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Characteristic of ordered Menger systems of multiplace functions

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Abstract

In this article the abstract characterization of Menger systems and some Menger T-systems of multiplace functions is given. These multiplace functions can have different number of variables.

1. Introduction

In his work [1] K. Menger has formulated the problem of abstract characterization of sets of functions of several variables on which the operation of superposition is given and the relation of continuation of functions is marked. This problem for functions with the number of variables $n = 1$ was solved by B. M. Schein [2] and for fixed $n \geq 2$ was examined by V. S. Trokhimenko. But in a general case, when n takes different natural values it is open until the present moment. In this article the author solves the word problem for the so-called Menger systems and some Menger T-systems of multiplace functions. The results of the work were partially presented during the Colloquium on Semigroups in Szeged (1994).

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2. Preliminaries

Let A be a set, n – natural number. Then, every partial mapping $f : A^n \to A$ is called a *n–place function*, where A^n is the n-th Cartesian power of A. The set of all n-place functions, which are considered on A, is denoted by $F_n(A)$. Now let $(F_n(A))_{n\in I}$ be some family of sets denoted above, where $I \subseteq N$ (N is the set of natural numbers). For any $n, m_1, ..., m_n \in I$ and $f \in F_n(A)$, $g_i \in F_{m_i}(A)$ $(i = 1, ..., n)$, by $f[g_1...g_n]$ a m-place function will be denoted, where $m = \max(m_1, ..., m_n)$ and for all $a_1, ..., a_m \in A$ the following identity holds:

$$f[g_1...g_n](a_1, ..., a_m) = f(g_1(a_1, ..., a_{m_1}), ..., g_n(a_1, ..., a_{m_n}))\,. \tag{1}$$

Doing so, we consider that the left and right members of (1) are determined or not determined simultaneously. The operation

$$(f, g_1, ..., g_n)\ \xi\ f[g_1...g_n]$$

will be denoted by $\mathcal{O}_n$. It is evident that the family of operations $(\mathcal{O}_n)_{n\in I}$ satisfies the so-called condition of *superassociativity*

$$f[g_1...g_n][h_1...h_m] = f[g_1[h_1 \ldots h_{m_1}]...g_n[h_1...h_{m_n}]]\,, \tag{2}$$

where

$$f \in F_n(A), \quad g_i \in F_{m_i}(A), \quad i = 1, ..., n, \quad m = \max(m_1, ..., m_n)\,.$$

Let $(\Phi_n)_{n\in I}$ be a family of subsets, which is stable regarding operations $(\mathcal{O}_n)_{n\in I}$, where $\Phi_n \subset F_n(A)$, $n \in I$, $(\zeta_{\Phi_n}, \chi_{\Phi_n})_{n\in I}$ is a family of pairs of binary relations such that

$$(f, g) \in \zeta_{\Phi_n} \Longleftrightarrow f \subset g \tag{3}$$

$$(f, g) \in \chi_{\Phi_n} \Longleftrightarrow \mathrm{Dom}\, f \subset \mathrm{Dom}\, g \tag{4}$$

for all $f, g \in \Phi_n$, where $\mathrm{Dom} f$ is the domain of definition for f. Then the systems $(\Phi_n, \mathcal{O}_n)_{n\in I}$ will denote the *Menger T-systems.* The forms $(\Phi_n, \mathcal{O}_n, \zeta_{\Phi_n}, \chi_{\Phi_n})_{n\in I}$, $(\Phi_n, \mathcal{O}_n, \zeta_{\Phi_n})_{n\in I}$ and $(\Phi_n, \mathcal{O}_n, \chi_{\Phi_n})_{n\in I}$ will denote (respectively) *fundamentally ordered projection (f.o.p.) Menger systems, fundamentally ordered (f.o.) Menger systems and projection quasi-ordered (p.q-o.) Menger systems* of multiplace functions. If in

(1) $m = m_1 = ... = m_n$, then we come to the notion of Menger system of multiplace functions in the sense of work [6]. We similarly introduce the definition of f.o.p. and p.q-o. Menger systems of multiplace functions.

An abstract *Menger T-system of rank I* will be called a family $(G_n, \mathcal{O}_n)_{n\in I}$, where $I \subset N$, $(G_n)_{n\in I}$ are non-empty sets, such that $G_n \cap G_m = \emptyset$ for any $n, m \in I$; for every $n \in I$ $\mathcal{O}_n$ is a mapping, which brings to conformity for every $(n+1)$-th of the elements (x, y_1, y_n) from $G_n \times G_{m_1} \times ... \times G_{m_n}$, $m_1, ..., m_n \in I$; the element $x[y_1, ..., y_n]$ is from G_m, where $m = \max(m_1, ..., m_n)$, and it satisfies the identity of superassociativity of the form (2). If $m = m_1 = ... = m_n$, then we obtain the definition of an abstract Menger system [3], [6].

The Menger system of rank I is called *weakly unitary* if for every $n \in I$ the set G_n contains such elements $e_1^n, ..., e_n^n$ that for every element g from G_n the identity $g[e_1^n, ..., e_n^n] = g$ is true.

By $(T_n)_{n\in I}$ we shall denote the family of sets of polynomials for the (weakly unitary) Menger $(T-)$ system $(G_n, \mathcal{O}_n)_{n\in I}$ that is defined in the following way: let $\{x_n : n \in I\}$ be a set of different subject variables, then we can consider:

a) $x_n \in T_n$ for every $n \in I$,

b) if $t \in T_m$, $g_1, ..., g_{i-1}, g_{i+1}, ..., g_n \in G_m$, $g \in G_n$, then
$$g[g_1 ... g_{i-1} t\, g_{i+1} ... g_n] \in T_m\,,$$
for every $i = 1, ..., n$ and $m \in I$.

If $t \in T_n$ and $g \in G_n$, where $n \in I$, then by $t(g)$ we shall denote the element from G_n which is obtained as a result of realization of all operations after the substitution of g for variable x_n in the polynomial t.

Let $(G_n, \mathcal{O}_n)_{n\in I}$ be the (weakly unitary) Menger $(T-)$ system of rank I, then the family of binary relations $(\rho_n)_{n\in I}$ such that $\rho_n \subset G_n \times G_n$, is called:

– *stable*, if for all $n, m \in I$, $x, y \in G_n$, $x_i, y_i \in G_m$, $i = 1, ..., n$
$$(x, y) \in \rho_n \wedge \{(x_1, y_1), ..., (x_n, y_n)\} \subset \rho_m \Rightarrow (x[x_1 ... x_n]\,,\, y[y_1 ... y_n]) \in \rho_m,$$
– *v-regular*, if for any $n, m \in I$, $u \in G_n$, $x_i, y_i \in G_m$, $i = 1, ..., n$
$$(x_1, y_1), ..., (x_n, y_n) \subset \rho_m \Rightarrow (u[x_1 ... x_n]\,,\, u[y_1 ... y_n]) \in \rho_m,$$

– *l-regular*, if for all $n, m \in I,\ x, y \in G_n,\ z_1, ..., z_n \in G_m$

$$(x, y) \in \rho_n \Rightarrow (x[z_1...z_n]\,,\, y[z_1...z_n]) \in \rho_m.$$

A family of subsets $(W_n)_{n\in I}$ (where $W_n \subset G_n$) is called a *l-ideal*, if for all $n, m \in I,\ g \in G_n,\ \bar{y} \in G_m^n,\ x \in W_m$ and $i = 1, ..., n$ it is true that

$$g[\bar{y}\,|_i\, x] \in W_m\,,$$

where $[\bar{y}\,|_i\, x]$ denotes $(y_1, ..., y_{i-1}, x, y_{i+1}, ..., y_n)$.

Let us consider two Menger T-systems of rank $I : \mathcal{G} = (G_n, \mathcal{O}_n)_{n\in I}$ and $\mathcal{G}' = (G'_n\,, \mathcal{O}'_{n\in I})$. By a homomorphism $\mathcal{G}$ on $\mathcal{G}'$ we'll denote the family $(P^n)_{n\in I}$, where P^n is a mapping G_n on G'_n for every $n \in I$ such that for all $n, m_1, ..., m_n \in I,\ g \in G_n,\ g_i \in G_m,\ i = 1, ..., n$ the following identity holds:

$$P^m(g[g_1...g_n]) = P^n(g)[p^{m_1}(g_1)...P^{m_n}(g_n)]\,, \tag{5}$$

where $m = \max(m_1, ..., m_n)$. If every mapping P^n is one-to-one then such a homomorphism is called a *proper* one (or *isomorphism*). A homomorphism of the Menger T-system $\mathcal{G}$ on some Menger system of multiplace functions is called a *representation of $\mathcal{G}$ by functions*. The notions of homomorphism, isomorphism and representation are defined similarly for the Menger systems.

Consider the family of pairs $(\mathcal{E}_n, W_n)_{n\in I}$ on Menger ($T-$)system $(G_n, \mathcal{O}_n)_{n\in I}$, where for every $n \in I$ $\mathcal{E}_n$ is a relation of equivalence on G_n, and W_n is either an empty set or a $\mathcal{E}_n$-class; the family $(\mathcal{E}_n)_{n\in I}$ is v-regular and $(W_n)_{n\in I}$ is an l-ideal. Let $(H_a)_{a\in A_n}$ denote the family of $\mathcal{E}_n$-classes that is different from W_n, and let it be one-to-one indexed by elements of a set A_n. We find that $A_n \cap A_m = \emptyset$ for any $n, m \in I$ if $n \neq m$. Let $\{C_n : n \in I\}$ be the set of different elements that do not fall into $A = \bigcup\limits_{n\in I} A_n$. For every $n \in I$ denote by I_n the set $\{m : m \in I \wedge m < n\}$, and by I'_n – the set $I \backslash (I_n \cup \{n\})$. Let

$$B_n = \prod_{m\in I_n} A_m \times \{C_n\} \times \prod_{m\in I'_n} A_m\,, \quad \overline{A} = \prod_{n\in I} A_n\,, \quad \Im_n = \overline{A}^{\,n} \cup B_n^n$$

(for the Menger T-systems we find that $B_n = \emptyset$ and $\Im_n = \overline{A}^{\,n}$). For every element $g \in G_n$, $n \in I$, we'll determine n-place function such that:

$$(\overline{a}_1,...,\overline{a}_n,\bar{b}\,)\in P^n(g)\Longleftrightarrow$$

$$(\overline{a}_1,...,\overline{a}_n)\in\Im_n\,\wedge\,(\forall i\in I)(g[H_{\overline{a_1}<i>}...H_{\overline{a_n}<i>}]\subset H_{\bar{b}<i>})\qquad(6)$$

where $\overline{a_k}<i>$ denotes the component of the vector $\overline{a}_k$ that belongs to the set A_i for $i\neq n$; and to the set $A_n\cup\{C_n\}$ for $i=n$. It can be shown that the family of mappings $(P^n)_{n\in I}$ (where $P^n:g\mapsto P^n(g)$) is a representation of Menger ($T-$)system $\mathcal{G}$ by multiplace functions which in future will be called the *simplest*.

3. Results

Let $(\Phi_n,\mathcal{O}_n)_{n\in I}$ be some Menger ($T-$) system of multiplace functions, $(\zeta_{\Phi_n})_{n\in I}$ and $(\chi_{\Phi_n})_{n\in I}$ be the family of binary relations that are defined by means of (3) and (4). In the future, instead of $(f,g)\in\zeta_{\Phi_n}$ and $(f,g)\in\chi_{\Phi_n}$ we'll write $f\subset_n g$ and $f\sqsubset_n g$ respectively.

Proposition 1. *On the Menger ($T-$) system $(\Phi_n,\mathcal{O}_n)_{n\in I}$ of multiplace functions set $(\zeta_{\Phi_n})_{n\in I}$ is the stable family of relations of order and set $(\chi_{\Phi_n})_{n\in I}$ is the l-regular family of quasi-order being for every $n\in I$; the inclusion $\zeta_{\Phi_n}\subset\chi_{\Phi_n}$ is true too.*

Proof. It is evident that ζ_{Φ_n} is an order and χ_{Φ_n} – a quasi-order on Φ_n, therefore it is necessary to verify only the conditions of stability and l-regularity.

Let $f\subset_n g$, $f_i\subset_m g_i$, $i=1,...,n$, and $(a_1,...,a_m,c)\in f[f_1...f_n]$. Then there will exist such elements $b_1,...,b_n$ that $(a_1,...,a_n,b_i)\in f_i$, $i=1,...,n$, and $(b_1,...,b_n,c)\in f$. Therefore, $(a_1,...,a_m,b_i)\in g_i$, $i=1,...,n$, and $(b_1,...,b_n,c)\in g$, whence $(a_1,...,a_m,c)\in g[g_1...g_n]$. And so

$$f[f_1...f_n]\subset_m g[g_1...g_n]\,.$$

Stability of $(\zeta_{\Phi_n})_{n\in I}$ is proved.

Now assume that $f\sqsubset_n g$, i.e. $\mathrm{Dom}\,f\subset\mathrm{Dom}\,g$ and $f,g\in\Phi_n$. Let $h_1,...,h_n\in\Phi_m$ and $(a_1,...,a_m)\in\mathrm{Dom}\,f[h_1...h_n]$. The latter means that there exists an element c, such that $(a_1,...,a_m,c)\in f[h_1...h_n]$. Then, for some $b_1,...,b_n$: $(a_1,...,a_m,b_i)\in h_i$, $i=1,...,n$, and $(b_1,...,b_n,c)\in f$; consequently $(b_1,...,b_n)\in\mathrm{Dom}\,f$. Therefore,

$(b_1, ..., b_n) \in \operatorname{Dom} g$. Thus, $(a_1, ..., a_m, b_i) \in h_i$, $i = 1, ..., n$, and $(b_1, ..., b_n, d) \in g$ for some element d.

The latter means that $(a_1, ..., a_m, d) \in g[h_1...h_n]$ for some d, i.e. $(a_1, ..., a_m,) \in \operatorname{Dom} g[h_1...h_n]$. Then

$$f[h_1...h_n] \leftharpoondown_m g[h_1...h_n] \,,$$

therefore, the l-regularity of family $(\chi_{\Phi_n})_{n\in I}$ is proved. The inclusion $\zeta_{\Phi_n} \subset \chi_{\Phi_n}$ is evident (for every $n \in I$). □

Proposition 2. *Let the families of relations $(\zeta_{\Phi_n})_{n\in I}$, $(\chi_{\Phi_n})_{n\in I}$ be defined on the Menger $(T-)$ system $(\Phi_n, \mathcal{O}_n)_{n\in I}$ of multiplace functions. Then they satisfy the following conditions:*

$$f_1 \subset_n f_2 \ \wedge\ g \subset_n t_1(f_1) \ \wedge\ g \subset_n t_2(g_2) \Rightarrow g \subset_n t_2(f_1)\,, \tag{7}$$

$$g_1 \subset_n f \ \wedge\ g_2 \subset_n f \ \wedge\ g_1 \leftharpoondown_n g_2 \Rightarrow g_1 \subset_n g_2\,, \tag{8}$$

$$g_1 \subset_n g_2 \ \wedge\ f \leftharpoondown_n g_1 \ \wedge\ f \leftharpoondown_n u[\overline{\omega}\,|_j\, g_2] \Rightarrow f \leftharpoondown_n u[\overline{\omega}\,|_j\, g_1]\,, \tag{9}$$

$$f[h_1...h_n] \leftharpoondown_m h_1\,, \tag{10}$$

for any $n, m \in I$, $i = 1, ..., n$, $f, f_1, f_2, g, g_1, g_2 \in \Phi_n$, $\overline{\omega} \in G_n^m$, $u, h_1, ..., h_n \in \Phi_m$, $j = 1, ..., m$, $t_1, t_2 \in T_n$.

Proof. Let the premise of condition (7) be valid, then, from $f_1 \subset_n f_2$, we obtain that $f_1 = f_2 \circ \pm_{Dom\, f_1}$ (the restriction of function f_2 on the domain of the function f_1 is denoted by $f_2 \circ \pm_{Dom\, f_1}$). From $g \subset_n t_1(f_1)$ follows $\operatorname{Dom} g \subset \operatorname{Dom} f_1$, therefore from $g \subset_n t_2(f_2)$ we obtain that

$$g = g \circ \pm_{Dom\, f_1} \subset t_2(f_2) \circ_{Dom\, f_1} = t_2(f_2 \circ \pm_{Dom\, f_1}) = t_2(f_1)\,.$$

Thus, $g \subset_n t_2(f_1)$. The condition (7) is proved.

Let now the premise of condition (8) be valid, then from $g_1 \subset_n f$ and $g_2 \subset_n f$ we have $g_1 = f \circ \pm_{Dom\, g_1}$ and $g_2 = f \circ \pm_{Dom\, g_2}$, respectively. Since $g_1 \leftharpoondown_n g_2$, then $\operatorname{Dom} g_1 \subset \operatorname{Dom} g_2$, therefore $f \circ \pm_{Dom\, g_1} \subset f \circ \pm_{Dom\, g_2}$, i.e. $g_1 \subset g_2$. The condition (8) is proved. The following condition (9) is proved similarly (7), therefore we must prove validity of (10).

Let $(a_1, ...a_m) \in \operatorname{Dom} f[h_1...h_n]$, therefore $(a_1, ..., a_m, c) \in f[h_1...h_n]$ for some element c, and there exists vector $(b_1, ..., b_n)$ for which

$(b_1,...,b_n,c) \in f$ and $(a_1,...,a_m,b_i) \in h_i$, $i = 1,...,n$. Therefore, we obtain $(a_1,...,a_m) \in \mathrm{Dom}\, h_i$. Thus, $\mathrm{Dom}\, f[h_1...h_n] \subset \mathrm{Dom}\, h_i$ which was needed to prove. □

Theorem. *A (weakly unitary) Menger ($T-$) system of the form $(G_n,\mathcal{O}_n,\zeta_n,\chi_n)_{n\in I}$, where ζ_n, χ_n are fixed binary relations on G_n, is isomorphic to some f.o.p. Menger ($T-$) system of multiplace functions if and only if it is a stable family of relations of order, $(\chi_n)_{n\in I}$ is a l-regular family of relations of quasi-order such that $\zeta_n \subset \chi_n$ for every $n \in I$, and the following conditions hold:*

$$g_1 \leq_n g \;\wedge\; g_2 \leq_n g \;\wedge\; g_1 \sqsubset_n g_2 \;\Rightarrow\; g_1 \leq_n g_2\,, \tag{11}$$

$$g[h_1...h_n] \sqsubset_m h_i\,, \tag{12}$$

$$g_1 \leq_n g_2 \;\wedge\; g \sqsubset_n g_1 \;\wedge\; g \sqsubset_n u[\overline{\omega}\,|_j\, g_2] \;\Rightarrow\; g \sqsubset_n u[\overline{\omega}\,|_j\, g_1]\,, \tag{13}$$

for all $n,m \in I$, $i = 1,...,n$, $j = 1,...,m$, $g,g_1,g_2 \in G_n$, $u,h_1,...,h_n \in G_m$, $\overline{\omega} = (\omega_1,...,\omega_m) \in G_n^m$, where $g_1 \leq_n g_2$, $g_1 \sqsubset_n g_2$ mean that $(g_1,g_2) \in \zeta_n$ and $(g_1,g_2) \in \chi_n$, respectively.

Proof. The necessity of conditions of the theorem follows from Propositions 1 and 2, therefore weŠll dwell on the proof of their sufficiency. So, let the Menger (weakly unitary $T-$) system of the form $(G_n,\mathcal{O}_n,\zeta_n,\chi_n)_{n\in I}$ satisfies all conditions of the theorem. Easily can be proved that for all $n \in I$, $g,g_1,g_2 \in G_n$, $t_1,t_2 \in T_n$ the conditions:

$$g_1 \leq_n g_2 \;\wedge\; g \sqsubset_n t_1(g_1) \;\wedge\; g \sqsubset_n t_2(g_2) \;\Rightarrow\; g \sqsubset_n t_2(g_1)\,, \tag{14}$$

$$g_1 \leq_n g_2 \;\wedge\; g \sqsubset_n t_1(g_1) \;\wedge\; g \sqsubset_n t_2(g_2) \;\Rightarrow\; g \sqsubset_n t_2(g_1)\,, \tag{15}$$

are valid.

Let $\overline{G}$ denote the Cartesian power of the sets of the family $(G_n)_{n\in I}$. For every $\overline{g}$ from $\overline{G}$ we shall assign a family of pairs of the form $(\mathcal{E}_{\overline{g}<n>},W_{\overline{g}<n>})_{n\in I}$, where $\mathcal{E}_{\overline{g}<n>} = \mathcal{E}^1_{\overline{g}<n>} \cap \mathcal{E}^2_{\overline{g}<n>}$, and $\mathcal{E}^1_{\overline{g}<n>}$, $\mathcal{E}^2_{\overline{g}<n>}$, $W_{\overline{g}<n>}$ are defined as follows:

$$(g_1,g_2) \in \mathcal{E}^1_{\overline{g}<n>} \Leftrightarrow (\forall t \in T_n)\,[\overline{g}\langle n\rangle \leq_n t(g_1) \Leftrightarrow \overline{g}\langle n\rangle \leq_n t(g_2)]\;, \tag{16}$$

$$(g_1,g_2) \in \mathcal{E}^2_{\overline{g}\langle n\rangle} \Leftrightarrow (\forall t \in T_n)\,[\overline{g}\langle n\rangle \sqsubset_n t(g_1) \Leftrightarrow \overline{g}\langle n\rangle \sqsubset_n t(g_2)] \tag{17}$$

$$W_{\overline{g}<n>} = G_n \backslash \chi_n \langle \overline{g}\langle n \rangle \rangle \,, \tag{18}$$

for all $g, g_1, g_2 \in G_n$, where

$$\chi_n \langle \overline{g}\langle n \rangle \rangle = \{x \,:\, \overline{g}\langle n \rangle \leftharpoondown_n x\} \,.$$

It is easy to see that $(\mathcal{E}_{\overline{g}<n>})_{n\in I}$ is the v-regular family of relations of equivalency and $(W_{\overline{g}<n>})_{n\in I}$ is an l-ideal family of $\mathcal{E}_{\overline{g}<n>}$-classes, if $W_{\overline{g}<n>} \neq \emptyset$ for every $n \in I$. Therefore, as it is shown in the previous part, the family $(\mathcal{E}_{\overline{g}<n>}, W_{\overline{g}<n>})_{n\in I}$ defines the simplest representation of the system $(G_n, \mathcal{O}_n)_{n\in I}$ with the help of multiplace functions, which we denote by $(P^n_{\overline{g}})_{n\in I}$. If $\overline{u}, \overline{v} \in \overline{G}$, $\overline{u} \neq \overline{v}$, then for every $g \in G_n$, $n \in I$ we'll find that the n-place functions $P^n_{\overline{u}}(g)$ and $P^n_{\overline{v}}(g)$ are given on disjoint sets. Let now

$$P^n(g) = \bigcup_{\overline{u}\in\overline{G}} P^n_{\overline{u}}(g),$$

then the family $(P^n)_{n\in I}$ is a representation of $(G_n, \mathcal{O}_n)_{n\in I}$ with the help of multiplace functions.

Let's prove that for every $n \in I$ and any $g_1, g_2 \in G_n$

$$g_1 \leftharpoondown_n g_2 \Longleftrightarrow \mathrm{Dom}\, P^n(g_1) \subset \mathrm{Dom}\, P^n(g_2) \,.$$

Indeed, if $\mathrm{Dom}\, P^n(g_1) \subset \mathrm{Dom}\, P^n(g_2)$, then it means that

for every $\overline{u} \in \overline{G}$:

$$\mathrm{Dom}\, P^n_{\overline{u}}(g_1) \subset \mathrm{Dom}\, P^n_{\overline{u}}(g_2)$$

for every $\overline{u} \in \overline{G}$.

The latter inclusion means that for $\overline{a_1}, ..., \overline{a_n} \in \overline{A}^n$ such that $(\overline{a_1}, ..., \overline{a_n}) \in \Im^n$, the implication

$$(\overline{a_1}, ..., \overline{a_n}) \in \mathrm{Dom}\, P^n_{\overline{u}}(g_1) \;\Rightarrow\; (\overline{a_1}, ..., \overline{a_n}) \in \mathrm{Dom}\, P^n_{\overline{u}}(g_2) \tag{19}$$

is valid. This condition, as it is easily to see, is equivalent to

$$(\forall\, \overline{b})\, (\exists\, \overline{c})\;\; (\overline{a_1}, ..., \overline{a_n}, \overline{b}) \in P^n_{\overline{u}}(g_1) \;\Rightarrow\; (\overline{a_1}, ..., \overline{a_n}, \overline{c}) \in P^n_{\overline{u}}(g_1) \,,$$

which, in its turn means

$$\begin{aligned}(\forall\, \overline{b})\, (\exists\, \overline{c})\; \Big[\, (\forall\, i \in I)\; \Big(g_1[H_{\overline{a_1}<i>} ... H_{\overline{a_n}<i>}] \subset H_{\overline{b}<i>} \Big) \;\Rightarrow \\ \Rightarrow (\forall k \in I)\; (\; g_2[H_{\overline{a_1}<k>} ... H_{\overline{a_n}<k>}] \subset H_{\overline{c}<k>}\;) \,\Big] \,. \end{aligned} \tag{20}$$

It can be proved that (20) is equivalent to the formula

$$(\forall\,(\overline{x_1},...,\overline{x_n})\in D_n)\,(\forall k\in I)$$
$$(\,g_1[\overline{x_1}\langle k\rangle...\overline{x_n}\langle k\rangle]\notin W_{\overline{u}<k>}\ \Rightarrow\ g_2[\overline{x_1}\langle k\rangle...\overline{x_n}\langle k\rangle]\notin W_{\overline{u}<k>}\,)\,, \tag{21}$$

where

$$D_n=\overline{G}^{\,n}\cup E_n^n\,,\quad E_n=\prod_{m\in I_n}G_m\times\{e_n\}\times\prod_{m\in I_n'}G_m\,,$$

(for the weakly unitary Menger T-system we suppose $E_n=\emptyset$), $e_n\notin G_n$ and $g[e_n...e_n]=g$ for every $g\in G_n$ by the definition. Let the condition (21) fulfill: $\overline{x_1},...,\overline{x_n}\in E_n^n$, $k=n$, and let $\overline{u}$ be an element from $\overline{G}$ such that $\overline{u}\langle n\rangle=g_1$, then we obtain:

$$g_1[e_n...e_n]\notin W_{g_1}\Rightarrow g_2[e_n...e_n]\notin W_{g_1}\,, \tag{22}$$

i.e.
$$g_1\notin W_{g_1}\Rightarrow g_2\notin W_{g_1}\,.$$

So $g_1\notin W_{g_1}$ is true for every $g_1\in G_n$, and (22) can be written as $g_2\in W_{g_1}$, i.e. $g_1\leftharpoondown_n g_2$.

Conversely, let

(a) $g_1\leftharpoondown_n\,g_2\,,$

(b) $g_1[\overline{x_1}\langle k\rangle...\overline{x_n}\langle k\rangle]\notin W_{\overline{u}<k>}$

for some $\overline{u}\in\overline{G}$, $k\in I$, $\overline{x_1},...,\overline{x_n}\in D_n$.

So (χ_n) is an l-regular family, and from (a) we obtain

$$g_1[\overline{x_1}\langle k\rangle...\overline{x_n}\langle k\rangle]\leftharpoondown_k\,g_2[\overline{x_1}\langle k\rangle...\overline{x_n}\langle k\rangle]\,.$$

The condition (b) means that

$$\overline{u}\langle k\rangle\leftharpoondown_k\,g_1[\overline{x_1}\langle k\rangle...\overline{x_n}\langle k\rangle]\,,$$

therefore, due to transitivity of χ_n we have

$$\overline{u}\langle k\rangle\leftharpoondown_k\,g_2[\overline{x_1}\langle k\rangle...\overline{x_n}\langle k\rangle]\,,$$

i.e. $g_2[\overline{x_1}\langle k\rangle...\overline{x_n}\langle k\rangle]\notin W_{\overline{u}<k>}$. Thus, (21) is proved. Hence,

$$\mathrm{Dom}\ P^n_{\overline{u}}(g_1)\subset\mathrm{Dom}\ P^n_{\overline{u}}(g_2)$$

for every $\overline{u}\in\overline{G}$, i.e. $\mathrm{Dom}\ P^n(g_1)\subset\mathrm{Dom}\ P^n(g_2)$. Let's prove now that for every $n\in I$ and any $g_1,g_2\in G_n$ the condition $g_1\leq_n(g_2)$ is valid if and only if the inclusion $P^n(g_1)\subset P^n(g_2)$ is true.

Indeed, if $P^n(g_1)\subset P^n(g_2)$, then for every $\overline{u}\in\overline{G}$ we have

$P^n_{\overline{u}}(g_1) \subset P^n_{\overline{u}}(g_2)$. This inclusion means that

$$(\overline{a_1},...,\overline{a_n},\overline{b}) \in P^n_{\overline{u}}(g_1) \Rightarrow (\overline{a_1},...,\overline{a_n},\overline{b}) \in P^n_{\overline{u}}(g_2) \tag{23}$$

for any $\overline{a_1},...,\overline{a_n},\overline{b} \in \overline{A}$, where $(\overline{a_1},...,\overline{a_n}) \in \Im_n$. According to the definition of the simplest representation the condition (22) can be rewritten as follows:

$$(\forall i \in I)\,(\,g_1[H_{\overline{a_1}<i>}...H_{\overline{a_n}<i>}] \subset H_{\overline{b}<i>}\,) \Rightarrow$$

$$\Rightarrow (\forall k \in I)\,(\,g_2[H_{\overline{a_1}<k>}...H_{\overline{a_n}<k>}] \subset H_{\overline{b}<k>}\,), \tag{24}$$

for all $\overline{a_1},...\overline{a_n},\overline{b} \in \overline{A}$, where $(\overline{a_1},...,\overline{a_n}) \in \Im_n$. One can check that (24) is equivalent to the formula:

$$(\forall(\overline{x_1},...,\overline{x_n} \in D_n)\,(\forall k \in I)\;(g_1[\overline{x_1}\langle k\rangle...\overline{x_n}\langle k\rangle] \notin W_{\overline{u}<k>} \Rightarrow$$
$$\Rightarrow g_1[\overline{x_1}\langle k\rangle...\overline{x_n}\langle k\rangle] \equiv g_2[\overline{x_1}\langle k\rangle...\overline{x_n}\langle k\rangle]\;(\mathcal{E}_{\overline{u}} < k >)\,) \tag{25}$$

Assume that $\overline{x_1},...,\overline{x_n} \in E^n_n$, $k = n$ in the condition (25) and let $\overline{u}$ be an element from $\overline{G}$, such that $\overline{u}\langle n\rangle = g_1$, then we obtain $g_1 \equiv g_2(\mathcal{E}_{g_1})$, whence it follows: $g_1 \equiv g_2(\mathcal{E}^1_{g_1})$. The latter according to formula (16), means that

$$(\forall t \in T_n)\,(\,g_1 \leq_n t(g_1) \Longleftrightarrow g_1 \leq_n t(g_2)\,). \tag{26}$$

Let t be the variable x_n, then from (26) it follows that $g_1 \leq_n g_2$.

Conversely, suppose that $g_1 \leq_n g_2$ and

$$g_1[\overline{x_1}\langle k\rangle...\overline{x_n}\langle k\rangle] \notin W_{\overline{u}<k>}$$

for any $\overline{u} \in \overline{G}$, $(\overline{x_1},...,\overline{x_n}) \in D_n$, $k \in I$. We must prove that

$$g_1[\overline{x_1}\langle k\rangle...\overline{x_n}\langle k\rangle] \equiv g_2[\overline{x_1}\langle k\rangle...\overline{x_n}\langle k\rangle]\;(\mathcal{E}_{\overline{u}<k>}) \tag{27}$$

is valid.

For this purpose we must check if the condition (27) is valid for every relation $\mathcal{E}^i_{\overline{u}<k>}$, $i = 1,2$, which is defined with the help of the formulas (16) and (17). Let

$$\overline{u}\langle k\rangle \leq_k t(g_1[\overline{x_1}\langle k\rangle...\overline{x_n}\langle k\rangle])$$

for some $t \in T_k$. Since the family $(\zeta_n)_{n\in I}$ is stable, then from $g_1 \leq_n g_2$

we obtain

$$t(g_1[\overline{x_1}\langle k\rangle ... \overline{x_n}\langle k\rangle]) \leq_k t(g_2[\overline{x_1}\langle k\rangle ... \overline{x_n}\langle k\rangle]),$$

therefore

$$\overline{u}\langle k\rangle \leq_k t(g_2[\overline{x_1}\langle k\rangle ... \overline{x_n}\langle k\rangle]).$$

Suppose now that

$$\text{(c)} \quad \overline{u}\langle k\rangle \leq_k t(g_2[\overline{x_1}\langle k\rangle ... \overline{x_n}\langle k\rangle])$$

is valid, where $t \in T_k$.

Since $g_1 \leq_n g_2$ and

$$g_1[\overline{x_1}\langle k\rangle ... \overline{x_n}\langle k\rangle] \notin W_{\overline{u}<k>}$$

then it is evident that

$$g_1[\overline{x_1}\langle k\rangle ... \overline{x_n}\langle k\rangle] \leq_k g_2[\overline{x_1}\langle k\rangle ... \overline{x_n}\langle k\rangle]$$

and

$$\overline{u}\langle k\rangle \leftharpoonup_k g_1[\overline{x_1}\langle k\rangle ... \overline{x_n}\langle k\rangle],$$

whence considering (c) and (15) we obtain

$$\overline{u}\langle k\rangle \leq_k t(g_1[\overline{x_1}\langle k\rangle ... \overline{x_n}\langle k\rangle]).$$

This means that (27) is true for $\mathcal{E}^1_{\overline{u}<k>}$.

Now, let

$$\text{(d)} \quad \overline{u}\langle k\rangle \leftharpoonup_k t(g_1[\overline{x_1}\langle k\rangle ... \overline{x_n}\langle k\rangle])$$

be valid for some $t \in T_k$.

Since $g_1 \leq_n g_2$ then as it has been stated above

$$t(g_1[\overline{x_1}\langle k\rangle ... \overline{x_n}\langle k\rangle]) \leq_k t(g_2[\overline{x_1}\langle k\rangle ... \overline{x_n}\langle k\rangle]),$$

therefore,

$$t(g_1[\overline{x_1}\langle k\rangle ... \overline{x_n}\langle k\rangle]) \leftharpoonup_k t(g_2[\overline{x_1}\langle k\rangle ... \overline{x_n}\langle k\rangle]),$$

hence,

$$\text{(e)} \quad \overline{u}\langle k\rangle \leftharpoonup_k t(g_2[\overline{x_1}\langle k\rangle ... \overline{x_n}\langle k\rangle]).$$

Conversely, let (c) be valid. From $g_1 \leq_n g_2$ we obtain

$$g_1[\overline{x_1}\langle k\rangle ... \overline{x_n}\langle k\rangle] \leq_k g_2[\overline{x_1}\langle k\rangle ... \overline{x_n}\langle k\rangle],$$

therefore, according to (14),

$$\overline{u}\langle k\rangle \leftharpoondown_k t(g_1[\overline{x_1}\langle k\rangle ... \overline{x_n}\langle k\rangle])$$

it will be true. Thus, (27) is true for $\mathcal{E}^2_{\overline{u}<k>}$. Thus, (25) is valid because, as it has been stated above, it is equivalent to $P^n_{\overline{u}}(g_1) \subset P^n_{\overline{u}}(g_2)$, where $\overline{u} \in \overline{G}$. Thus $P^n(g_1) \subset P^n(g_2)$, $n \in I$, $g_1, g_2 \in G_n$, which was needed to prove.

Finally, let us suppose that $P^n(g_1) = P^n(g_2)$ for some $g_1, g_2 \in G_n$. Then $P^n(g_1) \subset P^n(g_2)$ and $P^n(g_2) \subset P^n(g_1)$. Therefore $g_1 \leq_n g_2$ and $g_2 \leq_n g_1$, whence $g_1 = g_2$, since ζ_n is an order. So we have proved that the (weakly unitary $T-$) Menger system $\mathcal{G}$ is isomorphic to f.o.p. Menger ($T-$) system of multiplace functions. The theorem is proved. □

Corollary 1. *A (weakly unitary $T-$) Menger system of the form $(G_n, \mathcal{O}_n, \zeta_n)_{n\in I}$ (where $\zeta_n \subset G_n \times G_n$) is isomorphic to some f.o. ($T-$) Menger system of multiplace functions if and only if $(\zeta_n)_{n\in I}$ is a stable family of relations of order, satisfying the condition:*

$$g_1 \leq_n g_2 \ \wedge \ g \leq_n t_1(g_1) \ \wedge \ g \leq_n t_2(g_2) \ \Rightarrow \ g \leq_n t_2(g_1) \qquad (28)$$

for all $n \in I$, $g, g_1, g_2 \in G_n$, $t_1, t_2 \in T_n$.

Proof. Supposing $\chi_n = \delta_n \circ \zeta_n$, where $\delta_n \subset G_n \times G_n$ and

$$(g_1, g_2) \in \delta_n \Longleftrightarrow (\exists\, t \in T_n)\ g_1 = t(g_2)\,,$$

we come to the conclusion that the system $(G_n, \mathcal{O}_n, \zeta_n, \chi_n)_{n\in I}$ satisfies all the conditions of the theorem. □

Corollary 2. *A (weakly unitary $T-$) Menger system of the form $(G_n, \mathcal{O}_n, \chi_n)_{n\in I}$ (where $\chi_n \subset G_n \times G_n$) is isomorphic to some p.q-o. ($T-$) Menger system of multiplace functions if and only if $(\chi_n)_{n\in I}$ is a l-regular family of relations of quasi-order satisfying the condition* (12).

Proof. Supposing $\zeta_n = \Delta_{G_n}$ in the theorem, where Δ_{G_n} is the identical relation on G_n, we obtain the present corollary. □

We must remark that analogous results for semigroups [2] and for Menger algebras [4] may obtain from the proved theorem.

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