \geq n - 1$, $s + m \geq n - 1$. Then

$$d_1 ... d_s B b_1 ... b_m = d_1 ... d_{s+m+1-n} (d^s_{s+m+2-n} B b^m_1) = d_1 ... d_{s+m+1-n} B_1 \,,$$

where $B_1 = (d^s_{s+m+2-n} B b^m_1) \in \tau$ and

$$a_1 ... a_k A c_1 ... c_l = a_1 ... a_{k+l+1-n} (a^k_{k+l+2-n} A c^l_1) = a_1 ... a_{k+l+1-n} A_1 \,,$$

where $A_1 = (a^k_{k+l+2-n} A c^l_1) \in \tau$. By the case a) there exists $U \in \tau$ such that $d_1 ... d_s U b_1 ... b_m \subset a_1 ... a_k A c_1 ... c_l \cap d_1 ... d_s B b_1 ... b_m$. It is clear that $\cup\{B : B \in \mathcal{A}\} = \mathcal{S}$. But then there exists an unique topology τ_S on S such that $\mathcal{A}$ is a basis of open sets for this topology.

As $X^k = \cup\{x_1 ... x_{k-1} X : x^{k-1}_1 \in X^{k-1}\}$, then $X^k \in \tau_S$ for $k = 1, 2, .., n-1$. It is clear that $\tau \subset \tau_S$. Let $\mathcal{A} \ni \dashv_\infty ... \dashv_{\|} \mathcal{B} \lfloor_\infty ... \lfloor_\updownarrow \subset \mathcal{X}$. Then either $k = 0$, $l = 0$ and hence $\tau \ni B = a_1 ... a_k B b_1 ... b_l$, or $k + l = n - 1$ or $k + l = 2n - 2$ and then $a_1 ... a_k B b_1 ... b_l = (a^k_1 B b^l_1) \in \tau$, as the translations of $< X, (\,), \tau >$ are open. Hence, restrictions of τ_S to X coincide with τ.

Let $A \in \tau$, and let a^k_1, b^l_1, x^s_1 be the sequences of elements of X and $0 \geq k \geq n - 1$, $0 \geq l \leq n - 1$. Then the set $.x_1 ... x_s \{a_1 ... a_k A b_1 ... b_l\}$ is equal to $x_1 ... x_s a_1 ... a_k A b_1 ... b_l$, if $s + k \geq n - 1$ and it is equal to $(x^s_1 a^{n-s}_1) a_{n-s+1} ... a_k A b_1 ... b_l$, if $s + k > n - 1$, i.e. this set belongs to $\mathcal{A}$. Hence, the left translations of $< S, \bullet, \tau_s >$ are open. The right translations case is proved as above.

Let $a = a_1 ... a_k$, $b = b_1 ... b_l$, where $a_i, b_j \in X$, $1 \geq k \geq n - 1$, $1 \geq l \geq n - 1$. Let U be a neighbourhood of ab in the topology τ_S. We can suppose that $U = c_1 ... c_s B d_1 ... d_m$, where $0 \geq m, s \geq n - 1$ and $B \in \tau$. Let $k + l < n$ and $s + m < n - 1$. Then $k + l = s + m + 1$ and for $x \in X$ we have

$$(\overset{(n-k-l)}{x}\ a^k_1 b^l_1) \in (\overset{(n-k-l)}{x}\ c^s_1 B d^m_1).$$

As all translations of X are open, then the set $(\overset{(n-k-l)}{x}\ c^s_1 B d^m_1)$ is open in X. The translation $y \mapsto (\overset{(n-k-l)}{x}\ a^k_1 b^{l-1}_1 y)$ is continuous. Hence, there exists an open neighbourhood V of b_l such that

$$(\overset{(n-k-l)}{x}\ a^k_1 b^{l-1}_1 V) \subset (\overset{(n-k-l)}{x}\ c^s_1 B d^m_1) \,.$$

As S is a cancellative semigroup we have

$$a_1...a_kb_1...b_{l-1}V \subset c_1...c_sBd_1...d_m \subset U.$$

As $b_1...b_{l-1}V$ is neighbourhood of b in the topology τ_S, then left translations of $< S, \bullet, \tau_S >$ are continuous.

Let $k+l \geq n$, $s+m < n-1$. Then $k+l = s+m+n$ and for $x \in X$ we have $(\overset{(2n-k-l)}{x}\ a_1^kb_1^l) \in (\overset{(2n-k-l)}{x}\ c_1^sBd_1^m)$. As shown above we can see that left translations of $< S, \bullet, \tau_S >$ are continuous.

Let $k+l < n$ and $s+m \geq n-1$. Then $k+l+n-1 = s+m+1$. Therefore, for $x \in X$ we have $(\overset{(2n-s-m-2)}{x}\ a_1^kb_1^l) \in (\overset{(2n-s-m-2)}{x}\ c_1^sBd_1^m)$ and again as shown above we see that left translations of $< S, \bullet, \tau_S >$ are continuous.

If $k+l \geq n$, $s+m \geq n-1$ then we have $k+l = s+m+1$. Let $x \in X$. Then we have $(\overset{(2n-k-l-1)}{x}\ a_1^kb_1^l) \in (\overset{(2n-k-l-1)}{x}\ c_1^sBd_1^m)$ and again left translations of $< S, \bullet, \tau_S >$ will be continuous.

Let τ be a Hausdorff topology. Let $a, b \in S$, $a \neq b$. If $a \in X^k$, $b \in X^l$, $k \neq l$, $1 \geq k$, $l \geq n-1$, then $X^k \cap X^l = \emptyset$, X^k is the neighbourhood of a and X^l is the neighbourhood b. If $a, b \in X$, then the disjoint neighbourhoods of this points in the space X are the disjoint neighbourhoods of this points in the space $< S, \bullet, \tau_S >$. Let $a = a_1...a_k$, $b = b_1...b_k$, where $k \in \{2, ..., n-1\}$ and a_1^k, b_1^k are the sequences of points of X. If $x \in X$ then $(\overset{(n-k)}{x}\ a_1^k) \neq (\overset{(n-k)}{x}\ b_1^k)$, because S is a cancellative semigroup. Hence, open disjoint neighbourhoods of U and V of points $(\overset{(n-k)}{x}\ a_1^k)$ and $(\overset{(n-k)}{x}\ b_1^k)$, respectively, exist. Then $\lambda^{-1}(U) \cap \lambda^{-1}(V) = \emptyset$, where $\lambda(t) = x^{n-k}t$, $t \in S$, is the left translation of $< S, \bullet, \tau_S >$. Hence, $\lambda^{-1}(U)$ and $\lambda^{-1}(V)$ are neighbourhoods of points a and b, respectively, in the space $< S, \bullet, \tau_S >$. So the topology τ_S is Hausdorff.

Let τ be a locally compact topology. The mapping $x \mapsto a_1a_2...a_kx$, where $x \in X$, is homeomorphisme between the open subset U of X and open subset $a_1^kX \subset X^{k+1}$ of the space $< S, \bullet, \tau_S >$. Hence, each point of the space $< S, \bullet, \tau_S >$ has a compact neighbourhood. And so τ_S is a Hausdorff topology, therefore τ_S is a locally compact topology. □

Theorem 5. *Let* $< X, (\,), \tau >$ *be a topological n-semigroup such that for a certain* $p \in \{0, 1, ..., n-1\}$ *and for any* $c_1^{n-1} \in X^{n-1}$ *the trans-*

lation $x \mapsto (c_1^p x c_{p+1}^{n-1})$ *is open. If the universal enveloping semigroup* $< S, \bullet >$ *for n-semigroup* $< X, (\,) >$ *is cancellative semigroup, then the collection*

$$\mathcal{B} = \{A_1 ... A_k \,:\, A_i \in \tau, \quad i = 1, 2, ..., k, \quad k = 1, 2, ..., n-1\}$$

is the base of topology τ_S *on the enveloping semigroup* $< S, \bullet >$*, where each set* $X^i \subset S$ $(i = 1, 2, ..., n-1)$ *is open; restriction of* τ_S *to* X *is the same as* τ *and the semigroup's operation is continuous. If each translation of* $< X, (\,), \tau >$ *is open, then any translation of* $< S, \bullet, \tau_S >$ *is open too.*

Proof. Note, that $\cup\{B \,:\, B \in \mathcal{B}\} = S$. If $A = A_1 ... A_i \cap B_1 ... B_j \neq \emptyset$, where $1 \geq i, j \geq n-1$ and $A_1 ... A_i\,,\, B_1 ... B_j \in \tau$, then $i = j$. Let $g \in A$ and let a be a certain but fixed an element of X. Then $a_1 \in A_1\,, \,...\,,\, a_i \neq A_i$ are such that $g = a_1 ... a_i$. The set

$$(\overset{(l)}{a}\; B_1 ... B_i \;\overset{(n-i-l)}{a}) = \bigcup_{b_1 \in B_1\,,\,...\,,\,b_i \in B_i} (\overset{(l)}{a}\; b_1 ... b_{k-1} B_k b_{k+1} ... b_i \;\overset{(n-i-l)}{a})$$

is the open neighbourhood of $(\overset{(l)}{a}\; g \;\overset{(n-i-l)}{a})$, if k and l are chosen so $1 \geq k \geq j$, $l + k = p + 1$. As n-ary operation $(\,)$ is continuous, then for every $m = 1, 2, ..., i$, there exists the open neighbourhoods F_m of points a_m such that $(\overset{(l)}{a}\; F_1 ... F_i \;\overset{(n-i-l)}{a}) \subset (\overset{(l)}{a}\; B_1 ... B_i \;\overset{(n-i-l)}{a})$. Using the cancellation in S we have $F_1 ... F_i \subset B_1 ... B_i$. This proves that $a_m \in G_m = A_m \cap F_m \in \tau$ and $g \in G_1 ... G_i \subset A_1 ... A_i \cap B_1 ... B_j$. Hence, there exists a unique topology τ_S on S having $\mathcal{B}$ as a base of open sets. Notice that $X, X^2, ..., X_{n-1} \in \tau_S$ and $\tau \subset \tau_S$. As for any $U \in \tau_S$, $U \subset X$ we have $U \in \tau$, then restriction of τ_S to X coincides with τ.

Now we show that $< S, \bullet, \tau_S >$ is a topological semigroup. Let $s = a_1 ... a_i$, $t = b_1 ... b_j$, where $a_1, ..., a_i, b_1, ..., b_j \in X$ and $1 \geq i, j \geq n-1$, and let $g = st$. If $C \in \mathcal{B}$ and $g \in C$, then $C = C_1 ... C_k$, where any $C_l \in \tau$ and $C_l \neq \emptyset$. Let $c_l \in C_l$ be so that $g = c_1 ... c_k$. If $i + j < n$, then we have $ab = a_1 ... a_i b_1 ... b_j = c_1 ... c_k$ and by hypothesis of the theorem we conclude that $k = i + j$. Using cancellativity in S and continuity of n-ary operation $(\,)$ we conclude that there exists open neighbourhoods $A_1, ..., A_i$ of points $a_1, ..., a_i$ and open neighbourhoods $B_1, ..., B_j$ of points $b_1, ..., b_j$ such that $A_1 ... A_i B_1 ... B_j \subset C_1 ... C_k = C$.

Note that $A = A_1...A_i$ is open neighbourhood of s and $B_1...B_j$ is open neighbourhood of t. So $AB \subset C$.

Let $i + j \geq n$ and let $a = (a_1^i b_1^{n-i})$. Then $ab_{n-i+1}...b_j = c_1...c_k$. As shown above we conclude that $k = i + j - n - 1$ and that there exists open neighbourhoods F of point a and $B_{n-i+1}, ..., B_j$ of points $b_{n-i+1}, ..., b_j$ such that $FB_{n-i+1}...B_j \subset C_1...C_k = C$. As the n-ary operation $(\,)$ is continuous there exists open neighbourhoods $A_1, ..., A_i$ of points $a_1, ..., a_i$ and open neighbourhoods $B_1, ..., B_{n-i}$ of points $b_1, ..., b_{n-i}$ such that $(A_1...A_iB_1...B_{n-i}) \subset F$. Then for $A = A_1...A_i$ and for $B = B_1...B_j$ we have $AB \subset C$, $s \in A \in \tau_S$ and $t \in B \in \tau_S$. Hence, $< S, \bullet, \tau_S >$ is the topological semigroup.

Let $B_1, ..., B_j \in \tau$, $a_1, ..., a_i \in X$. If $i + j < n$, then for $a \in X$ we have

$$a_1...a_iB_1...B_j = \lambda^{-1}\Big((\overset{(n-i-j)}{a}\ a_1...a_iB_1...B_j)\Big)\,,$$

where λ is a left translation of S on element a^{n-i-j}. It follows from the fact that λ is injective mapping and

$$\lambda(a_1...a_iB_1...B_j) = (\overset{(n-i-j)}{a}\ a_1...a_iB_1...B_j)\,.$$

Let each translation of $< X, (\,), \tau >$ be an open mapping. Then the set $(\overset{(n-i-j)}{a}\ a_1...a_iB_1...B_j)$ is open in X and therefore it is open in S. Hence, the set $a_1...a_iB_1...B_j$ is open in S, as λ is the continuous mapping from S to S.

If $i + j > n - 1$, then the set

$$a_1...a_iB_1...B_j = a...a_{i+j-n}(a_{i+j-n+1}...a_iB_1...B_j)$$

is open in $< S, \bullet, \tau_S >$ as the set $(a_{i+j-n+1}...a_iB_1...B_j)$ is open in $< X, (\,), \tau >$. Hence, each left translation of $< S, \bullet, \tau_S >$ is open mapping. It is similarly shown that each right translation of $< S, \bullet, \tau_S >$ is an open mapping. □

Remark. The Theorem 6 is extension of the theorem of G. Čupona [2] on the case of topological n-semigroups.

4. Reversible n-semigroups with topology

We say that an n-semigroup $< X, () >$ is *right (left) reversible* n-semigroup if for any sequences a_1^{n-1} and b_1^{n-1} of elements of X there exists x and z from X such that

$$(xa_1^{n-1}) = (zb_1^{n-1}) \qquad (\text{ respectively}: \ (a_1^{n-1}x) = (b_1^{n-1}z)\ ).$$

An n-semigroup X is said to be *algebraically imbedded* in a binary group G if there exists an injective homomorphism from X to G. An n-semigroup X with the topology is said to be *topologically imbedded* in a binary group G with a topology if there exists an injective bicontinuous homomorphism from X to G.

Theorem 6. *Let $< S, \bullet >$ be an enveloping semigroup for a right (left) reversible n-semigroup $< X, () >$. Then $< S, \bullet >$ is a right (left) reversible semigroup.*

Proof. Let $< X, () >$ be a right reversible n-semigroup and $a, b \in S$. Then $a = a_1 ... a_k$, $b = b_1 ... b_l$, where $a_1^k \in X^k$, $b_1^l \in X^l$, $1 \geq k, l \geq n-1$. Let $c \in X$. Then there exists $x, z \in X$ such that $(x \overset{(n-k-l)}{c} a_1^k) = (z \overset{(n-l-1)}{c} b_1^l)$. As $fa = gb$, where $f = xc^{n-k-1} \in S$, $g = yc^{n-l-1} \in S$, then the binary semigroup $< S, \bullet >$ is right reversible. It is similarly proved that $< S, \bullet >$ is left reversible semigroup. □

Corollary. *For a right (left) reversible cancellative n-semigroup there exists an universal enveloping right (left) reversible cancellative semigroup and therefore right (left) reversible cancellative n-semigroup can be algebraically imbedded in a binary group.* □

Theorem 7. *Let $< X, (), \tau >$ be a right (left) reversible cancellative n-semigroup with a locally compact topology τ such that each translation of $< X, (), \tau >$ is continuous and open. Then $< X, (), \tau >$ is a topological n-semigroup and it can be topologically imbedded in a locally compact binary group as open subset.*

Proof. By the Theorems 4 and 6 there exists an universal enveloping right (left) reversible cancellation semigroup $< S, \bullet >$ for n-semigroup $< X, ( ) >$ and the locally compact topology τ_S such that $X \in \tau_S$, restriction of τ_S to X is the same as τ. By the Theorem 3 from [9] semigroup $< S, \bullet, \tau_S >$ can be topologically imbedded as open subsemigroup of a locally compact topological group. □

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