Quasigroups and related systems, **3** (1996), 55–72

On ternary semigroups of lattice homomorphisms

Antoni Chronowski

Abstract

The notion of a ternary semigroup of lattice homomorphisms is introduced. Some properties of the ternary semigroup homomorphisms of Boolean algebras are studied. Necessary and sufficient conditions for a certain characterization of lattices by means of ternary semigroups of lattice homomorphisms are given.

1. Introduction

In providing a setting for this paper, one notes that there exist many papers concerned with the study of the semigroups of endomorphisms of algebraic, ordered, topological structures (e.g. [1], [2]). In the present paper we introduce the notion of a ternary semigroup of lattice homomorphisms. This ternary semigroup is the counterpart of the semigroup of lattice endomorphisms. At the beginning of the paper we study some properties of the ternary semigroup homomorphisms of Boolean algebras. In the main theorem of this paper we give necessary and sufficient conditions for a certain characterization of lattices by means of ternary semigroups of lattice homomorphisms.

1991 Mathematics Subject Classification: 20N10, 06B05, 06E25
Keywords: ternary semigroup, lattice, lattice homomorphism

2. Basic definitions

Definition 2.1. (cf. [3]). A *ternary semigroup* is an algebraic structure (A, f) such that A is a nonempty set and $f : A^3 \to A$ is a ternary operation satisfying the following associative law:

$$f(f(x_1, x_2, x_3), x_4, x_5) = f(x_1, f(x_2, x_3, x_4), x_5) = f(x_1, x_2, f(x_3, x_4, x_5))$$

for all $x_1, x_2, x_3, x_4, x_5 \in A$.

Definition 2.2. (cf. [3]). A nonempty subset $I \subset A$ is called an *ideal* of a ternary semigroup (A, f) if $f(I, A, A) \subset I$, $f(A, I, A) \subset I$, $f(A, A, I) \subset I$.

Definition 2.3. An element $x_0 \in A$ is said to be a *left zero* of a ternary semigroup (A, f) if $f(x_0, x_1, x_2) = x_0$ for all $x_1, x_2 \in A$.

Throughout this paper the letter f will be reserved to denote the ternary operation in ternary semigroups.

Definition 2.4. A mapping $p : X \to Y$ is said to be a *lattice homomorphism* of lattices $(X, \vee, \wedge)$ and $(Y, \vee, \wedge)$ if

(i) $\quad p(x_1 \vee x_2) = p(x_1) \vee p(x_2)\,,$

(ii) $\quad p(x_1 \wedge x_2) = p(x_1) \wedge p(x_2)$

for all $x_1, x_2 \in X$. A one-to-one lattice homomorphism p is called a *lattice isomomorphism*.

Let $(X, \vee, \wedge)$ and $(Y, \vee, \wedge)$ be lattices. Let $H(X, Y)$ be the set of all lattice homomorphisms from the lattice $(X, \vee, \wedge)$ to the lattice $(Y, \vee, \wedge)$. Put

$$H[X, Y] = H(X, Y) \times H(Y, X).$$

Define the ternary operation $f : H[X, Y]^3 \to H[X, Y]$ by the rule:

$$f((p_1, q_1), (p_2, q_2), (p_3, q_3)) = (p_1 \circ q_2 \circ p_3\,, q_1 \circ p_2 \circ q_3)$$

for all (p_i, q_i) where $i = 1, 2, 3$. The algebraic structure $(H[X, Y], f)$ is a ternary semigroup.

Definition 2.5. The ternary semigroup $(H[X,Y],f)$ is called the *ternary semigroup homomorphisms of the lattices* X *and* Y.

3. Some properties of the ternary semigroup of lattice homomorphisms

Consider the Boolean algebras $(X,\vee,\wedge,',0,1)$ and $(Y,\vee,\wedge,',0,1)$. Let $H[X,Y]$ be the ternary semigroup of lattice homomorphisms of Boolean algebras X and Y. Put $P[X,Y]=X\times Y$. Define the ternary operation

$$f\,:\,(P[X,Y]\times H[X,Y])^3\to P[X,Y]\times H[X,Y]$$

by the rule:

$$\begin{aligned}&f(((x_1,y_1),(p_1,q_1)),((x_2,y_2),(p_2,q_2)),((x_3,y_3),(p_3,q_3)))=\\&\qquad((x_1\vee q_1(y_2)\vee q_1(p_2(x_3)),y_1\vee p_1(x_2)\vee p_1(q_2(y_3)),\\&\qquad\qquad(p_1\circ q_2\circ p_3,q_1\circ p_2\circ q_3))\end{aligned}$$

for all $((x_i,y_i),(p_i,q_i))\in P[X,Y]\times H[X,Y]$, where $i=1,2,3$.

Denote the obtained algebraic structure $(P[X,Y]\times H[X,Y],f)$ by $P[X,Y]\otimes H[X,Y]$. We will prove that $P[X,Y]\otimes H[X,Y]$ is a ternary semigroup. Assume that $((x_i,y_i),(p_i,q_i))\in P[X,Y]\otimes H[X,Y]$ for $i=1,...,5$. We have:

$$\begin{aligned}&f(f(((x_1,y_1),(p_1,q_1)),((x_2,y_2),(p_2,q_2)),((x_3,y_3),(p_3,q_3))),\\&\qquad((x_4,y_4),(p_4,q_4)),((x_5,y_5),(p_5,q_5)))=\\&f(((x_1\vee q_1(y_2)\vee q_1(p_2(x_3)),y_1\vee p_1(x_2)\vee p_1(q_2(y_3)),\\&\qquad(p_1\circ q_2\circ p_3,q_1\circ p_2\circ q_3)),((x_4,y_4),(p_4,q_4)),((x_5,y_5),(p_5,q_5)))=\\&((x_1\vee q_1(y_2)\vee q_1(p_2(x_3))\vee q_1(p_2(q_3(y_4)))\vee q_1(p_2(q_3(p_4(x_5)))),\\&\qquad y_1\vee p_1(x_2)\vee p_1(q_2(y_3))\vee p_1(q_2(p_3(x_4)))\vee p_1(q_2(p_3(q_4(y_5))))),\\&\qquad\qquad(p_1\circ q_2\circ p_3\circ q_4\circ p_5,q_1\circ p_2\circ q_3\circ p_4\circ q_5)),\end{aligned}$$

and on the other hand

$$f(((x_1,y_1),(p_1,q_1)),f(((x_2,y_2),(p_2,q_2)),((x_3,y_3),(p_3,q_3)),$$
$$((x_4,y_4),(p_4,q_4))),((x_5,y_5),(p_5,q_5))) =$$

$$f(((x_1,y_1),(p_1,q_1)),((x_2 \vee q_2(y_3) \vee q_2(p_3(x_4)),y_2 \vee p_2(x_3) \vee p_2(q_3(y_4))),$$
$$(p_2 \circ q_3 \circ p_4, q_2 \circ p_3 \circ q_4)),((x_5,y_5),(p_5,q_5))) =$$

$$(x_1 \vee q_1(y_2 \vee p_2(x_3) \vee p_2(q_3(y_4))) \vee q_1(p_2(q_3(p_4(x_5)))),$$
$$y_1 \vee p_1(x_2 \vee q_2(y_3) \vee q_2(p_3(x_4))) \vee p_1(q_2(p_3(q_4(y_5))))),$$
$$(p_1 \circ q_2 \circ p_3 \circ q_4 \circ p_5, q_1 \circ p_2 \circ q_3 \circ p_4 \circ q_5)) =$$

$$= ((x_1 \vee q_1(y_2) \vee q_1(p_2(x_3)) \vee q_1(p_2(q_3(y_4))) \vee q_1(p_2(q_3(p_4(x_5)))),$$
$$y_1 \vee p_1(x_2) \vee p_1(q_2(y_3)) \vee p_1(q_2(p_3(q_4(y_5))))),$$
$$(p_1 \circ q_2 \circ p_3 \circ q_4 \circ p_5, q_1 \circ p_2 \circ q_3 \circ p_4 \circ q_5)).$$

Similarly

$$f(((x_1,y_1),(p_1,q_1)),((x_2,y_2),(p_2,q_2)),$$
$$f((x_3,y_3),(p_3,q_3)),((x_4,y_4),(p_4,q_4)),((x_5,y_5),(p_5,q_5)))) =$$

$$f(((x_1,y_1),(p_1,q_1)),((x_2,y_2),(p_2,q_2)),((x_3 \vee q_3(y_4) \vee q_3(p_4(x_5)),$$
$$y_3 \vee p_3(x_4) \vee p_3(q_4(y_5))),(p_3 \circ q_4 \circ p_5, q_3 \circ p_4 \circ q_5))) =$$

$$((x_1 \vee q_1(y_2) \vee q_1(p_2(x_3 \vee q_3(y_4) \vee q_3(p_4(x_5)))),$$
$$y_1 \vee p_1(x_2) \vee p_1(q_2(y_3 \vee p_3(x_4) \vee p_3(q_4(y_5))))),$$
$$(p_1 \circ q_2 \circ p_3 \circ q_4 \circ p_5, q_1 \circ p_2 \circ q_3 \circ p_4 \circ q_5)) =$$

$$(x_1 \vee q_1(y_2) \vee q_1(p_2(x_3)) \vee q_1(p_2(q_3(y_4))) \vee q_1(p_2(q_3(p_4(x_5)))),$$
$$y_1 \vee p_1(x_2) \vee p_1(q_2y_3)) \vee p_1(q_2(p_3(x_4))) \vee p_1(q_2(p_3(q_4(y_5))))),$$
$$(p_1 \circ q_2 \circ p_3 \circ q_4 \circ p_5, q_1 \circ p_2 \circ q_3 \circ p_4 \circ q_5)).$$

This proves that the algebraic structure $P[X,Y] \otimes H[X,Y]$ is a ternary semigroup.

Consider the sets

$$H_0(X,Y) = \{p \in H(X,Y) : p(0) = 0\}$$

and

$$H_0(Y,X) = \{q \in H(Y,X) : q(0) = 0\}.$$

Put

$$H_0[X,Y] = H_0(X,Y) \times H_0(Y,X).$$

It is easy to notice that $P[X,Y] \otimes H_0[X,Y]$ is a ternary subsemigroup of the ternary semigroup $P[X,Y] \otimes H[X,Y]$. Assume that $p \in H(X,Y)$. Set $z_p = p(0)$. Define the mapping $g_p : X \to Y$ by the rule:

$$g_p(x) = p(x) \wedge z'_p$$

for every $x \in X$. It is easy to check that $g_p \in H_0(X,Y)$. Assume that $q \in H(Y,X)$. Similarly, $g_q \in H_0(Y,X)$. Define the mapping

$$F : H[X,Y] \to P[X,Y] \otimes H_0[X,Y]$$

by the rule:

$$F(p,q) = ((z_q, z_p), (g_p, g_q))$$

for every pair $(p,q) \in H[X,Y]$. Define the mapping

$$G : P[X,Y] \otimes H_0[X,Y] \to H[X,Y]$$

by the formula:

$$G((x,y),(p,q)) = (\overline{p}, \overline{q})$$

for every pair $((x,y),(p,q)) \in P[X,Y] \otimes H_0[X,Y]$, where

$$\overline{p}(x_1) = p(x_1) \vee y, \quad x_1 \in X,$$
$$\overline{q}(y_1) = q(y_1) \vee x, \quad y_1 \in Y.$$

Clearly $\overline{p} \in H(X,Y)$ and $\overline{q} \in H(Y,X)$. We will prove that G is an epimorphism of the ternary semigroup $P[X,Y] \otimes H_0[X,Y]$ onto the ternary semigroup $H[X,Y]$. Assume that

$$((x_i, y_i), (p_i, q_i)) \in P[X,Y] \otimes H_0[X,Y]$$

for $i = 1,2,3$. Therefore,

$$f(((x_1,y_1),(p_1,q_1)),((x_2,y_2),(p_2,q_2)),((x_3,y_3),(p_3,q_3))) =$$
$$= (x_1 \vee q_1(y_2) \vee q_1(p_2(x_3)), y_i \vee p_1(x_2) \vee p_1(q_2(y_3))),$$
$$(p_1 \circ q_2 \circ p_3, q_1 \circ p_2 \circ q_3)) \in P[X,Y] \otimes H_0[X,Y].$$

Note that

$$(\overline{p_1} \circ \overline{q_2} \circ \overline{p_3})(x) = \overline{p_1}(\overline{q_2}(p_3(x) \vee y_3) \vee x_2) =$$
$$= p_1(q_2(p_3(x))) \vee p_1(q_2(y_3)) \vee p_1(x_2) \vee y_1 =$$
$$= (p_1 \circ q_2 \circ p_3)(x) \vee (y_1 \vee p_1(x_2) \vee p_1(q_2(y_3))) = (\overline{p_1 \circ q_2 \circ p_3})(x)$$

for every $x \in X$. Thus

$$\overline{p_1 \circ q_2 \circ p_3} = \overline{p_1} \circ \overline{q_2} \circ \overline{p_3}.$$

Similarly,

$$\overline{q_1 \circ p_2 \circ q_3} = \overline{q_1} \circ \overline{p_2} \circ \overline{q_3}\,.$$

Therefore,

$$G(f(((x_1,y_1),(p_1,q_1)),((x_2,y_2),(p_2,q_2)),((x_3,y_3),(p_3,q_3)))) =$$

$$= G((x_1 \vee q_1(y_2) \vee q_1(p_2(x_3)), y_1 \vee p_1(x_2) \vee p_1(q_2(y_3))),$$

$$(p_1 \circ q_2 \circ p_3, q_1 \circ p_2 \circ q_3)) = (\overline{p_1 \circ q_2 \circ p_3}, \overline{q_1 \circ p_2 \circ q_3}) =$$

$$= (\overline{p_1} \circ \overline{q_2} \circ \overline{p_3}, \overline{q_1} \circ \overline{p_2} \circ \overline{q_3}) = f((\overline{p_1},\overline{q_1}),(\overline{p_2},\overline{q_2}),(\overline{p_3},\overline{q_3})) =$$

$$= f(G((x_1,y_1),(p_1,q_1)), G((x_2,y_2),(p_2,q_2)), G((x_3,y_3),(p_3,q_3)))\,.$$

Assume that $p \in H(X,Y)$. Thus

$$p(x) = p(x \vee 0) = p(x) \vee z_p$$

for every $x \in X$. Notice that

$$g_p(x) \vee z_p = (p(x) \vee z'_p) \vee z_p =$$
$$(p(x) \vee z_p) \wedge (z'_p \vee z_p) = p(x) \vee z_p = p(x)$$

for every $x \in X$. Hence $p(x) = g_p(x) \vee z_p$ for every $x \in X$. Similarly, if $q \in H(Y,X)$ then $q(y) = g_q(y) \vee z_q$ for every $y \in Y$.

We will show that

$$G \circ F = id_{H[X,Y]}.$$

Indeed,

$$(G \circ F)(p,q) = G((z_q,z_p),(g_p,g_q)) = (\overline{g_p},\overline{g_q})$$

for every pair $(p,q) \in H[X,Y]$. Notice that

$$\overline{g}_p(x) = g_p(x) \vee z_p = p(x)$$

for $x \in X$,

$$\overline{g}_q(y) = g_q(y) \vee z_q = q(y)$$

for $y \in Y$. Hence $(G \circ F)(p,q) = (p,q)$ for every $(p,q) \in H[X,Y]$. Therefore, F is an injection and G is an epimorphism.

We define the mapping

$$\varphi : P[X,Y] \otimes H_0[X,Y]/Ker\,G \to H[X,Y]$$

by the rule:

$$\varphi([((x,y),(p,q))]_{Ker\,G}) = G((x,y),(p,q))$$

for every

$$[((x,y),(p,q))]_{Ker\,G} \in P[X,Y] \otimes H_0[X,Y]/Ker\,G\,.$$

Of course, φ is an isomorphism of the ternary semigroups $P[X,Y]\otimes H_0[X,Y]/Ker\,G$ and $H[X,Y]$. Let

$$k\,:\,P[X,Y]\otimes H_0[X,Y]\to P[X,Y]\otimes H_0[X,Y]/Ker\,G$$

be the canonical epimorphism. We will show that $k\circ F=\varphi^{-1}$. Assume that $(p,q)\in H[X,Y]$. We have

$$(k\circ F)(p,q)=[F(p,q)]_{Ker\,G}.$$

We know that $(p,q)=G((x,y),(p_0,q_0))$ for a pair

$$((x,y),(p_0,q_0))\in P[X,Y]\otimes H_0[X,Y]\,.$$

Obviously, $G(F(p,q))=(p,q)$. Hence

$$(((x,y)\,,(p_0,q_0))\,,F(p,q))\in Ker\,G\,.$$

Therefore,

$$\varphi^{-1}(p,q)=\varphi^{-1}(G((x,y)\,,(p_0,q_0)))=[((x,y)\,,(p_0,q_0))]_{Ker\,G}=$$
$$=[F(p,q)]_{Ker\,G}=(k\circ F)(p,q)\,.$$

We have obtained the following

Theorem 3.1. *For arbitrary Boolean algebras X and Y the ternary semigroups $H[X,Y]$ and $P[X,Y]\otimes H_0[X,Y]/Ker\,G$ are isomorphic.*

Theorem 3.1. provides a certain characterization of the ternary semigroup $H[X,Y]$ of all lattice homomorphisms of Boolean algebras X and Y by means of all lattice homomorphisms of X and Y by means of all lattice homomorphisms of X and Y which preserve zero elements.

Consider the further properties of the mappings F and G. Assume that $(p_i,q_i)\in H[X,Y]$ for $i=1,2,3$. Notice that

$$G(F(f((p_1,q_1),(p_2,q_2),(p_3,q_3))))=f((p_1,q_1),(p_2,q_2),(p_3,q_3))$$

and

$$G(f(F(p_1,q_1)\,,F(p_2,q_2)\,,F(p_3,q_3)))=$$
$$=f(G(F(p_1,q_1))\,,G(F(p_2,q_2))\,,G(F(p_3,q_3)))=$$
$$=f((p_1,q_1)\,,(p_2,q_2)\,,(p_3,q_3)).$$

Therefore,

$$(F(f((p_1,q_1),(p_2,q_2),(p_3,q_3)),f(F(p_1,q_1),F(p_2,q_2),F(p_3,q_3))))$$

belongs to $Ker\,G$. We may say that

$$F : H[X,Y] \to P[X,Y] \otimes H_0[X,Y]$$

is a monomorphism modulo $Ker\,G$ from the ternary semigroup $H[X,Y]$ into $P[X,Y] \otimes H_0[X,Y]$.

For all $(p_1,q_1),(p_2,q_2) \in H[X,Y]$ if

$$(F(p_1,q_1),F(p_2,q_2)) \in Ker\,G\,,$$

then

$$G(F(p_1,q_1)) = G(F(p_2,q_2))\,,$$

and so $(p_1,q_1) = (p_2,q_2)$. Assume that

$$[((x,y),(p_0,q_0))]_{Ker\,G} \in P[X,Y] \otimes H_0[X,Y]/Ker\,G$$

is an arbitrary equivalence class. Notice that

$$G(F(G((x,y),(p_0,q_0)))) = G((x,y),(p_0,q_0))\,.$$

Hence

$$F(G((x,y),(p_0,q_0))) \in [((x,y),(p_0,q_0))]_{Ker\,G}\,.$$

Thus, each equivalence class from the set $P[X,Y] \otimes H_0[X,Y]/KerG$ has exactly one element of the set $F(H[X,Y])$.

We have obtained the following

Proposition 3.1. *The set $F(H[X,Y])$ is a selector of the family of equivalence classes $P[X,Y] \otimes H_0[X,Y]/Ker\,G$.* □

4. Main result

Let $(X,\vee,\wedge)$ and $(Y,\vee,\wedge)$ be lattices. In the sequel lattice homomorphisms (isomomorphisms) will often be referred to as homomorphisms (isomomorphisms). Let $H[X,Y]$ be a ternary semigroup of homomorphisms of lattices X and Y.

Consider the following sets:

$$H_c(X,Y) = \{p \in H(X,Y) : \exists y_0 \in Y\ \forall x \in X\ p(x) = y_0\}$$
$$H_c(Y,X) = \{q \in H(Y,X) : \exists x_0 \in X\ \forall y \in Y\ p(y) = x_0\}$$

The such homomorphisms $p \in H_c(X,Y)$ and $q \in H_c(Y,X)$ that their single values are $y_0 \in Y$ and $x_0 \in X$ we denote by p_{y_0} and q_{x_0}, respectively. Put

$$H_c[X,Y] = H_c(X,Y) \times H_c(Y,X)\,.$$

It is easy to notice that $H_c[X,Y]$ is a ternary subsemigroup of $H[X,Y]$.

Define two binary operations $\vee$ and $\wedge$ in the set $H_c(X,Y)$ by the rules:

$$p_{y_1} \vee p_{y_2} = p_y \Longleftrightarrow y_1 \vee y_2 = y\,,$$
$$p_{y_1} \wedge p_{y_2} = p_y \Longleftrightarrow y_1 \wedge y_2 = y$$

for $p_{y_1}, p_{y_2}, p_y \in H_c(X,Y)$.

Similarly, define two binary operations $\vee$ and $\wedge$ in the set $H_c(Y,X)$ by the rules:

$$q_{x_1} \vee q_{x_2} = q_x \Longleftrightarrow x_1 \vee x_2 = x,$$
$$q_{x_1} \wedge q_{x_2} = q_x \Longleftrightarrow x_1 \wedge x_2 = x$$

for $q_{x_1}, q_{x_2}, q_x \in H_c(Y,X)$.

Notice that $(H_c(X,Y), \vee, \wedge)$ and $(H_c(Y,X), \vee, \wedge)$ are lattices.

Lemma 4.1. *Let X and Y be lattices. A pair of homomorphisms (p,q) is a left zero of the ternary semigroup $H[X,Y]$ if and only if $(p,q) \in H_c[X,Y]$.*

Proof. Let (p,q) be a left zero of $H[X,Y]$. By Definition 2.3 we have

$$f((p,q)\,,(p_1,q_1)\,,(p_2,q_2)) = (p,q)$$

for all $(p_1,q_1)\,,(p_2,q_2) \in H[X,Y]$. Put $(p_1,q_1) = (p_{y_0}, q_{x_0})$ for some $x_0 \in X$, $y_0 \in Y$. Hence

$$f((p,q)\,,(p_{y_0},q_{x_0})\,,(p_2,q_2)) = (p,q)$$

and $p = p \circ q_{x_0} \circ p_2$, $q = q \circ p_{y_0} \circ q_2$. Therefore,

$$\forall x \in X \; p(x) = p(x_0)$$

and

$$\forall y \in Y \; q(y) = q(y_0),$$

and so $(p,q) \in H_c[X,Y]$.

Conversely, suppose that $(p,q) \in H_c[X,Y]$. Consequently $p = p_{y_0}$ and $q = q_{x_0}$ for some $x_0 \in X$, $y_0 \in Y$. For any $(p_1,q_1)\,,(p_2,q_2) \in H[X,Y]$ we have:

$$f((p,q),(p_1,q_1),(p_2,q_2)) = f((p_{y_0},p_{x_0}),(p_1,q_1),(p_2,q_2)) =$$
$$= (p_{y_0} \circ q_1 \circ p_2 , q_{x_0} \circ p_1 \circ q_2) = (p_{y_0}, q_{x_0}) = (p,q).$$

Therefore, the pair (p,q) is a left zero of $H[X,Y]$.

Proposition 4.1. *The set $H_c[X,Y]$ is the smallest ideal of the ternary semigroup $H[X,Y]$.*

Proof. It is easy to check that $H_c[X,Y]$ is an ideal of $H[X,Y]$. Put $I_c = H_c[X,Y]$. Let $I \subset H[X,Y]$ be an ideal of $H[X,Y]$. By Lemma 4.1 $f(I_c,I,I) = I_c$. On the other hand, $f(I_c,I,I) \subset I$. Hence $I_c \subset I$, which completes our proof. □

Lemma 4.2. *Let X_i and Y_i $(i = 1,2)$ be lattices. Let*
$$F : H[X_1,Y_1] \to H[X_2,Y_2]$$
be an epimorphism of the ternary semigroups $H[X_1,Y_1]$ and $H[X_2,Y_2]$. Then
$$F(H_c[X_1,Y_1]) = H_c[X_2,Y_2].$$

Proof. Suppose that $(p,q) \in H_c[X_1,Y_1]$. By Lemma 4.1 we have
$$f((p,q),(p_1,q_1),(p_2,q_2)) = (p,q)$$
for all $(p_1,q_1),(p_2,q_2) \in H[X_1,Y_1]$. Therefore,
$$f(F(p,q),F(p_1,q_1),F(p_2,q_2)) = F(p,q)$$
for all $(p_1,q_1),(p_2,q_2) \in H[X_1,Y_1]$. Again by Lemma 4.1 $F(p,q) \in H_c[X_2,Y_2]$.

Conversely, suppose that $(r,s) \in H_c[X_2,Y_2]$. This implies that $r = r_{y_2}$ and $s = s_{x_2}$ for some $x_2 \in X_2$, $y_2 \in Y_2$. There exists a such pair $(p',q') \in H[X_1,Y_1]$ that $F(p',q') = (r_{y_2}, s_{x_2})$. Assume that $(p'_{y_1}, q'_{x_1}) \in H_c[X_1,Y_1]$ is an arbitrary fixed pair and $(p_1,q_1) \in H[X_1,Y_1]$. Put
$$(p,q) = f((p',q'),(p'_{y_1},q'_{x_1}),(p_1,q_1)).$$
Hence $p = p' \circ q_{x_i} \circ p_1$ and $q = q' \circ p_{y'_1} \circ q_1$. Set $y_1 = p'(x'_1)$ and $x_1 = q'(y'_1)$. Thus $p = p_{y_1}$ and $q = q_{x_1}$, hence $(p,q) \in H_c[X_1,Y_1]$. We have

$$F(p,q) = f(F(p',q'), F(p_{y_1'}, q_{x_1'}), F(p_1,q_1)) =$$
$$= f((r_{y_2}, s_{x_2}), F(p_{y_1'}, q_{x_1})) = (r_{y_2}, s_{x_2}) = (r,s).$$

Therefore, there exists a such pair $(p,q) \in H_c[X_1,Y_1]$ that $F(p,q) = (r,s)$.

Notice that a mapping

$$F_0 : H_c[X_1,Y_1] \to H_c[X_2,Y_2]$$

is an isomorphism of the ternary semigroups $H_c[X_1,Y_1]$ and $H_c[X_2,Y_2]$ if and only if F_0 is a bijection.

Let X_i and Y_i $(i = 1,2)$ be lattices. Suppose that $f_1 : X_1 \to X_2$ and $f_2 : Y_1 \to Y_2$ are lattice isomorphism. Define the mapping

$$F : H[X_1,Y_1] \to H[X_2,Y_2]$$

by the rule:

$$F(p,q) = (f_2 \circ p \circ f_1^{-1}, f_1 \circ q \circ f_2^{-1}) \tag{1}$$

for every $(p,q) \in H[X_1,Y_1]$. It is easy to check that F is an isomorphism of the ternary semigroups $H[X_1,Y_1]$ and $H[X_2,Y_2]$.

Definition 4.1. The mapping F defined by the formula (1) is called the isomorphism of the ternary semigroups $H[X_1,Y_1]$ and $H[X_2,Y_2]$ induced by the pair of lattice isomorphisms (f_1, f_2).

An isomorphism

$$F : H[X_1,Y_1] \to H[X_2,Y_2]$$

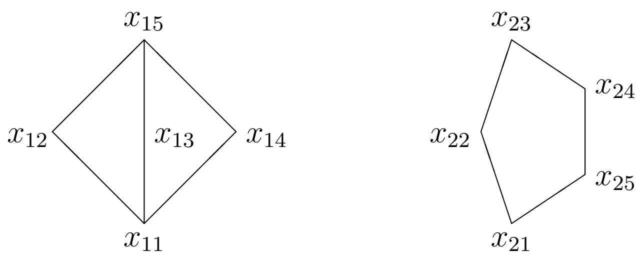
of the ternary semigroups $H[X_1,Y_1]$ and $H[X_2,Y_2]$ need not imply the existence of isomorphisms $f_1 : X_1 \to X_2$ and $f_2 : Y_1 \to Y_2$ of the lattices X_1, X_2, Y_1, Y_2.

The following example illustrates the above statement.

Example. Consider the following sets:

$$X_1 = \{x_{11}, x_{12}, ..., x_{15}\}, \quad Y_1 = \{y_1\},$$
$$X_2 = \{x_{21}, x_{22}, ..., x_{25}\}, \quad Y_2 = \{y_2\}.$$

Assume that Y_1 and Y_2 are trivially ordered sets. Define the partial orders in the sets X_1 and X_2 by the following diagrams:



The sets X_1, X_2, Y_1, Y_2 equipped with the foregoing orders are lattices. Thus we have:

$$H(X_1, Y_1) = \{p_{y_1}\}\,, \qquad H(Y_1, X_1) = \{q_{x_{11}}, q_{x_{15}}\}\,,$$
$$H(X_2, Y_2) = \{p_{y_2}\}\,, \qquad H(Y_2, X_2) = \{q_{x_{21}}, q_{x_{25}}\}\,,$$

Hence

$$H[X_1, Y_1] = \{(p_{y_1}, q_{x_{11}})\,,\ \ldots,\ (p_{y_1}, q_{x_{15}})\}\,,$$
$$H[X_2, Y_2] = \{(p_{y_2}, q_{x_{21}})\,,\ \ldots,\ (p_{y_2}, q_{x_{25}})\}\,,$$

Therefore,

$$H[X_1, Y_1] = H_c[X_1, Y_1]$$

and

$$H[X_2, Y_2] = H_c[X_2, Y_2]$$

Define the mapping

$$F : H[X_1, Y_1] \to H[X_2, Y_2]$$

by the formula

$$F(p_{y_1}, q_{x_{11}}) = (p_{y_2}, q_{x_{21}})\,,\ \ldots,\ F(p_{y_1}, q_{x_{15}}) = (p_{y_2}, q_{x_{25}})\,.$$

The mapping F is an isomorphism of the ternary semigroups $H[X_1, Y_1]$ and $H[X_2, Y_2]$. However, the lattices X_1 and X_2 are not isomorphic.

Let X_i and Y_i $(i = 1, 2)$ be lattices. Let

$$F : H[X_{1,Y_1}] \to H[X_2, Y_2]$$

be an isomorphism of the ternary semigroup $[X_1, Y_1]$ and $H[X_2, Y_2]$ induced by a pair of lattice isomorphisms (f_1, f_2). Assume that $p_{y_1}, p_{y'_1} \in H_c(X_1, Y_1)$ and $q, q' \in H_c(Y_1, X_1)$. We have

$$F(p_{y_i}, q) = (f_2 \circ p_{y_1} \circ f_1^{-1}\,, f_1 \circ q \circ f_2^{-1})\,,$$
$$F(p_{y'_i}, q') = (f_2 \circ p_{y'_1} \circ f_1^{-1}\,, f_1 \circ q' \circ f_2^{-1})\,,$$

Notice that $f_2 \circ p_{y_1} \circ f_1^{-1} = r_{f_2(y_1)}$ and $f_2 \circ p_{y'_1} \circ f_1^{-1} = r_{f_2(y'_1)}$, it means that

$$r_{f_2(y_1)}, r_{f_2(y_1')} \in H_c(X_2, Y_2).$$

If $p_{y_1} \leq p_{y_1'}$, then $y_1 \leq y_1'$. Since $f_2(y_1) \leq f_2(y_1')$, it follows that $r_{f_2(y_1)} \leq r_{f_2(y_1')}$. Conversely, suppose that $r_{f_2(y_1)} \leq r_{f_2(y_1')}$. Hence $f_2(y_1) \leq f_2(y_1')$, and so $y_1 \leq y_1'$. This means that $p_{y_1} \leq p_{y_1'}$. Let us denote by π_1 and π_2 the projections of Cartesian product. From the foregoing we have obtained the following condition:

$$\begin{gathered}\forall p, p' \in H_c(X_1, Y_1)\ \forall q, q' \in H_c(Y_1, X_1)\\ [p \leq p' \Leftrightarrow \pi_1(F(p,q)) \leq \pi_1(F(p',q'))]\end{gathered} \qquad (W_1)$$

A similar argument yields the following condition:

$$\begin{gathered}\forall p, p' \in H_c(X_1, Y_1)\ \forall q, q' \in H_c(Y_1, X_1)\\ [q \leq q' \Leftrightarrow \pi_2(F(p,q)) \leq \pi_2(F(p',q'))]\end{gathered} \qquad (W_2)$$

Notice that the isomorphism

$$F : H[X_1, Y_1] \to H[X_2, Y_2]$$

defined in the previous example does not satisfy the condition (W_2).

Theorem 4.1. *Let X_i and Y_i $(i = 1, 2)$ be lattices. An isomorphism*

$$F : H[X_1, Y_1] \to H[X_2, Y_2]$$

of the ternary semigroups $H[X_1, Y_1]$ and $H[X_2, Y_2]$ is induced by a pair of lattice isomorphisms (f_1, f_2) if and only if the isomorphism F satisfies the conditions (W_1) and (W_2).

Proof. We have proved that the isomorphism F induced by the pair of lattice isomorphisms (f_1, f_2) satisfies the conditions (W_1) and (W_2).

Let us assume that

$$F : H[X_1, Y_1] \to H[X_2, Y_2]$$

is an isomorphism of the ternary semigroups $H[X_1, Y_1]$ and $H[X_2, Y_2]$ such that the conditions (W_1) and (W_2) are satisfied.

In view of Lemma 4.2 we can define the mapping

$$F^* : X_1 \times Y_1 \to X_2 \times Y_2$$

by the formula:

$$F^*(x_1, y_1) = (x_2, y_2) \iff F(p_{y_1}, q_{x_1}) = (r_{y_1}, s_{x_2}) \tag{2}$$

for $(x_1, y_1) \in X_1 \times Y_1$ and $(x_2, y_2) \in X_2 \times Y_2$. It is easy to notice that F^* is a bijection. Let $y_0 \in Y_1$ be an arbitrary fixed element. We define the mapping $f_1 : X_1 \to X_2$ by the rule:

$$f_1(x_1) = x_2 \iff \pi_1(F^*(x_1, y_0)) = x_2 \tag{3}$$

for $x_1 \in X$, $x_2 \in X_2$.

We will prove that

$$f_1(x_1) = x_2 \iff \forall y_1 \in Y_1 \;\; \pi_1(F^*(x_1, y_1)) = x_2$$

for $x_1 \in X_1$, $x_2 \in X_2$.

Suppose that $F^*(x_1, y_0)$ and $F^*(x_1, y_1) = (x_2', y_2')$ for an arbitrary fixed element $y_1 \in Y_1$. Thus we have

$$F^*(x_1, y_0) = (x_2, y_2) \iff F(p_{y_0}, q_{x_1}) = (r_{y_2}, s_{x_2}),$$
$$F^*(x_1, y_1) = (x_2', y_2') \iff F(p_{y_1}, q_{x_1}) = (r_{y_2'}, s_{x_2'}).$$

In view of the condition (W_2) we infer that $s_{x_2} = s_{x_2'}$, and so $x_2 = x_2'$. Therefore,

$$f_1(x_1) = x_2 \iff \forall y_1 \in Y_1 \;\; \pi_1(F^*(x_1, y_1)) = x_2 \tag{4}$$

for $x_1 \in X_1$, $x_2 \in X_2$. It is easy to verify that

$$f_1(x_1) = x_2 \iff \exists y_1 \in Y_1 \;\; \pi_1(F^*(x_1, y_1)) = x_2 \tag{5}$$

for $x_1 \in X_1$, $x_2 \in X_2$.

Next we will prove that $f_1 : X_1 \to X_2$ is a bijection. Suppose that $x_2 \in X_2$. Let us take an arbitrary fixed element $y_2 \in Y_2$. Thus there exists a such pair $(x_1, y_1) \in X_1 \times Y_1$ that $F^*(x_1, y_1) = (x_2, y_2)$. Therefore using the condition (5) we obtain $f_1(x_1) = x_2$, and so f_1 is a surjection. Suppose that $f_1(x_1) = f_1(x_1')$ for x_1, $x_1' \in X_1$. Hence $f_1(x_1) = x_2$ and $f_1(x_1') = x_2$ for some $x_2 \in X_2$. By (3) it follows that $F^*(x_1, y_0) = (x_2, y_2)$ and $F^*(x_1', y_0) = (x_2, y_2')$ for some $y_2, y_2' \in Y_2$. Hence we have

$$F(p_{y_0}, q_{x_1}) = (r_{y_2}, s_{x_2})$$

and

$$F(p_{y_0}, q_{x_1'}) = (r_{y_2'}, s_{x_2})\,.$$

Using the condition (W_2) we get $q_{x_1} = q_{x_1'}$, and so $x_1 = x_1'$. Therefore f_1 is an injection. We will prove that

$$\forall x_1, x_1' \in X_1 [x_1 \leq x_1' \iff f(x_1) \leq f(x_1')] \tag{6}$$

Suppose that $x_1 \leq x_1'$ for $x_1, x_1' \in X_1$. Set $f_1(x_1) = x_2$ and $f_1(x_1') = x_2'$ where $x_2, x_2' \in X_2$. Hence

$$f_1(x_1) = x_2 \iff \pi_1(F^*(x_1, y_0)) = x_2,$$
$$f_1(x_1') = x_2' \iff \pi_1(F^*(x_1, y_0)) = x_2'.$$

Thus $F^*(x_1, y_0) = (x_2, y_2)$, and $F^*(x_1', y_0) = (x_2', y_2')$ for some $y_2, y_2' \in Y_2$. We have

$$F^*(x_1, y_0) = (x_2, y_2) \iff F(p_{y_0}, q_{x_1}) = (r_{y_2}, s_{x_2})\,,$$
$$F^*(x_1', y_0) = (x_2', y_2') \iff F(p_{y_0}, q_{x_1'}) = (r_{y_2'}, s_{x_2'})\,.$$

Since $q_{x_1} \leq q_{x_1'}$, the condition (W_2) yields $s_{x_2} \leq s_{x_2'}$, that is $x_2 \leq x_2'$, and so $f_1(x_1) \leq f_1(x_1')$. It is easy to notice that $f_1(x_1) \leq f_1(x_1')$ implies $x_1 \leq x_1'$ for all $x_1, x_2 \in X_1$. Therefore we have proved the condition (6).

Summarizing, the mapping $f_1 : X_1 \to X_2$ is an isomorphism of the lattices X_1 and X_2.

Let $x_0 \in X_1$ be an arbitrary fixed element. We define the mapping $f_2 : Y_1 \to Y_2$ by the formula:

$$f_2(y_1) = y_2 \iff \pi_2(F^*(x_0, y_1)) = y_2 \tag{7}$$

for all $y_1 \in Y_1$ and $y_2 \in Y_2$.

The analogous argument applied to the mapping f_2 allows to prove that

$$f_2(y_1) = y_2 \iff \forall x_1 \in X_1\ \pi_2(F^*(x_1, y_1)) = y_2\,, \tag{8}$$

$$f_2(y_1) = y_2 \iff \exists x_1 \in X_1\ \pi_2(F^*(x_1, y_1)) = y_2\,, \tag{9}$$

for all $y_1 \in Y_1$ and $y_2 \in Y_2$. We can similarly show that the mapping $f_2 : Y_1 \to Y_2$ is an isomorphism of the lattices Y_1 and Y_2. By the conditions (4) and (8) we get

$$F^*(x_1, y_1) = (\pi_1(F^*(x_1, y_1)\,, \pi_1(F^*(x_1, y_1)))) = (f_1(x_1), f_2(y_1))$$

for every $(x_1, y_1) \in X_1 \times Y_1$. Consequently,

$$F^* = (f_1, f_2)\,. \tag{10}$$

We will prove that the isomorphism F is induced by the pair of lattice isomorphisms (f_1, f_2). First, we will show that the following condition is satisfied

$$\forall x_1 \in X_1\ \forall y_1 \in Y_1\ \forall (p,q) \in H[X_1, Y_1]$$
$$F(p,q)(f_1(x_1), f_2(y_1)) = (f_2(p(x_1)), f_1(q(y_1)))\,. \tag{11}$$

Suppose that $x_1 \in X_1$, $y_1 \in Y_1$ and $(p,q), (p_1, q_1) \in H[X_1, Y_1]$. Hence

$$f((p,q)\,,\,(p_{y_1}, q_{x_1})\,,\,(p_{y_1}, q_{x_1})\,,\,(p_1,q_1)) =$$
$$= (p \circ q_{x_1} \circ p_1\,,\, q \circ p_{y_1} \circ q_1) = (p_{p(x_1)}\,,\, q_{q(y_1)})\,.$$

We have

$$F(p_{p(x_1)}\,,\, q_{q(y_1)}) =$$
$$= F(f((p,q)\,,\,(p_{y_1}, q_{x_1})\,,\,(p_1,q_1))) = f(F(p,q)\,,\,(p_{y_1}, q_{x_1})\,,\,(p_1,q_1)).$$

Set $F(p,q) = (r,s)$ and $F(p_1,q_1) = (r_1,s_1)$. By Lemma 4.2 we get $F(p_{y_1}, q_{x_1}) = (r_{y_2}, s_{x_2})$ for some $x_2 \in X_2$, $y_2 \in Y_2$. By (10)

$$F(p_{y_1}, q_{x_1}) = (r_{y_2}, s_{x_2}) \iff F^*(x_1, y_1) = (x_2, y_2) \iff$$
$$\iff (f_1(x_1), f_2(y_1)) = (x_2, y_2) \iff (x_2 = f_1(x_1) \wedge y_2 = f_2(y_1)).$$

Therefore,

$$F(p_{p(x_1)}\,,\, q_{q(y_1)}) = f((r,s)\,,\,(r_{f_2(y_1)}\,,\, s_{f_1(x_1)}), (r_1,s_1)) =$$
$$= (r \circ s_{f_1(x_1)} \circ r_1\,,\, s \circ r_{f_2(y_1)} \circ s_1) = (r_{r(f_1(x_1))}\,,\, s_{s(f_2(y_1))})\,.$$

On the other hand,

$$F(p_{p(x_1)}\,,\, q_{q(y_1)}) = (r_{y_1}, s_{x_2})$$

for some $x_2 \in X_2$, $y_2 \in Y_2$. By (10)

$$F(p_{p(x_1)}\,,\, q_{q(y_1)}) = (r_{y_2}, s_{x_2}) \iff F^*(q(y_1), p(x_1)) = (x_2, y_2) \iff$$
$$(f_1(q(y_1))\,,\, f_2(p(x_1))) = (x_2, y_2) \iff (x_2 = f_1(q(y_1)) \wedge y_2 = f_2(p(x_1)))\,.$$

Therefore,

$$F(p_{p(x_1)}\,,\, q_{q(y_1)}) = (r_{f_2(p(x_1))}\,,\, s_{f_1(q(y_1))})\,.$$

Consequently, $f(f_1(x_1)) = f_2(p(x_1))$ and $s(f_2(y_1)) = f_1(q(y_1))$. Thus,

$$F(p,q)(f_1(x_1),f_2(y_1)) = (r,s)(f_1(x_1),f_2(y_1)) =$$
$$= (r(f_1(x_1)),s(f_2(y_1))) = (f_2(p(x_1)),f_1(q(y_1)))\,.$$

Therefore, we have obtained the formula (11). For $x_2 \in X_2$ and $y_2 \in Y_2$ there exist such $x_1 \in X_1$ and $y_1 \in Y_1$ that $f_1(x_1) = x_2$ and $f_2(y_1) = y_2$. Hence $x_1 = f_1^{-1}(x_2)$ and $y_1 = f_2^{-1}(y_2)$. Using the formula (11) we obtain

$$F(p,q)(x_2,y_2) = ((f_2 \circ p \circ f_1^{-1})(x_2)\,,\,(f_1 \circ q \circ f_2^{-1})(y_2)) =$$
$$= (f_2 \circ p \circ f_1^{-1},\ f_1 \circ q \circ f_2^{-1})(x_2,y_2)$$

for any pair $(p,q) \in H[X_1,Y_1]$. Therefore,

$$F(p,q) = (f_2 \circ p \circ f_1^{-1}, f_1 \circ q \circ f_2^{-1})$$

for every $(p,q) \in H[X_1,Y_1]$. Finally, we conclude that the isomorphism F is induced by the pair of lattice isomorphisms (f_1,f_2) defined by the formulas (3) and (7). □

Definition 4.2. Let X_i and Y_i $(i = 1,2)$ be lattices. The ternary semigroups $H[X_1,Y_1]$ and $H[X_2,Y_2]$ are called *W-isomorphic* if there exists an isomorphism $F : H[X_1,Y_1] \to H[X_2,Y_2]$ of the ternary semigroups $H[X_1,Y_1]$ and $H[X_2,Y_2]$ fulfilling the conditions (W_1) and (W_2).

From Theorem 4.1 we deduce the following two corollaries.

Corollary 4.1. *Let X_i and Y_i $(i = 1,2)$ be lattices. The ternary semigroups $H[X_1,Y_1]$ and $H[X_2,Y_2]$ are W-isomorphic if and only if the lattices X_1 and X_2 are isomorphic and the lattices Y_1 and Y_2 are isomorphic.* □

Corollary 4.2. *Let X_i and Y_i $(i = 1,2)$ be lattices. Let*

$$G : H[X_1,Y_1] \to H[X_2,Y_2]$$

be an isomorphic of the ternary semigroups $H[X_1,Y_1]$ and $H[X_2,Y_2]$. The lattices X_1 and X_2 are isomorphic and the lattices Y_1 and Y_2 are isomorphic if and only if there exists a such automorphism μ of the ternary semigroup $H[X_1,Y_1]$ that the isomorphism $F = G \circ \mu$ satisfies the conditions (W_1) and (W_2). □

References

[1] **C. J. Maxon, P. Natarajan**: *Lattice endomorphisms of* 2^x, Czech. Math. J. **27**(102) (1977), 663 – 671.

[2] **B. M. Schein**: *Ordered sets, semilattices, distributive lattices and Boolean algebras with homomorphic endomorphism semigroups*, Fund. Math. **68** (1970), 31 – 49.

[3] **F. M. Sioson**: *Ideal theory in ternary semigroups*, Math. Japon. **10** (1965), 63 – 84.

Institute of Mathematics
Pedagogical University
ul. Podchorążych 2
30-084 Kraków
POLAND

Received September 20, 1994