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IK-loops

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Abstract

A loop $\mathcal{Q}(\cdot)$ is called a K-loop, if the identities:

$$(x \cdot yIx) \cdot xz = x \cdot yz\,, \qquad (y \cdot x) \cdot (I^{-1}xz \cdot x) = yz \cdot x$$
$$(\;Ix = x^{-1}\,, \quad I^{-1}x = {}^{-1}x\,, \quad I^{-1}x \cdot z = {}^{-1}x \cdot z\;)$$

hold. A K-loop is called an *IK*-loop if the substitution I is an automorphism of the loop. It is proved that: a K-loop generated by one element is solvable; in a *IK*-loop the center $\mathcal{Z}(\mathcal{Q})$ and the nucleus $\mathcal{N}$ coincide and every *IK*-loop is nilpotent. Examples of K-loops, generated by one element are given.

In [1] and [2] the following result is obtained: in a K-loop $\mathcal{Q}(\cdot)$ the nucleus $\mathcal{N}$ is a nontrivial ($\mathcal{N} \neq \{e\}$) normal subloop and the quotient loop $\mathcal{Q}/\mathcal{N}(\cdot)$ is an abelian group. If a K-loop $\mathcal{Q}(\cdot)$ is not a group, then the nucleus $\mathcal{N}$ of this loop has a nontrivial center $\mathcal{Z}(\mathcal{N})$.

Proposition 1. *If a loop $\mathcal{Q}(\cdot)$ has a nontrivial nucleus $\mathcal{N}$, which is a normal subloop of $\mathcal{Q}(\cdot)$ and (x,y,z) is the associator of elements $x,y,z \in \mathcal{Q}$, then $(x,y,z)n = n(x,y,z)$, where $n \in \mathcal{N}$.*

Proof. For every $x,y,z \in \mathcal{Q}$ and $n \in \mathcal{N}$ we have

$$xy \cdot zn = (xy \cdot z) \cdot n = (x \cdot yz) \cdot (x,y,z)n\,. \tag{1}$$

Since $\mathcal{N}$ is a normal subloop of $\mathcal{Q}(\cdot)$, then for every $x \in \mathcal{Q}$ and $n \in \mathcal{Q}$ there exist $n', n'' \in \mathcal{N}$ such that

$$xn = n'x\,, \qquad nx = xn''\,. \tag{2}$$

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Applying (2) to $xy \cdot zn$, we get

$$xy \cdot zn = xy \cdot n_1 z = xyn_1 \cdot z = (x \cdot n_2 y) \cdot z = (xn_2 \cdot y) \cdot z =$$
$$(n_3 x \cdot y) \cdot z = n_3(xy \cdot z) = (xn_2 \cdot yz) \cdot (x,y,z) = (x \cdot n_2 yz) \cdot (x,y,z) =$$
$$= (x \cdot yn_1 z) \cdot (x,y,z) = (x \cdot yz)n \cdot (x,y,z) = (x \cdot yz) \cdot n(x,y,z)$$

that is

$$xy \cdot zn = (x \cdot yz) \cdot n(x,y,z)\,. \tag{3}$$

If follows from (1) and (3) that $(x,y,z)n = n(x,y,z)$, which was to be proved. □

Corollary 1. *If a (nongroup) loop $\mathcal{Q}(\cdot)$ has a nontrivial nucleus $\mathcal{N}$ which is a normal subloop of $\mathcal{Q}(\cdot)$ and the associator of any three elements of $\mathcal{Q}$ belongs to $\mathcal{N}$, then $\mathcal{N}$ has a nontrivial center $\mathcal{Z}(\mathcal{N})$.* □

In [2] it is proved that in a K-loop $\mathcal{Q}(\cdot)$ the nucleus $\mathcal{N}$ contains the associator of any three elements of $\mathcal{Q}$.

Corollary 2. (Theorem 3 from [1]) *If a K-loop $\mathcal{Q}(\cdot)$ is not a group, then the nucleus $\mathcal{N}$ of $\mathcal{Q}(\cdot)$ has a nontrivial center.* □

Proposition 2. *The center $\mathcal{Z}(\mathcal{N})$ of the nucleus $\mathcal{N}$ of a K-loop $\mathcal{Q}(\cdot)$ is a normal subloop of $\mathcal{Q}(\cdot)$.*

Proof. In a K-loop $\mathcal{Q}(\cdot)$ the nucleus $\mathcal{N}$ is a normal subloop of $\mathcal{Q}(\cdot)$, therefore, $L_x^{-1}R_x c \in \mathcal{N}$ for every $c \in \mathcal{N}$ and every $x \in \mathcal{Q}$.

If $z \in \mathcal{Z}(\mathcal{N})$, then

$$z \cdot L_x^{-1}R_x c = L_x^{-1}R_x c \cdot z\,. \tag{4}$$

From the definition of a K-loop we have the autotopy

$$T = (R_x^{-1}L_x\,,\, L_x\,,\, L_x)\,. \tag{5}$$

Applying (5) to the equality (4), we get

$$R_x^{-1}L_x z \cdot L_x L_x^{-1}R_x c = L_x(L_x^{-1}R_x c \cdot z)$$

or

$$(R_x^{-1}L_x z\cdot c) = (L_x L_x^{-1}(cz\cdot x)Ix)$$

or

$$R_x^{-1}L_x z\cdot c = c\cdot(x\cdot zI_x)\,,$$

hence $R_x^{-1}L_x z\cdot c = c\cdot L_x R_{Ix} z$. Every K-loop is an Osborn loop where $R_{Ix} = L_x^{-1}R_x^{-1}L_x$ and then

$$R_x^{-1}L_x z\cdot c = c\cdot L_x L_x^{-1}R_x^{-1}L_x z$$

or

$$R_x^{-1}L_x z\cdot c = c\cdot R_x^{-1}L_x z\,,$$

which proves that $R_x^{-1}L_x z\in\mathcal{Z}(\mathcal{N})$. □

Proposition 3. *If a K-loop $\mathcal{Q}(\cdot)$ is not a group, the quotient loop $\mathcal{Q}/\mathcal{Z}(\mathcal{N})$ is a group.*

Proof. From Proposition 2 it follows that $\mathcal{Z}(\mathcal{N})$ is a normal subloop of $\mathcal{Q}(\cdot)$, hence there exists the quotient loop $\mathcal{Q}/\mathcal{Z}(\mathcal{N})$, in which

$$\begin{aligned} &a\mathcal{Z}(\mathcal{N})\cdot(b\mathcal{Z}(\mathcal{N})\cdot c\mathcal{Z}(\mathcal{N})) = \\ &\qquad = a\mathcal{Z}(\mathcal{N}) = (ab\cdot c)\mathcal{Z}(\mathcal{N}) = (ab\cdot c)\cdot(a,b,c)\mathcal{Z}\mathcal{N})\,. \end{aligned}$$

As $(a,b,c)\in\mathcal{Z}(\mathcal{N})$, we have

$$\begin{aligned} &(ab\cdot c)\cdot(a,b,c)\mathcal{Z}(\mathcal{N}) = (ab\cdot c)\mathcal{Z}(\mathcal{N}) = ab\mathcal{Z}(\mathcal{N})\cdot c\mathcal{Z}\mathcal{N}) = \\ &\qquad = (a\mathcal{Z}(\mathcal{N})\cdot b\mathcal{Z}(\mathcal{N}))\cdot c\mathcal{Z}(\mathcal{N})\,. \end{aligned}$$

Thus,

$$a\mathcal{Z}(\mathcal{N})\cdot(b\mathcal{Z}(\mathcal{N})\cdot c\mathcal{Z}(\mathcal{N})) = (a\mathcal{Z}(\mathcal{N})\cdot b\mathcal{Z}(\mathcal{N})\cdot c\mathcal{Z}(\mathcal{N}))\,,$$

so the operation $(\cdot)$ on $\mathcal{Q}/\mathcal{Z}(\mathcal{N})$ is associative. □

Definition 1. The loop $\mathcal{Q}(\cdot)$ is called *solvable* if it has a series of the form

$$\mathcal{Q} = \mathcal{Q}_0 \supseteq \mathcal{Q}_1 \supseteq \mathcal{Q}_2 \supseteq \ldots \supseteq \mathcal{Q}_m = E\,,$$

where $\mathcal{Q}_i$ is a normal subloop of $\mathcal{Q}_{i-1}$ and the quotient loop $\mathcal{Q}_{i-1}/\mathcal{Q}_i$ is an abelian group.

Theorem 1. *A K-loop generated by one element is solvable.*

Proof. Let an element $a \in \mathcal{Q}$ generates the K-loop $\mathcal{Q}(\cdot)$. From Proposition 3 we obtain that $\mathcal{Q}/\mathcal{Z}(\mathcal{N})$ is a group. If φ is a homomorphism of $\mathcal{Q}(\cdot)$ on $\mathcal{Q}/\mathcal{Z}(\mathcal{N})$, then the group $\mathcal{Q}/\mathcal{Z}(\mathcal{N})$ is also generated by an element, namely by $\varphi(a)$. But a group generated by an element is cyclic and since $\mathcal{Z}(\mathcal{N})$ is an abelian group, the loop $\mathcal{Q}(\mathcal{N})$ is solvable. □

Corollary. *Every subloop of a K-loop generated by one element is solvable.* □

Example 1. ([3], p.193). Let $\mathcal{F}$ be a field, $\mathcal{F}'$ be the set of nonzero elements of $\mathcal{F}$. Define on the set $\mathcal{Q} = \mathcal{F}' \times \mathcal{F}$ the operation $(\cdot)$ as follows:

$$(a, x) \cdot (b, y) = (a \cdot b\,,\ (a^{-1} - 1) \cdot (b^{-1} - 1) + b^{-1}x + y)\,.$$

Then $\mathcal{Q}(\cdot)$ is a K-loop. The nucleus $\mathcal{N}$ of this loop consists of pairs $(1, x), x \in \mathcal{F}$. The operation $(\cdot)$ is commutative on $\mathcal{N}$. Indeed,

$$(1, x) \cdot (1, y) = (1\,,\ x + y) = (1\,,\ y + x) = (1, y) \cdot (1, x)$$

hence, $\mathcal{N}$ is an abelian group. But then the loop $\mathcal{Q}(\cdot)$ from this example is solvable (for any field $\mathcal{F}$).

For $\mathcal{F} = \mathcal{Z}_3$ we get a K-loop consisting of six elements:

$*$	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	3	1	6	4	5
3	3	1	2	5	6	4
4	4	5	6	2	3	1
5	5	6	4	1	2	3
6	6	4	5	3	1	2

This loop is generated by any of elements 4, 5, 6, so by Theorem 1 it is solvable. □

Example 2. Let $\mathcal{R}$ be a commutative ring (which is not $\mathcal{Z}_2$ and the zero ring). Define on $\mathcal{Q} = \mathcal{R} \times \mathcal{R}$ the operation $(\cdot)$

$$(a,x)\cdot(b,y)=(a+b\,,\;x+y+ab^2\,)$$

for any $(a,x),(b,y)\in\mathcal{Q}$. Then $\mathcal{Q}(\cdot)$ is a K-loop. If $\mathcal{R}=\mathcal{Z}_3$, we get a loop of 9 elements:

$\bullet$	1	2	3	4	5	6	7	8	9
1	1	2	3	4	5	6	7	8	9
2	2	3	1	5	6	4	8	9	7
3	3	1	2	6	4	5	9	7	8
4	4	5	6	8	9	7	3	1	2
5	5	6	4	9	7	8	1	2	3
6	6	4	5	7	8	9	2	3	1
7	7	8	9	2	3	1	6	4	5
8	8	9	7	3	1	2	4	5	6
9	9	7	8	1	2	3	5	6	4

This loop is one generated by each of the elements 4, 5, 6, 7, 8, 9. By Theorem 1 it is solvable.

Note that in this example the permutation I $(\,Ix=x^{-1}\,)$ is an automorphism of $\mathcal{Q}(\cdot)$. □

Definition 2. A K-loop is called an *IK-loop* if the permutation I is an automorphism of $\mathcal{Q}(\cdot)$, i.e. $I(x\cdot y)=Ix\cdot Iy$ for every $x,y\in\mathcal{Q}$.

Proposition 4. *If $\mathcal{N}$ is the nucleus of the loop $\mathcal{Q}(\cdot)$, then for any $x\in\mathcal{Q}$ and $c\in\mathcal{N}$ the equalities*

$$I(c\cdot x)=Ix\cdot Ic\,,\qquad I(x\cdot c)=Ic\cdot Ix \tag{6}$$

hold up.

Proof. Directly from the equality $cx\cdot I(c\cdot x)=1$ it follows that $x\cdot I(c\cdot x)=x^{-1}$ or $I(c\cdot x)=L_x^{-1}Ic$ or $I(c\cdot x)=L_{Ix}L_{Ix}^{-1}L_x^{-1}Ic$. But

$$L_xL_{Ix}c=x\cdot Ixc=(x\cdot Ix)\cdot c=c\,.$$

Hence, $L_{Ix}^{-1}L_x^{-1}Ic=Ic$ and then $I(c\cdot x)=L_{Ix}Ic=Ix\cdot Ic$. The second equality can be proved similarly. □

Proposition 5. *The center $\mathcal{Z}(\mathcal{Q})$ and the nucleus $\mathcal{N}$ of an IK-loop $\mathcal{Q}(\cdot)$ coincide.*

Proof. Let $\mathcal{Q}(\cdot)$ be an *IK*-loop. Then the permutation I is an automorphism of $\mathcal{Q}(\cdot)$ and $I(x \cdot y) = Ix \cdot Iy$ for any $x, y \in \mathcal{Q}$. In particular, if $x \in \mathcal{Q}$ and $c \in \mathcal{N}$, then

$$I(c \cdot x) = Ic \cdot Ix\,. \qquad (7)$$

From (6) and (7) it follows that

$$Ix \cdot Ic = Ic \cdot Ix\,. \qquad (8)$$

From (8) and $c \in \mathcal{N}$ we obtain $c \in \mathcal{Z}(\mathcal{Q})$, therefore

$$\mathcal{N} \subseteq \mathcal{Z}(\mathcal{Q})\,. \qquad (9)$$

But from the definition of the center of a loop it follows that

$$\mathcal{Z}(\mathcal{Q}) \subseteq \mathcal{N}\,. \qquad (10)$$

Thus, from (9) and (10) we get $\mathcal{Z}(\mathcal{Q}) = \mathcal{N}$. □

Definition 3. A loop $\mathcal{Q}(\cdot)$ is *nilpotent* if it has a finite invariant series

$$\mathcal{Q} = \mathcal{Q}_0 \supseteq \mathcal{Q}_1 \supseteq \mathcal{Q}_2 \supseteq ... \supseteq \mathcal{Q}_k = E\,,$$

where every quotient loop $\mathcal{Q}_{i-1}/\mathcal{Q}_i$ is contained in the center of the loop $\mathcal{Q}/\mathcal{Q}_i$ $(i = 1, 2, ..., k)$.

Theorem 2. *Every IK-loop $\mathcal{Q}(\cdot)$ is nilpotent.*

Proof. Let $\mathcal{Q}(\cdot)$ be a nongroup *IK*-loop, then $\mathcal{Q}(\cdot)$ has a nontrivial nucleus $\mathcal{N}$, which by Proposition 5 coincides with the center of $\mathcal{Q}(\cdot)$, i.e. $\mathcal{N} = \mathcal{Z}(\mathcal{Q})$. Hence, for the loop $\mathcal{Q}(\cdot)$ there is a series of normal subloops

$$\mathcal{Q} = \mathcal{Q}_0 \supseteq \mathcal{Q}_1 \supseteq \mathcal{Q}_2 = E\,,$$

satisfying the condition: $\mathcal{Q}_{i-1}/\mathcal{Q}_i \subseteq \mathcal{Z}(\mathcal{Q}/\mathcal{Q}_i)\,,$ $i = 1, 2\,,$ and this means that $\mathcal{Q}(\cdot)$ is nilpotent.

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