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Monoquasigroups isotopic to groups

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Abstract

In this work the quasigroups isotopic to groups are considered. The necessary and sufficient conditions are found which the isotopy must satisfy so that the corresponding group isotope be: monogenic quasigroup, momoquasigroup.

1. Introduction

The algebraic systems generated by one element (monogenic systems) are the simplest in the lattice-theoretic sense in every class of algebraic systems (in some cases, such as quasigroups, semigroups, in order to be able always to talk about lattice, we need to consider the empty set as a subsystem). These systems are contained as subsystems in some other systems. More precisely, every non-empty system of some class of algebraic systems includes some monogenic systems of this class as its subsystems. Hence the structure of algebraic systems depends on the structure of their monogenic systems.

In such classes of algebraic systems as groups and semigroups the monogenic systems are the cyclic groups and semigroups, respectively, which are completely described, as it is well known. In other classes it is very difficult to describe monogenic systems.

Definition 1. A quasigroup generated by one its element is called a *monogenic quasigroup*.

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Definition 2. A quasigroup generated by every its element is called a *monoquasigroup.*

From the definition it is clear that monoquasigroups have no nontrivial subquasigroups and, hence, its lattice of subquasigroups consists of one element. A nontrivial (or proper) subquasigroup is a subquasigroup different from the empty subquasigroup and the quasigroup itself [1].

Researching different kinds of functional completeness of universal algebras, A. V. Kuznetzov and A. F.Danilichenko A. F. announced during *The First All-Union Symposium on the Theory of Quasigroups and its Applications* (Suchumi, 1968) that for every positive integer n there exists a monoquasigroup with $|\mathcal{Q}| = n$, where by $|\mathcal{Q}|$ we denote the order of $\mathcal{Q}$.

A quasigroup $(\mathcal{Q}, \cdot)$ is said to be *without congruences* if it has no congruences except $\varepsilon = \mathcal{Q} \times \mathcal{Q} = \mathcal{Q}^2$ (the complete relation on $\mathcal{Q}$) and $\omega = \{(a, a) : a \in \mathcal{Q}\}$ (the equality relation on $\mathcal{Q}$, sometimes called the *diagonal* of $\mathcal{Q}^2$).

T. Kepka has proved in [4] that a quasigroup $(\mathcal{Q}, \cdot)$ such that $3 \leq |\mathcal{Q}| \leq \aleph_0$ is isotopic to a monoquasigroup. In [3] is proved

Theorem 1.
a) Every quasigroup $(\mathcal{Q}, \cdot)$ *such that* $3 \leq |\mathcal{Q}| \leq \aleph_0$ *is isotopic to a monoquasigroup without congruences.*
b) Every quasigroup $(\mathcal{Q}, \cdot)$ *such that* $5 \leq |\mathcal{Q}| \leq \aleph_0$ *is isotopic to a monoquasigroup without congruences and automorphisms.* □

A quasigroup without automorphisms is a quasigroup with the unitary group of automorphisms. There exist no monoquasigroups with more than the countable order, because a finitely generated free algebra with a finite set of operations has at most the countable order [5]. A quasigroup $(\mathcal{Q}, \cdot)$ with $|\mathcal{Q}| = 1$ satisfies Theorem 1a and a quasigroup $(\mathcal{Q}, \cdot)$ with $|\mathcal{Q}| = 2$ is a group and does not satisfies Theorem 1a.

In [3] a class of order $2^{\aleph_0}$ of pairwise non-isomorphic monoquasigroups without congruences and without automorphisms is given. This fact allows us to assert that it is not very probably to describe monoquasigroups or monogenic quasigroups. Therefore to describe monoquasigroups we shall restrict ourselves to particular classes. In the class of all idempotent quasigroups and the class of all loops there are no monoquasigroups except single-element groups. In other classes it is rather difficult to give examples of monoquasigroups.

Below we consider only quasigroups with the order greater than 2 and smaller than $\aleph_0$.

2. Preliminaries

A groupoid (i.e. a set $(\mathcal{Q},\cdot)$ with a binary operation "∃" on $\mathcal{Q}$) is called a *quasigroup* if equations $ax=b$, $ya=b$ have unique solutions for any elements $a,b\in\mathcal{Q}$. In a quasigroup $(\mathcal{Q},\cdot)$ a mapping $x\rightarrow ax$ is called the *left translation by* a and is denoted by L_a. The *right translation by* a is the mapping $x\rightarrow xa$ that is denoted by R_a. For any $a\in\mathcal{Q}$ the translations R_a and L_a are permutations on the set $\mathcal{Q}$ and belong to the permutation group $\mathcal{S}(\mathcal{Q})$.

A non-empty subset H of the quasigroup $(\mathcal{Q},\cdot)$ is a subquasigroup of $(\mathcal{Q},\cdot)$ provided $(H,\cdot)$ is a quasigroup with respect to the operation "∃". The empty set $\emptyset$ will be considered as a subquasigroup iff the intersection of all non-empty subquasigroups of $(\mathcal{Q},\cdot)$ is $\emptyset$. The set-theoretic intersection of all subquasigroups of $(\mathcal{Q},\cdot)$ containing a subset M of $\mathcal{Q}$ is a subquasigroup that will be denoted by $\langle M\rangle$ and will be called the subquasigroup generated by M. The class $L(\mathcal{Q},\cdot)$ of all subquasigroups of a quasigroup $(\mathcal{Q},\cdot)$ is a complete lattice with respect to the set-theoretic intersection and "generate" operation. The last element in $L(\mathcal{Q},\cdot)$ is the intersection of all non-empty subquasigroups and the greatest element is $(\mathcal{Q},\cdot)$.

Let " $*$ " and " $\circ$ " be two operations defined on $\mathcal{Q}$. The operation " $*$ " is said *to be isotopic* to " $\circ$ " if there exist three permutations $\alpha,\beta,\gamma\in\mathcal{S}(\mathcal{Q})$ such that

$$x*y=\gamma^{-1}(\alpha x\circ\beta y) \qquad (1)$$

for all $x,y\in\mathcal{Q}$.

We also say that $(\mathcal{Q},*)$ and $(\mathcal{Q},\circ)$ are *isotopic*, or that $(\mathcal{Q},*)$

is an *isotop* of $(\mathcal{Q}\,,\circ)$ of the form (1). Shortly we write this as

$$(\mathcal{Q}\,,*) : x*y=\gamma^{-1}(\alpha x\circ\beta y), \quad \alpha,\beta,\gamma\in\mathcal{S}(\mathcal{Q})\,, \quad x,y\in\mathcal{Q}\,.$$

Then triple (α,β,γ) of permutations such that the relation (1) holds is called the *isotopy* of $(\mathcal{Q}\,,\circ)$.

If in (1) γ is the identical permutation ϵ, then $(\mathcal{Q}\,,*)$ is said to be a *principal isotope* of $(\mathcal{Q}\,,\circ)$.

If in (1) $\alpha=\beta=\gamma$, then

$$x*y=\gamma^{-1}(\gamma x\circ\gamma y)\,,$$

which means that γ is an isomorphism between $(\mathcal{Q},*)$ and $(\mathcal{Q},\circ)$. The equality (1) is equivalent to

$$x*y=\gamma^{-1}(\alpha\gamma^{-1}\gamma\, x\circ\beta\gamma^{-1}\gamma\, y)\,.$$

Whence we have proved the following:

Theorem 2 ([1] Theorem 1.2)**.** *An isotope* $(\mathcal{Q}\,,*)$ *such that* $x*y=\gamma^{-1}(\alpha x\circ\beta y)$, $\alpha,\beta,\gamma\in\mathcal{S}(\mathcal{Q})$, $x,y\in\mathcal{Q}$ *is isomorphic to the principal isotope* $(\mathcal{Q}\,,\otimes)$, *where* $x\otimes y=\alpha\gamma^{-1}x\circ\beta\gamma^{-1}y$, $\alpha,\beta,\gamma\in\mathcal{S}(\mathcal{Q})$, $x,y\in\mathcal{Q}$, *and* γ *is the isomorphism between them.* □

3. Group isotopes

Let $(\mathcal{Q},\cdot)$ be a group with the unit element e. We will find the necessary and sufficient conditions which the isotopy must satisfy in order that the corresponding isotope of $(\mathcal{Q},\cdot)$ be a monogenic group; monoquasigroup (Theorem 3). Since an isomorphism keeps the number of generators, then taking into consideration Theorem 2 it is sufficient to find these conditions for principal isotopes of a group $(\mathcal{Q}\,,\cdot)$.

Lemma 1. *For a principal isotope*

$$(\mathcal{Q},*)\ :\ x*y=\varphi x\cdot\psi y, \quad \varphi,\psi\in\mathcal{S}(\mathcal{Q}), \quad x,y\in\mathcal{Q}$$

of a group $(\mathcal{Q}\,,\cdot)$ *with the unit* e *there exist permutations* $\alpha,\beta\in\mathcal{S}(\mathcal{Q})$ *such that* $\beta e=e$ *and* $x*y=\alpha x\cdot\beta y$, *i.e.*

$$(\mathcal{Q}\,,*)\ :\ x*y=\alpha x\cdot\beta y, \quad \alpha,\beta\in\mathcal{S}(\mathcal{Q}), \quad \beta e=e, \quad x,y\in\mathcal{Q}\,. \qquad (2)$$

Proof. For every $x, y \in \mathcal{Q}$ we have

$$x * y = \varphi x \cdot \psi y = \varphi x \cdot \psi e \cdot (\psi e)^{-1} \psi y = R_{\psi e} \varphi x \cdot L_{(\psi e)^{-1}} \psi y = \alpha x \cdot \beta y\,,$$

where $\alpha = R_{\psi e} \varphi$ and $\beta = L_{(\psi e)^{-1}} \psi$. Moreover,

$$\beta e = L_{(\psi e)^{-1}} \psi e = (\psi e)^{-1} \psi e = e\,,$$

which completes the proof. □

For all $\alpha \in \mathcal{S}(\mathcal{Q})$, $H \subseteq \mathcal{Q}$ put

$$\alpha H = \{\, \alpha h \,:\, h \in H \,\}\,.$$

Lemma 2. *For an isotope*

$$(\mathcal{Q}\,, *) \,:\, x * y = \alpha x \cdot \beta y, \quad \alpha, \beta \in \mathcal{S}(\mathcal{Q}), \quad \beta e = e, \quad x, y \in \mathcal{Q}$$

of a group $(\mathcal{Q}\,, \cdot)$ *with the unit* e *the following conditions are equivalent*

a) $e \in (H\,, *) \in L(\mathcal{Q}\,, *)\,,$

b) $\alpha H = H = \beta H$ *and* $(H\,, \cdot) \in L(\mathcal{Q}\,, \cdot)\,.$

Proof. Let $e \in (H\,, *) \in L(\mathcal{Q}\,, *)$. Then for any $x \in H$, we have $x * e \in (H\,, *)$ and $x * e = \alpha x \cdot \beta e = \alpha x \cdot e = \alpha x$. Hence $\alpha x \in H$ and, as x is an arbitrary element, we have $\alpha H \subseteq H$. For any $x \in H$ there exists $y \in H$ such that $x = y * e$, since $(H\,, *)$ is a subquasigroup of $(\mathcal{Q}\,, *)$ and $e \in H$. From the last equality we get $y = \alpha^{-1} x$ and $\alpha^{-1} H \in H$ since x is an arbitrary element of H. Therefore $H \subseteq \alpha H$ and we have $H = \alpha H$. Let $h \in H$ be such that $\alpha h = e$. For any $x \in H$ we have $h * x \in H$ and $h * x = \alpha h \cdot \beta x = e \cdot \beta x = \beta x$. Therefore $\beta x \in H$ for any $x \in H$, so $\beta H \subseteq H$. There exists $y \in H$ such that $h * y = x$ for any $x \in H$. Then

$$h * y = \alpha h \cdot \beta y = e \cdot \beta y = \beta y = x$$

and $y = \beta^{-1}(x)$. Hence $\beta^{-1} H \subseteq H$, $H \subseteq \beta H$ and finally we have $\beta H = H$. So, the restrictions of α and β to H are permutations on H, and $(H, \cdot)$ is an associative quasigroup isotopic to the quasigroup $(H, *)$ since $x \cdot y = \alpha^{-1} x * \beta^{-1} y$, i.e. $(H, \cdot) \in L(\mathcal{Q}, *)$. Therefore we have proved that $e \in (H,*) \in L(\mathcal{Q},*)$ implies $\alpha H = H = \beta H$ and $(H,\cdot) \in L(\mathcal{Q},\cdot)$.

The converse implication is trivial. □

For any $\varphi \in \mathcal{S}(\mathcal{Q})$ put

$$Stab_{L(\mathcal{Q},\cdot)}\varphi = \{\, H \subseteq \mathcal{Q} \,:\, (H,\cdot) \in L(\mathcal{Q},\cdot) \quad \text{and} \quad \varphi H = H \,\}\,.$$

Lemma 3. *A quasigroup*

$$(\mathcal{Q},*) \,:\, x*y = \alpha x \cdot \beta y, \quad \alpha,\beta \in \mathcal{S}(\mathcal{Q}), \quad \beta e = e, \quad x,y \in \mathcal{Q}$$

which is isotopic to a group $(\mathcal{Q},\cdot)$ *with the unit* e *is generated by* e *if and only if*

$$Stab_{L(\mathcal{Q},\cdot)}\alpha \cap Stab_{L(\mathcal{Q},\cdot)}\beta = \{\mathcal{Q}\}\,. \tag{3}$$

Proof. Let $(\mathcal{Q},*)$ be generated by the unit e and

$$H \in Stab_{L(\mathcal{Q},\cdot)}\alpha \cap Stab_{L(\mathcal{Q},\cdot)}\beta\,.$$

Then $(H,\cdot) \in L(\mathcal{Q})$ and $\alpha H = H = \beta H$. By Lemma 2 we get $(H,*) \in L(\mathcal{Q},*)$ and $(H,*) = (\mathcal{Q},*)$ as $e \in (H,*)$. Hence

$$Stab_{L(\mathcal{Q},\cdot)}\alpha \cap Stab_{L(\mathcal{Q},\cdot)}\beta = \{\mathcal{Q}\}\,.$$

Conversely, let the relation (3) holds and $(H,*) \in L(\mathcal{Q},*)$, where $e \in (H,*)$. By Lemma 2 we have $\alpha H = H = \beta H$ and from (3) we get $H = \mathcal{Q}$. Therefore $(\mathcal{Q},*)$ is generated by the unit e. □

Directly from Lemmas 1 and 3 we get

Corollary 1. *A quasigroup*

$$(\mathcal{Q},*) \,:\, x*y = \alpha x \cdot \beta y, \quad \alpha,\beta \in \mathcal{S}(\mathcal{Q}), \quad x,y \in \mathcal{Q}\,,$$

which is isotopic to a group $(\mathcal{Q},\cdot)$ *with the unit* e *is generated by* e *if and only if*

$$Stab_{L(\mathcal{Q},\cdot)}R_{\beta e}\alpha \cap Stab_{L(\mathcal{Q},\cdot)}L_{\beta e}^{-1}\beta = \{\mathcal{Q}\}\,. \tag{4}$$

Proof. By Lemma 1 we get the equalities $x * y = R_{\beta e}\alpha x \cdot L_{\beta e}^{-1}\beta y$ and $L_{\beta e}^{-1}\beta e = e$. The equality (4) follows from Lemma 3.

Remark that relation (4) gives $\alpha e \cdot \beta e \neq e$, otherwise the set $\{e\}$ is a subquasigroup of $(\mathcal{Q}\,, *)$, contrary to (4). □

Now we will find the condition for a quasigroup

$$(\mathcal{Q}\,, *)\ :\ x * y = \alpha x \cdot \beta y, \quad \alpha, \beta \in \mathcal{S}(\mathcal{Q}), \quad x, y \in \mathcal{Q}$$

to be generated by any its element $a \in \mathcal{Q}\,,\ \ a \neq e$.

Let us consider the isotope

$$(\mathcal{Q}\,, \circ)\ :\ x \circ y = x \cdot a^{-1} y, \quad x, y \in \mathcal{Q}$$

for any fixed element $a \in \mathcal{Q}$. This isotope is a group with the unit a and the left translation L_a of a group $(\mathcal{Q}\,, \cdot)$ is an isomorphism between groups $(\mathcal{Q}\,, \cdot)$ and $(\mathcal{Q}\,, \circ)$, i.e. we have $L_a(x \cdot y) = L_a x \circ L_a y$. Then the equality

$$L_a^{-1}(x \circ y) = L_a^{-1} x \cdot L_a^{-1} y$$

and implications

$$(H, \cdot) \in L(\mathcal{Q}\,, \cdot)\ \Rightarrow\ (L_a H, \circ) \in L(\mathcal{Q}\,, \circ)\,,$$

$$(H, \circ) \in L(\mathcal{Q}\,, \circ)\ \Rightarrow\ (L_a^{-1} H, \cdot) \in L(\mathcal{Q}\,, \cdot) \tag{5}$$

hold. The quasigroup $(\mathcal{Q}\,, *)$ is an isotope of a group $(\mathcal{Q}\,, \circ)$ with the unit a since we have $x * y = \alpha x \cdot \beta y = \alpha x \circ L_a \beta y$. By Corollary 1 the quasigroup $(\mathcal{Q}\,, *)$ is generated by a if and only if the equality

$$Stab_{L(\mathcal{Q}\,,\circ)} \hat{R}^{-1}_{L_a \beta a}\alpha \,\cap\, Stab_{L(\mathcal{Q}\,,\circ)} \hat{L}^{-1}_{L_a \beta a} L_a \beta = \{\mathcal{Q}\} \tag{6}$$

holds, where by $\hat{R}_x\,, \hat{L}_x$ we denote the translations by x on the group $(\mathcal{Q}\,, \circ)$.

Remark. If the equality (6) holds, then we have $\alpha a \cdot \beta a \neq a$. The following equalities hold for any $a, u \in \mathcal{Q}$:

$$L_u^{-1} = L_{u^{-1}}\,,$$
$$\hat{L}_u = L_u L_{a^{-1}} = L_{ua^{-1}}\,, \quad \hat{L}_u^{-1} = L_a L_{u^{-1}} = L_{au^{-1}}\,,$$
$$\hat{R}_u = R_u R_{a^{-1}} = R_{a^{-1}u}\,, \quad \hat{R}_u^{-1} = R_a R_{u^{-1}} = R_{u^{-1}a}\,,$$

where a^{-1}, u^{-1} are the inverses of a, u in $(\mathcal{Q}\,,\cdot)$. Then

$$\hat{R}_{L_a\beta a}\alpha = \hat{R}_{a\beta a}\alpha = R_{a^{-1}\cdot a\beta a}\alpha = R_{\beta a}\alpha\,,$$

$$\hat{L}^{-1}_{L_a\beta a}L_a\beta = \hat{L}^{-1}_{a\beta a}L_a\beta = L_{a(\beta a)^{-1}a^{-1}}L_a\beta = L_{a(\beta a)^{-1}}\beta\,.$$

Now the equality (6) can be rewritten in the following way:

$$Stab_{L(\mathcal{Q}\,,\circ)}R_{\beta a}\alpha \cap Stab_{L(\mathcal{Q}\,,\circ)}L_{a(\beta a)^{-1}}\beta = \{\mathcal{Q}\}\,. \tag{7}$$

Lemma 4. *For any* $\varphi \in \mathcal{S}(\mathcal{Q})$ *we have*

$$Stab_{L(\mathcal{Q}\,,\circ)}\varphi = L_a(Stab_{L(\mathcal{Q}\,,\cdot)}L_{a^{-1}}\varphi L_a)$$

and

$$Stab_{L(\mathcal{Q}\,,\circ)}\varphi = \{\, L_aH \,:\, (H,\cdot) \in L(\mathcal{Q}\,,\cdot) \text{ and } L_{a^{-1}}\varphi L_a H = H\}\,.$$

Proof. If $(H,\cdot) \in L(\mathcal{Q}\,,\cdot)$ and $L_{a^{-1}}\varphi L_aH = H$, then $\varphi L_aH = L_aH$ and $(L_aH,\circ) \in Stab_{L(\mathcal{Q}\,,\circ)}\varphi$ since $(L_aH,\circ) \in L(\mathcal{Q}\,,\circ)$. Hence we have

$$Stab_{L(\mathcal{Q}\,,\circ)}\varphi \supseteq L_a(Stab_{L(\mathcal{Q}\,,\cdot)}L_{a^{-1}}\varphi L_a)\,.$$

Conversely, let $(\hat{H},\circ) \in Stab_{L(\mathcal{Q},\circ)}\varphi$. Then $\varphi\hat{H} = \hat{H}$ and we can write

$$\varphi L_a {L_a}^{-1}\hat{H} = L_a {L_a}^{-1}\hat{H}$$

which get

$$L_{a^{-1}}\varphi L_a L_a^{-1}\hat{H} = L_a^{-1}\hat{H}\,.$$

Hence,

$$(L_a^{-1}\hat{H},\cdot) \in Stab_{L(\mathcal{Q}\,,\cdot)}L_{a^{-1}}\varphi L_a$$

as $(L_a^{-1}\hat{H},\,\cdot) \in L(\mathcal{Q}\,,\,\cdot)$ by (5). Therefore

$$Stab_{L(\mathcal{Q},\circ)}\varphi \subseteq L_a(Stab_{L(\mathcal{Q},\cdot)}L_{a^{-1}}\varphi L_a)$$

and the statement of the lemma is proved. □

Now the equality (7) can be rewritten as

$$L_a(Stab_{L(\mathcal{Q},\cdot)}L_{a^{-1}}R_{\beta a}\alpha L_a) \cap L_a(Stab_{L(\mathcal{Q},\cdot)}L_{(\beta a)^{-1}}\beta L_a) = \{\mathcal{Q}\}\,,$$

from which we have

$$Stab_{L(\mathcal{Q},\cdot)}L_{a^{-1}}R_{\beta a}\alpha L_a \cap Stab_{L(\mathcal{Q},\cdot)}L_{(\beta a)^{-1}}\beta L_a = \{\mathcal{Q}\}\,. \qquad (8)$$

So, we have proved the following:

Theorem 3. *A quasigroup*

$$(\mathcal{Q}\,,*) \;\; : \;\; x*y=\alpha x\cdot\beta y, \quad \alpha,\beta\in\mathcal{S}(\mathcal{Q}), \quad x,y\in\mathcal{Q}$$

isotopic to a group $(\mathcal{Q}\,,\,\cdot)$ *with the unit* e *is:*
a) generated by an element $a\in\mathcal{Q}$ *iff the equality (8) holds,*
b) a monoquasigroup iff the equality (8) holds for any $a\in\mathcal{Q}$ *.* □

Corollary 2. *If a quasigroup* $(\mathcal{Q}\,,*)$ *isotopic to a group* $(\mathcal{Q}\,,\cdot)$ *with the unit* e *is a monoquasigroup, then* $\alpha x\cdot\beta x\neq x$ *for all* $x\in\mathcal{Q}$.

Proof. In fact, if $\alpha a\cdot\beta a=a$ for some $a\in\mathcal{Q}$, then a is an idempotent. Thus $(\mathcal{Q}\,,*)$ is not generated by a, contrary to the assertion of the corollary. □

Proposition 1. *The order of a subquasigroup of a group isotope* $(\mathcal{Q}\,,\oplus)$ *divides the order of the quasigroup* $(\mathcal{Q},\oplus)$.

Proof. Let $(H,\oplus)$ be a subquasigroup of a group isotope $(\mathcal{Q}\,,\oplus)$ and $\emptyset\neq H\neq\mathcal{Q}$. By Albert's Theorem the isotope

$$(\mathcal{Q}\,,\bullet) \;\; : \;\; x\bullet y=R_a^{-1}x\oplus L_a^{-1}y$$

is a group for every element $a\in H$ and $(H,\bullet)$ is a subgroup of $(\mathcal{Q}\,,\bullet)$ as $a\in H$. In a group the order of subgroup divides the order of the group. □

Corollary 3. *Every proper subquasigroup of a group isotope of prime order is a single-element set.* □

Corollary 4. *The isotope*

$$(\mathcal{Q}\,,*) \;\; : \;\; x * y = \alpha x \cdot \beta y, \quad \alpha, \beta \in \mathcal{S}(\mathcal{Q}), \quad x, y \in \mathcal{Q}$$

of a group $(\mathcal{Q}\,,*)$ *of a prime order is a monoquasigroup if and only if* $\alpha x \cdot \beta x \neq x$ *for all* $x \in \mathcal{Q}$.

Proof. Let $\alpha x \cdot \beta x \neq x$ for all $x \in \mathcal{Q}$ and $|\mathcal{Q}|$ be a prime number. The isotope $(\mathcal{Q}\,,*)$ has not single-element subquasigroups since $x* = \alpha x \cdot \beta x \neq x$ for all $x \in \mathcal{Q}$. From Corollary 3 we obtain that $(\mathcal{Q}\,,*)$ is generated by any its element, i.e. is a monoquasigroup. □

By Corollary 2 the relation $\alpha x \cdot \beta x \neq x$ holds in $(\mathcal{Q}\,,*)$ for all $x \in \mathcal{Q}$. We will adopt some of the above results for right loops principally isotopic to groups, i.e. we will find the necessary and sufficient conditions for a right loop isotopic to a group to be generated by any its non-unit element, therefore to have no proper subloops. A results can be obtained for left loops isotopic to groups.

Recall that a *right (left) loop* is a quasigroup $(\mathcal{Q}\,,*)$ with the right (left) unit f (respectively - (e)) i.e. such elements that $x * f = x$ $(e * x = x)$ for all $x \in \mathcal{Q}$. The sets $\{f\}$, $\{e\}$ and $\mathcal{Q}$ are right (left) subloops of $(\mathcal{Q}\,,*)$ called *improper subloops.* All other subloops are called *proper subloops.*

For all $a \in \mathcal{Q}$ put

$$\mathcal{S}_a(\mathcal{Q}) = \{\psi \in \mathcal{S}(\mathcal{Q}) \;\; : \;\; \psi a = a\}\,.$$

Proposition 2. *A right loop isotopic to a group* $(\mathcal{Q}\,,\cdot)$ *with the unit* e *is isomorphic to some right loop*

$$(\mathcal{Q}\,,\circ) \;\; : \;\; x \circ y = x \cdot \varphi y, \quad \varphi \in \mathcal{S}_e(\mathcal{Q}), \quad x, y \in \mathcal{Q}$$

with the unit e.

Proof. To prove that, by Theorem 2 it is sufficient to consider right loops principally isotopic to groups. Let

$$(\mathcal{Q}\,,*) \;\; : \;\; x * y = \alpha x \cdot \beta y, \quad \alpha, \beta \in \mathcal{S}(\mathcal{Q})$$

be a right loop isotopic to a group $(\mathcal{Q},\cdot)$ and f be the right unit of $(\mathcal{Q},*)$. For every $x\in\mathcal{Q}$ we have $x=x*f=\alpha x\cdot\beta f$ and we obtain

$$\alpha=R^{-1}_{(\beta f)}=R_{(\beta f)^{-1}}\,.$$

Therefore

$$x*y=R_{(\beta f)^{-1}}x\cdot\beta y=x(\beta f)^{-1}\beta y=x\cdot L_{(\beta f)^{-1}}\beta y$$

for all $x,y\in\mathcal{Q}$. Let us consider the isotope

$$(\mathcal{Q},\circ)\quad:\quad x\circ y=x\cdot L_{(\beta f)^{-1}}\beta L_f y,\qquad x,y\in\mathcal{Q}\,.$$

Remark that $L_{(\beta f)^{-1}}\beta L_f e=e$, i.e. $(\mathcal{Q},\circ)$ is a right loop with the unit e. The loop $(\mathcal{Q},\circ)$ is isomorphic to $(\mathcal{Q},*)$ since

$$L_f(x\circ y)=L_f(x\cdot L_{(\beta f)^{-1}}\beta L_f y)=L_f x\cdot L_{(\beta f)^{-1}}\beta L_f y=L_f x*L_f y$$

for all $x,y\in\mathcal{Q}$. □

The following assertion can be proved in a way analogous to that used in Theorem 3.

Lemma 5. *A right loop*

$$(\mathcal{Q},*)\quad:\quad x*y=x\cdot\alpha y,\qquad \alpha\in\mathcal{S}_e(\mathcal{Q}),\qquad x,y\in\mathcal{Q}$$

isotopic to a group $(\mathcal{Q},\cdot)$ *with the unit* e *has no proper subloops if and only if*

$$Stab_{L(\mathcal{Q},\cdot)}\alpha=\{\{e\},\,\mathcal{Q}\}.$$ □

Lemma 6. *A right loop*

$$(\mathcal{Q},*)\quad:\quad x*y=x\cdot\alpha y,\qquad \alpha\in\mathcal{S}(\mathcal{Q}),\qquad \alpha f=e,\qquad x,y\in\mathcal{Q}$$

isotopic to a group $(\mathcal{Q},\cdot)$ *with the unit* e *has no proper subloops if and only if*

$$Stab_{L(\mathcal{Q},\cdot)}\alpha L_e=\{\{e\},\mathcal{Q}\}$$

Proof. The right loop $(\mathcal{Q},*)$ is isomorphic to the right loop

$$(\mathcal{Q}\,,\circ) \;\;:\;\; x\circ y = x\cdot \alpha L_f y, \quad x,y\in\mathcal{Q}\,.$$

In fact

$$L_f(x\circ y) = L_f(x\cdot\alpha L_f y) = L_f x\cdot \alpha L_f y = L_f x * L_f y$$

for all $x,y\in\mathcal{Q}$. Then the right loop $(\mathcal{Q}\,,*)$ has no proper subloops provided that $(\mathcal{Q}\,,\circ)$ has no ones. Now we apply Lemma 5 to the right loop $(\mathcal{Q}\,,\circ)$ and this completes the proof of our Lemma. □

Corollary 5. *The loop isotopic to a group of prime order has no proper right subloops.*

Proof. The assertion follows from Lemma 6 taking into account that a group of a prime order has no proper subgroups. □

Theorem 4. *Let $(\mathcal{Q}\,,\cdot)$ be a group with the unit e. The right loop*

$$(\mathcal{Q}\,,*) \;\;:\;\; x*y = \gamma^{-1}(\alpha x\cdot\beta y), \quad \alpha,\beta,\gamma\in\mathcal{S}(\mathcal{Q}), \quad x,y\in\mathcal{Q}$$

with the right unit f has no proper right subloops if and only if

$$Stab_{L(\mathcal{Q},\cdot)} L_{(\beta f)^{-1}}\beta\gamma^{-1}L_{\gamma f} = \{\{e\},\,\mathcal{Q}\}$$

Proof. By Theorem 2 the right loop $(\mathcal{Q}\,,*)$ is isomorphic to the right loop

$$(\mathcal{Q}\,,*) \;\;:\;\; x*y = \gamma^{-1}(\alpha x\cdot\beta y), \quad \alpha,\beta,\gamma\in\mathcal{S}(\mathcal{Q}), \quad x,y\in\mathcal{Q}$$

with the unit γf. For every $x\in\mathcal{Q}$ we have

$$x = x\circ\gamma f = \alpha\gamma^{-1}x\cdot\beta f\,,$$

hence $\alpha\gamma^{-1} = R_{(\beta f)^{-1}}$. Therefore

$$x\circ y = R_{(\beta f)^{-1}}x\cdot\beta\gamma^{-1}y = x\cdot(\beta f)^{-1}\beta\gamma^{-1}y = x\cdot L_{(\beta f)^{-1}}\beta\gamma^{-1}y\,.$$

Let us consider the isotope

$$(\mathcal{Q}\,,\times) \;\;:\;\; x\times y = x\cdot L_{(\beta f)^{-1}}\beta\gamma^{-1}L_{(\gamma f)}y, \quad x,y\in\mathcal{Q}\,.$$

This isotope is a right loop with the right unit e and the translation $L_{\gamma f}$ is an isomorphism between $(\mathcal{Q}\,,\times)$ and $(\mathcal{Q}\,,\circ)$. Now we apply Lemma 5 to the right loop $(\mathcal{Q}\,,\times)$ and this completes the proof. □

4. Examples

1. Let $(\mathcal{Q}\,,\,\cdot) = \langle h\rangle$ be a cyclic group generated by $h \in \mathcal{Q}$ and $|\mathcal{Q}| = r$, $3 \le r \le \aleph_0$. Let e be the unit of the group $(\mathcal{Q}\,,\,\cdot)$, I be the permutation of $(\mathcal{Q}\,,\,\cdot)$ defined by $Ix = x^{-1}$, $\alpha = (e\,h)$ be the transposition of elements e and h. Then the isotope

$$(\mathcal{Q}\,,*) \quad : x * y = \alpha x \cdot Iy, \quad x, y \in \mathcal{Q}$$

is a monoquasigroup which satisfies the identity $x * (x * y) = y$. In fact

$$x * (x * y) = \alpha x \cdot I(\alpha x \cdot Iy) = \alpha x \cdot I\alpha x \cdot y = y\,.$$

Recall that every element of the group $(\mathcal{Q}\,,\,\cdot) = \langle h\rangle$ is a power of the element h and every its subgroup is the cyclic subgroup generated by some power h^k of h. To prove that $(\mathcal{Q}\,,*)$ is a monoquasigroup, by Lemma 3, it is sufficient to prove the equality

$$Stab_{L(\mathcal{Q},\cdot)}\alpha \cap Stab_{L(\mathcal{Q},\cdot)}I = \{\mathcal{Q}\}\,.$$

This equality holds since we have $\alpha e = h$ and the group $(\mathcal{Q}\,,\,\cdot)$ is the unique subgroup of $(\mathcal{Q}\,,\,\cdot)$ which contains h.

Now we will prove that every subquasigroup $(H, *)$ of $(\mathcal{Q}\,,*)$ contains e. Really, if $h^k \in (H, *)$ for $k \neq 1$, then $e = h^k \cdot h^{-k} \in (H, *)$. If $h \in (H, *)$, then $h^{-1} = e \cdot h^{-1} = \alpha h \cdot Ih = h * h \in (H, *)$ and $e \in (H, *)$ as it is proved above.

2. Let $(\mathcal{Q}\,,\cdot) = \langle h\rangle$ be a cyclic group with the unit e generated by $h \in \mathcal{Q}$, $|\mathcal{Q}| = n$, $3 \le n \le \aleph_0$ and α be a cyclic permutation $(hh^2h^3...h^{n-1})$. Then the isotope

$$(\mathcal{Q}\,,*) \quad : \quad x * y = x \cdot \alpha y, \quad x, y \in \mathcal{Q}$$

is a right loop with the unit e and has no proper right subloops.

Really, we have $x * e = x \cdot \alpha e = x \cdot e = x$ for all $x \in \mathcal{Q}$ and thus $(\mathcal{Q}\,,*)$ is a right loop with the unit e. If $1 \le k \le n-2$, then $\alpha h^k = h^{k+1}$ and $\alpha h^{-1} = \alpha h^{n-1} = h$. Now, if a subgroup of the cyclic group $\langle h\rangle$ contains elements h^k and h^{k+1}, then it contains h as a solution of the equation $h^k \cdot x = h^{k+1}$, and thus it coincides with $\langle h\rangle$. Hence, we have

$$Stab_{L(\mathcal{Q},\cdot)}\alpha = \{\{e\}, \mathcal{Q}\},$$

and thus $(\mathcal{Q}, *)$ has no proper right subloops.

3. Let $(\mathcal{Q}, \cdot) = \langle h \rangle$ be the infinite cyclic group generated by the element $h \in \mathcal{Q}$. Let e be the unit of $(\mathcal{Q}, \cdot)$ and α be the following permutation: $\alpha e = e$, $\alpha h^{-1} = h$, $\alpha h^k = h^{k+1}$ for all $k \neq 1$. Like that in the example 2 we can prove that the right loop

$$(\mathcal{Q}, *) \quad : \quad x * y = x \cdot \alpha y, \quad x, y \in \mathcal{Q}$$

has no proper right subloops.

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