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ON DISTRIBUTIVE n-ARY GROUPS

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Abstract

The classes of medial n-groups, distributive n-groups and autodistributive n-groups are described. These are the classes of n-ary groups $(n \geq 3)$ in which the unary operation $^{-} : x \to \bar{x}$ ($z = \bar{x}$ is a unique solution of the equation $f(x, x, \dots x, z) = x$ in an n-ary group) plays an important role.

1. Introduction

As it is well known [11], [8], [4], an n-ary group $(n \geq 3)$ may be defined as an n-ary semigroup (G, f) with a special unary operation $^{-} : x \to \bar{x}$, i.e. as an universal algebra $(G, f, ^{-})$ of type $(n,1)$. Since the equation

$$f(x, x, \dots, x, z) = x$$

has in any n-ary group (G, f) a unique solution $z = \bar{x}$, then the operation $^{-} : x \to \bar{x}$ is a uniquely defined by the operation f. The element $z = \bar{x}$ is called *skew to* x. Obviously $x = \bar{x}$ iff x is an idempotent. In general $\bar{x} \neq \bar{y}$, but in some n-ary groups (G, f) there exists an element z such that $z = \bar{x}$ for all $x \in G$. All such n-ary groups are derived (cf. [7]) from a binary group of the exponent $k|(n-2)$.

In this paper we describe some classes of n-ary groups in which the operation $^{-} : x \to \bar{x}$ plays a very important role.

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Because for $n=2$ such groups are trivial, we consider only the case $n \geq 3$. Used terminology and notion are standard.

2. Medial n-groups

From the proof of **Theorem 3** in [10] it follows that any medial n-ary group satisfies the identity

$$\overline{f(x_1^n)} = f(\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n). \tag{1}$$

Hence an n-ary group (G, f) is medial iff it is Abelian as an algebra $(G, f, \bar{\ })$ of type $(n,1)$. On the other hand, one can prove (cf. [4]) that n-ary group (G, f) is medial iff there exists $a \in G$ such that

$$f(x, \overset{(n-2)}{a}, y) = f(y, \overset{(n-2)}{a}, x)$$

for all $x, y \in G$, i.e. iff the binary retract of (G, f) is commutative (cf. [4], [6]).

Note that the identity (1) is satisfied also in some non-medial n-ary groups. For example, (1) holds in the 8-group derived from the group S_3. It is also satisfied in all idempotent n-ary groups.

Let $x = \bar{x}^{(0)}$ and let $\bar{x}^{(s+1)}$ be the skew element to $\bar{x}^{(s)}$, where $s \geq 0$. In other words, let $\bar{x}^{(1)} = \bar{x}$, $\bar{x}^{(2)} = \bar{\bar{x}}$ etc. For example, in a 4-group (G, f) derived from the additive group Z_8, we have

$$\bar{x} = 6x (\mathrm{mod}\ 8), \quad \bar{\bar{x}} = 4x (\mathrm{mod}\ 8), \quad \bar{x}^{(s)} = 0$$

for every $s \geq 3$. But in the n-ary group (Z, f) derived from the additive group of integers we have $\bar{x}^{(s)} \neq \bar{x}^{(t)}$ for all $s \neq t$.

If $\bar{x}^{(s)} = x$, then

$$ord_n(x) = ord_n(\overline{x}^{(t)})$$

for any natural t, where $ord_n(x)$ denotes the n-ary order of x, i.e. the minimal natural number p (if it exists) such that $x^{<p>} = x$. By $x^{<s>}$ we mean x if $s = 0$, and $f(x^{<s-1>}, x, \ldots, x)$ if $s > 1$ (cf. [3] or [5]).

One can prove (cf. [3]) that $\overline{x}^{(m)} = x$ iff $ord_n(x)$ divides

$$\frac{1-(2-n)^m}{n-1} = \sum_{k=0}^{m-1}(2-n)^k.$$

In particular $\overline{\overline{x}} = x$ iff $ord_n(x)$ divides $n-3$. Hence in any ternary group (G, f) we have $\overline{\overline{x}} = x$ for all $x \in G$. Note also that if the n-ary order of x is finite, then

$$ord_n(x) = ord_n(\overline{x})$$

iff $ord_n(x)$ and $n-2$ are relatively prime.

It is clear that if an n-ary group (G, f) satisfies (1), then for all $s \geq 0$ it satisfies also

$$\overline{f(x_1^n)}^{(s)} = f(\overline{x}_1^{(s)}, \overline{x}_2^{(s)}, \ldots, \overline{x}_n^{(s)}).$$

Therefore if an n-ary group (G, f) satisfies (1), then the mapping ϕ_s defined by the formula

$$\phi_s(x) = \overline{x}^{(s)}$$

is an n-ary endomorphism of (G, f). Obviously $\phi_s\phi_t = \phi_{s+t}$ and ϕ_k is the identity endomorphism of (G, f) iff $\overline{x}^{(k)} = x$ for all $x \in G$. Thus the set of all ϕ_s forms the cyclic subsemigroup of the semigroup $End(G, f)$.

Moreover, the relation ρ_s defined on (G, f) by the formula $(x, y) \in \rho_s$ iff $\overline{x}^{(s)} = \overline{y}^{(s)}$, i.e. iff $\phi_s(x) = \phi_s(y)$ is a congruence on (G, f). Obviously, $\rho_0 \leq \rho_s \leq \rho_t$ for any $s \leq t$.

If the set

$$E_s = \{x \in G \mid x = \phi_s(x)\}$$

is non-empty, then it is an n-ary subgroup of an n-ary group (G,f) with (1). It is clear that $E_1 \subset E_s \subset E_{st}$ and $E_s \cap E_{s+1} = E_1$.

Similarly, it is not difficult to verify that if (1) holds in (G,f), then for any s the set

$$G^{(s)} = \{\overline{x}^{(s)} \mid x \in G\}$$

is an n-ary subgroup of (G,f). Moreover, $G^{(s+t)} = (G^{(s)})^{(t)}$ for all $s,t \in N$ and $G \supseteq G^{(1)} \supseteq G^{(2)} \supseteq \ldots$ Obviously, for any finite n-ary group there exists $t \in N$ such that $G^{(s)} = G^{(t)}$ for all $s \geq t$. On the other hand, the n-ary group (G,f) derived from the additive group of all integers is an example of an n-ary group with $G^{(s)} \neq G^{(t)}$ for $s \neq t$ ($G^{(s)}$ contains all integers which are divided by $(n-2)^s$).

3. Distributive n-groups

Let (G,f) be an n-ary group in which the n-ary operation f is distributive with respect to itself , i.e. an n-ary group in which the identity

$$f(x_1^{i-1}, f(y_1^n), x_{i+1}^n) = f(f(x_1^{i-1}, y_1, x_{i+1}^n), f(x_1^{i-1}, y_2, x_{i+1}^n), \ldots, f(x_1^{i-1}, y_n, x_{i+1}^n)),$$

holds for all $i = 1,2,\ldots,n$. Such groups are called *autodistributive n-groups* (cf. [5]). One can prove (cf. [5], **Theorem 3**) that every autodistributive n-group (G,f) satisfies

$$\overline{f(x_1^n)} = f(x_1^{i-1}, \overline{x}_i, x_{i+1}^n), \tag{2}$$

where $i=1,2,\ldots,n$. An n-ary group (G,f) satisfying (2) will be called *distributive*.

Let (G,f) be an n-ary semigroup with a unary operation ϕ such that $f(x,x,\ldots x,\phi(x))=x$ for all $x\in G$. If for any $i=1,2,\ldots,n$ holds also the identity

$$\phi(f(x_1^n))=f(x_1^{i-1},\phi(x_i),x_{i+1}^n),$$

then (G,f,ϕ) is a (f/ϕ)-algebra in the sense of H.J.Hoehnke [12]. If (G,f) is an n-ary group, then we have a distributive n-group, because $\phi(x)=\overline{x}$.

Proposition 1. *Let (G,f) be an n-ary semigroup with the above defined unary operation ϕ. Then (G,f) is a distributive n-group iff it is a cancellative n-semigroup.*

Proof. Suppose that an n-semigroup (G,f) is cancellative. Then

$$f(x_1^{i-1},a,x_{i+1}^n)=f(x_1^{i-1},b,x_{i+1}^n)$$

implies $a=b$ (cf. [5]). If ϕ is distributive with respect to f, then

$$f(\overset{(n-2)}{x},\phi(x),x)=\phi(f(\overset{(n)}{x}))=f(\overset{(n-1)}{x},\phi(x))=x.$$

Thus for $y\in G$ we have

$$f(y,\overset{(n-3)}{x},\phi(x),x)=f(y,\overset{(n-3)}{x},\phi(x),f(\overset{(n-2)}{x},\phi(x),x))=$$
$$=f(f(y,\overset{(n-3)}{x},\phi(x),x),\overset{(n-3)}{x},\phi(x),x),$$

which (by cancellation) gives

$$f(y,\overset{(n-3)}{x},\phi(x),x)=y.$$

Similarly

$$f(\overset{(n-2)}{x},\phi(x),y)=f(f(\overset{(n-2)}{x},\phi(x),x),\overset{(n-3)}{x},\phi(x),y))=$$
$$=f(\overset{(n-2)}{x},\phi(x),f(\overset{(n-2)}{x},\phi(x),y)),$$

implies

$$f(\overset{(n-2)}{x},\phi(x),y)=y$$

Hence for all $x,y\in G$ we have

$$f(y,\overset{(n-3)}{x},\phi(x),x)=f(\overset{(n-2)}{x},\phi(x),y)=y,$$

which proves (cf. [4], [8]) that (G,f) is a distributive n-group and $\phi(x)=\bar{x}$. The converse is obvious. □

As it is well known (cf. [4], [8]) for $n>2$ an n-ary group may be defined as an n-semigroup (G,f) with a unary operation $^{-}: x\to\bar{x}$ in which so-called **Dornte**'s identities

$$f(y,\overset{(n-j-1)}{x},\bar{x},\overset{(j-1)}{x})=f(y,\overset{(i-1)}{x},\bar{x},\overset{(n-i-1)}{x},y)=y \tag{3}$$

hold for all $i,j=1,2,\dots,n-1$.

Using these identities and (2) it is not difficult to verify that the following lemma is true.

Lemma 1. *In any distributive n-group $x=\bar{x}^{(n-1)}$ and $\bar{x}=\bar{x}^{(n)}$.*

Corollary 1. *Any distributive n-group satisfies (1). Moreover, for $x\in G$ we have also $x=f(\bar{x},\bar{x},\dots,\bar{x})=\overline{f(x,x,\dots,x)}$.*

Lemma 2. *In distributive n-groups*

(*a*) $x^{<k>}=\bar{x}^{(n-1-k)}$,

(*b*) $\bar{x}^{<k>}=(\bar{x}^{<k+1>})^{<1>}$,

$$(c)\quad f(x_1^{i-1}, x_i^{<k>}, x_{i+1}^n) = (f(x_1^n))^{<k>}$$

for all $k = 0,1,\dots,n-1$ *and* $i = 1,2,\dots,n$.

Proof. We prove only *(a)*. For $k=0$ this condition is obvious. If it holds for some $t < n-1$, then for $t+1$ we have

$$x^{<t+1>} = f(x^{<t>}, \overset{(n-1)}{x}) = f(\bar{x}^{(n-1-t)}, \overset{(n-1)}{x}) =$$
$$= f(\bar{x}^{(n-2-t)}, \overset{(n-2)}{x}, \bar{x}) = \bar{x}^{(n-1-(t+1))},$$

which completes the proof of *(a)*.

The condition *(c)* is a simple consequence of (2) and *(a)*. □

Corollary 2. *If in a distributive n-group* (G,f) $ord_n(x) = p$ *for some* $x \in G$, *then* $\bar{x}^{(p)} = x$ *and* $x^{<k>} = \bar{x}^{(p-k)}$ *for* $k = 0,1,\dots,p$.

Corollary 3. *If* p *is a minimal natural number such that* $x = \bar{x}^{(p)}$ *for some element* x *of a distributive n-group, then* $ord_n(x) = p$.

Proof. See the proof of **Corollary 10** from [5]. □

Lemma 3. *All elements of a distributive n-group have the same finite n-ary order which divides* $n-1$.

Proof. As a simple consequence of **Lemma 2** (a) we obtain $x^{<n-1>} = x$. This shows that all elements of a distributive n-group have a finite n-ary order which is a divisor of $n-1$ (cf. [3]).

Now, if $ord_n(x) = t$, $ord_n(y) = s$, then

$$x = f(x, \bar{y}, \overset{(n-2)}{y}) = f(x, \bar{y}, \overset{(n-3)}{y}, y^{<s>}) = (f(x, \bar{y}, \overset{(n-3)}{y}, y))^{<s>} = x^{<s>},$$

by (3) and **Lemma 2** . Therefore $t|s$. Similarly we obtain $y = y^{<t>}$ and $s|t$. Hence $s = t$, which proves that all elements have the same n-ary order. □

Theorem 1. *Any distributive n-group is a set-theoretic union of disjoint cyclic and isomorphic autodistributive n-groups without proper subgroups.*

Proof. Let $ord_n(x) = t$ and let C_x be an n-ary subgroup generated by x. Then $C_x = \{x, x^{<1>}, x^{<2>}, \ldots, x^{<t-1>}\}$. Since all elements have the same n-ary order, then C_x has no proper subgroups and any two subgroups C_x and C_y are isomorphic. Such subgroups are autodistributive by **Theorem 4** *from* [5]. (This fact follows also from our **Corollary 12**). □

Corollary 4. *A distrivutive n-group is idempotent or has no any idempotents.*

Theorem 2. *Let x be an arbitrary element of a distributive n-group (G,f). Then C_x is the normal subgroup of the retract $ret_x(G,f)$ and every coset of C_x in $ret_x(G,f)$ is an n-ary subgroup of (G,f) isomorphic to (C_x,f).*

Proof. Let $(G,\bullet) = ret_x(G,f)$, i.e. let $a \bullet b = f(a, \overset{(n-2)}{x}, b)$ for all $a,b \in G$ (cf. [9]). Then $x^{<k>} = x^{k+1}$ in $(G,\bullet)$. Moreover, $C_x = \{x, x^2, \ldots, x^t\}$ and C_x is a cyclic subgroup of the order $t = ord_n(x)$ in $(G,\bullet)$. It is normal, because by (2) and (3) we get

$$a \bullet x = f(a, \overset{(n-1)}{x}) = f(\bar{x}, \overset{(n-2)}{x}, f(a, \overset{(n-1)}{x})) =$$
$$= f(\overset{(n-1)}{x}, f(a, \overset{(n-2)}{x}, \bar{x})) = f(\overset{(n-1)}{x}, a) = x \bullet a$$

for all $a \in G$.

Moreover, for every $k = 1,2,\dots,t$ we have also

$$a \bullet x^{(k)} = f(a, \overset{(n-2)}{x}, \bar{x}^{(k)}) = f(\bar{a}^{(k-1)}, \overset{(n-2)}{x}, \bar{x}) = \bar{a}^{(k-1)},$$

which gives $a \bullet C_x = C_a$ for every $a \in G$. This completes our proof.

Since by **Corollary 2** $C_x = \left\{x, \bar{x}, \bar{x}^{(2)}, \dots, x^{(t-1)}\right\}$, where $t = ord_n(x)$. Then $\bar{x}^{(s)} = \bar{y}^{(s)}$ implies $\bar{x}^{(t-1)} = \bar{y}^{(t-1)}$ and, in the consequence, $x = y$. This proves that in a distributive n-group any endomorphism ϕ x $\bar{x}^{( )}$ is one-to-one and there is only t different endomorphisms ϕ_s. Obviously any such endomorphism is also "onto" because for every $x \in G$ there exists $y = \bar{x}^{(t-s)} \in G$ such that $x = \phi_s(y)$. Thus $\phi_0, \phi_1, \dots, \phi_{t-1}$ form a cyclic subgroup in the group $Aut(G,f)$ of all automorphism of (G,f). Since $\phi(\bar{x}) = \overline{\phi(x)}$ for all automorphisms of an arbitrary n-ary group , then this subgroup is invariant in the group $Aut(G,f)$. Obviously any ϕ_s is a splitting-automorphism in the sense of Plonka [13].

Thus we obtain the following result.

Proposition 2. *If (G,f) is a distributive n-group , then the operation $\bar{\ }$: $x \to \bar{x}$ induces the cyclic subgroup in the group $Aut(G,f)$ of automorphisms of (G,f). Moreover, this subgroup is invariant in the group $Aut(G,f)$ and in the group of all splitting-automorphisms of (G,f).*

From the above results it follows also that $G = E_s = G^{(s)}$ for any distributive n-group (G,f). Thus the class V_s of n-ary groups (G,f) such that $G = E_s$ (cf. [6], **Problem 4**) contains the class of distributive n-groups. The class of distributive n-groups is also contained in the class of all n-ary groups satisfying the descending chain condition for $G^{(s)}$ (cf. [6], **Problem 5**).

The class of all n-ary groups (for fixed n) is a variety (cf. [11], [8]). The class of all distributive n-groups is a subvariety of this variety. From **Theorem 1** it follows that any free n-group in this subvariety is a set-theoretic union of disjoint cyclic autodistributive n-groups with $n-1$ elements which have no proper subgroups, but in general this n-group is not autodistributive.

Theorem 3. *An n-ary group (G,f) is distributive iff it has the form*

$$f(x_1^n) = x_1 \bullet \theta x_2 \bullet \theta^2 x_3 \bullet \ldots \bullet \theta^{n-2} x_{n-1} \bullet x_n \bullet b, \tag{4}$$

where b is a fixed central element of a group $(G,\bullet)$ with the identity e, $b^{n-1} = e$, θ is an automorphism of $(G,\bullet)$, $\theta b = b$, $x \bullet \theta x \bullet \theta^2 x \bullet \ldots \bullet \theta^{n-2} x = e$ and $\theta^{n-1} x = x$ for all $x \in G$.

Proof. According to the well known **Gluskin-Hosszu theorem** any n-ary group has the form $(G,f) = der_{\theta,b}(G,\bullet)$ (cf. for example [9]), i.e. for any n-ary group (G,f) there exist a group $(G,\bullet)$, $b \in G$, and an automorphism θ of $(G,\bullet)$ such that $\theta b = b$, $\theta^{n-1} x = b \bullet x \bullet b^{-1}$ for all $x \in G$ and

$$f(x_1^n) = x_1 \bullet \theta x_2 \bullet \theta^2 x_3 \bullet \ldots \bullet \theta^{n-1} x_n \bullet b.$$

If θ and b are as in our theorem, then direct computations show that $\bar{x} = x \bullet b^{n-2}$ for all $x \in G$ and $(G,f) = der_{\theta,b}(G,\bullet)$ is a distributive n-group.

Conversely, if $(G,f) = der_{\theta,b}(G,\bullet)$ is a distributive n-group and e is the identity of $(G,\bullet)$, then (2) and (3) imply

$$x \bullet b = f(x, \overset{(n-1)}{e}) = f(\bar{e}, \overset{(n-2)}{e}, f(x, \overset{(n-1)}{e})) =$$
$$= f(\overset{(n-1)}{e}, f(x, \overset{(n-2)}{e}, \bar{e})) = f(\overset{(n-1)}{e}, x) = \theta^{n-1}x \bullet b,$$

which shows that θ^{n-1} is an identity mapping and $x \bullet b = b \bullet x$ for all $x \in G$.

Since $e = f(\bar{e}, \overset{(n-1)}{e}) = \bar{e} \bullet b$, then b^{-1} is skew to e. Thus

$$e = f(\bar{x}, \overset{(n-2)}{x}, e) = f(\overset{(n-1)}{x}, \bar{e}) = x \bullet \theta x \bullet \theta^2 x \bullet \ldots \bullet \theta^{n-2} x,$$

and in particular $e = b^{n-1}$, which completes our proof. □

Corollary 5. *If* $(G,f) = der_{\theta,b}(G,\bullet)$ *is a distributive* n*-group, then* $ord_n(x) = ord_2(b)$ *for all* $x \in G$.

Proof. Since $e = x \bullet \theta x \bullet \theta^2 x \bullet \ldots \bullet \theta^{n-2} x$ for all $x \in G$, then

$$x^{<k>} = f(x^{<k-1>}, \overset{(n-1)}{x}) = x^{<k-1>} \bullet \theta(x \bullet \theta x \bullet \theta^2 x \bullet \ldots \bullet \theta^{n-2} x) \bullet b =$$
$$= x^{<k-1>} \bullet b = \ldots = x \bullet b^k,$$

then $x^{<k>} = e$ iff $b^k = e$. Hence $ord_n(x) = ord_2(b)$. □

Corollary 6. *A distributive* n*-group* (G,f) *is idempotent iff* $(G,f) = der_{\theta,e}(G,\bullet)$.

Corollary 7. *An n-ary group* $(G,f)=der_{\xi,b}(G,\bullet)$, *where* ξ *is an identity mapping, is distributive iff the exponent of* $(G,\bullet)$ *divides* $n-1$.

Corollary 8. *A ternary group* (G,f) *is distributive iff there exist a commutative group* $(G,\bullet)$ *and an element* $b\in G$ *such that* $b=b^{-1}$ *and* $f(x,y,z)=x\bullet y^{-1}\bullet z\bullet b$.

Proof. If a ternary group $(G,f)=der_{\theta,b}(G,\bullet)$ is distributive, then $b^2=e$ and $x\bullet\theta x=e$. Hence $\theta x=x^{-1}$ and $(G,\bullet)$ is a commutative group. The converse is obvious. □

Corollary 9. *The class of all distributive 3-groups is a proper subvariety of a variety of medial 3-groups.*

Theorem 4. *For any* $n\geq 3$ *there exists a medial distributive n-group which is not derived from any group of the arity* $k<n$.

Proof. Let Z_p be the additive group of rests modulo $p=t^{n-1}-1$, where $t\geq 2$ and $(t-1)|(n-1)$. Then $\theta x\equiv tx(\text{mod}\, p)$ is an automorphism of the group Z_p such that $\theta^{n-1}x=x$ for all $x\in Z_p$ and $\theta b=b$ for $b=1+t+t^2+...+t^{n-2}$. The n-ary group $(G,f)=der_{\theta,b}(Z_p,+_p)$ is medial, because the creasing group Z_p is commutative (cf. [4], [6]). Since, in this n-group $\bar{x}\equiv x-b(\text{mod}\, p)$ for all $x\in Z_p$, then it is also distributive.

Suppose now that our n-ary group (Z_p, f) is derived from some k-ary group (Z_p, g). Then $n = s(k-1)+1$,

$$f(x_1^{s(k-1)+1}) = g(\ldots g(g(x_1^k), x_{k+1}^{2k+1}), \ldots, x_{(s-1)(k-1)+2}^{s(k-1)+1}), \tag{5}$$

and for $a = 0$ there exists $d \in Z_p$ (d is skew to 0 in (Z_p, g)) such that for all $b \in Z_p$ we have (cf. (3))

$$g(b, \overset{(k-2)}{0}, d) = g(d, \overset{(k-2)}{0}, b)$$

and

$$f(b, \overset{(n-2)}{0}, d) = g(d, \overset{(k-2)}{0}, b, \overset{(n-k)}{0}).$$

For $b = 1$ the last identity gives $(1+d) \equiv (d + t^{k-1}) (\mathrm{mod}\, p)$, i.e. $(t^{k-1} - 1) \equiv 0 (\mathrm{mod}\, p)$, which for $t \geq 2$ and $2 \leq k < n$ is impossible. Obtained contradiction proves that our n-group is not derived from any k-group of the arity $k < n$, which finish the proof □

Observe that for $t = 2$ the n-ary group constructed in the above proof is idempotent. Thus the following statement is true.

Corollary 10. *For any $n \geq 3$ there exists a medial idempotent distributive n-group which is not derived from any group of the arity $k < n$.*

4. Autodistributive n-groups

Any commutative autodistributive n-ary group (G,f) may be considered as an algebra (G,f,f) of type (n,n). In this case it is an (n,n)-ring in the sense of G.Cupona [2] and G.Crombez [1]. It is also a special case of (f/g)-algebras described by H.J.Hoehnke [12].

Since a commutative idempotent n-ary group is autodistributive, then for any natural $n \geq 3$ there exists an (n,n)-ring in which all elements are identities of this (n,n)-ring.

Theorem 5. *An n-group (G,f) is autodistributive iff it has the form*

$$f(x_1^n) = x_1 \bullet \theta x_2 \bullet \theta^2 x_3 \bullet \ldots \bullet \theta^{n-2} x_{n-1} \bullet x_n \bullet b,$$

where b is a fixed element of a commutative group $(G,\bullet)$ with the identity e, θ is an automorphism of $(G,\bullet)$ such that $\theta b = b$, $x \bullet \theta x \bullet \theta^2 x \bullet \ldots \bullet \theta^{n-2} x = e$ and $\theta^{n-1} x = x$ for all $x \in G$.

Proof. Direct computations show that any n-group $(G,f) = der_{\theta,b}(G,\bullet)$, where $(G,\bullet)$, θ and b are as in our theorem, is autodistributive.

Conversely, if (G,f) is an autodistributive n-group, then (by **Theorem 3** from [5]) it is also distributive and has the form described in our **Theorem 3**.

Moreover, the autodistributivity of (G,f) implies

$$\theta x \bullet b \bullet \theta y \bullet b = f(f(e,x,\overset{(n-2)}{e}),y,\overset{(n-2)}{e}) =$$

$$= f(f(e,y,\overset{(n-2)}{e}),f(x,y,\overset{(n-2)}{e}),f(e,y,\overset{(n-2)}{e}),\ldots,f(e,y,\overset{(n-2)}{e})) =$$

$$= \theta y \bullet b \bullet \theta x \bullet \theta^2 y \bullet b \bullet \ldots \bullet \theta^n y \bullet b \bullet b =$$

$$= \theta y \bullet b \bullet \theta x \bullet \theta^2 (y \bullet \theta y \bullet \ldots \bullet \theta^{n-2} y) \bullet b^n =$$

$$= \theta y \bullet b \bullet \theta x \bullet b,$$

which gives the commutativity of $(G,\bullet)$. This completes our proof. □

Corollary 11. *Any autodistributive n-group is medial.*

Comparing the above result and **Theorem 3** we obtain

Corollary 12. *A distributive n-group is autodistributive iff it is $<\theta, b>$-derived from a commutative group, i.e. iff it is medial.*

This together with **Corollary 7** gives the following characterization of autodistributive n-groups which are b-derived from a some binary group

Corollary 13. *An n-ary group $(G,f) = der_{\xi,b}(G,\bullet)$, where ξ is an identity mapping, is autodistributive iff the group $(G,\bullet)$ is commutative and its exponent divides $n-1$.*

Thus for $n<7$ all distributive n-groups b-derived from a some binary group are autodistributive. For $n \geq 7$ there are distributive b-derived n-groups which are not autodistributive. As an example of such 7-groups we may consider 7-groups b-derived from the symmetric group S_3.

Observe that **Corollaries 9** and **12** give the following connection between distributive and autodistributive 3-groups.

Corollary 14. *Any distributive 3-group is autodistributive and vice versa.*

Theorem 6. *For any $n>3$ there exists a non-reducible idempotent distributive n-group which is not autodistributive.*

Proof. Let C be the field of complex number. It is not difficult to verify that $G=C^3$ with the multiplication defined by the formula

$$(x,y,z)\bullet(a,b,c)=(x+a,y+b,z+c)$$

is a non-commutative group with the identity $e=(0,0,0)$. The map $\theta(x,y,z)=(\alpha x,\alpha^2 y,\alpha z)$, where α is a primitive $(n-1)$th root of unity, is an automorphism of $(G,\bullet)$ such that $\theta^{n-1}x=x$ and $x\bullet\theta x\bullet\theta^2 x\bullet\ldots\bullet\theta^{n-2}x=e$ for all $x\in G$. Thus an n-group $(G,f)=der_{\theta,e}(G,\bullet)$ is idempotent and distributive (**Theorem 3**). Obviously it is not autodistributive (**Corollary 12**).

Now we prove that this n-group is not derived from any group of the arity $k<n$. Indeed, if an n-group $(G,f)=der_{\theta,e}(G,\bullet)$ is derived from a some binary group with the identity c, then for all $x\in G$ we have

$$f(x,\overset{(n-1)}{c})=f(c,x,\overset{(n-2)}{c})=x,$$

which implies $x\bullet\theta c=c\bullet\theta x$. This for $x=e$ gives $\theta c=c$. Hence $c=e$ and $\theta x=x$ for every $x\in G$, which is incompatible with the definition of θ. Thus this n-group is not derived from a binary group.

If it is derived from some k-ary ($k>2$) group (G,g), then $n=s(k-1)+1$, $s\geq 2$ and (5) holds. Moreover, **Dornte**'s identities for (G,g) and (5) show that

$$f(\bar{x},\overset{(k-2)}{x},x,\overset{(k-2)}{x},\bar{x},\ldots,\overset{(k-2)}{x},\bar{x})=f(x,\bar{x},\overset{(k-3)}{x},x,\overset{(k-2)}{x},\bar{x},\ldots,\overset{(k-2)}{x},\bar{x})$$

for all $x\in G$, where $\bar{x}$ denotes the skew element in (G,g). Hence $\bar{x}\bullet\theta x=x\bullet\theta\bar{x}$, which for $x=e$ gives $\bar{e}=\theta\bar{e}$. Therefore $\bar{e}=e$ and $\theta x=x$, which is incompatible with the definition of θ. This contradiction completes the proof. □

Theorem 7. *There exist non-reducible and non-idempotent distributive n-groups which are not autodistributive.*

Proof. Let K be a fixed field of the characteristic $p\neq 0$. As in the proof of the previous theorem, it is not difficult to verify that $G=K^3$ with the multiplication

$$(x,y,z)\bullet(a,b,c)=(x+a,y+xc+b,z+c)$$

is a non-commutative group with the identity $e=(0,0,0)$. Moreover, for any natural $m\geq 2$ such that $p\nmid m$, the map $\theta(x,y,z)=(\alpha x,y,\beta z)$, where $\alpha\beta=1$ and α is a primitive mth root of unity of **K**, is an automorphism of $(G,\bullet)$ such that $\theta b=b$ for $b=(0,1,0)$. Since $\theta^m x=x$, $x\bullet\theta x\bullet\theta^2 x\bullet\ldots\bullet\theta^{n-2}x=e$ and $b\bullet x=x\bullet b$ for all $x\in G$ and $n=pm+1$, then an n-group $(G,f)=der_{\theta,b}(G,\bullet)$ is distributive but not autodistributive.

In a similar way as in the previous proof we can see that this n-group is not derived from any binary group. It is not derived from any k-ary ($k>2$) group, too. Indeed , if it is derived from a some k-ary group (G,g), then as in the previous proof $n=s(k-1)+1$, $s\geq 2$ and $\bar{x}\bullet\theta x=x\bullet\theta\bar{x}$ for all $x\in G$. From this identity it follows that $\bar{e}=(0,y,0)$ and $\bar{a}=(1,u,1)$ for $a=(1,0,1)$. Therefore

$$f(\overset{(k-2)}{a},\bar{a},e,\overset{(k-2)}{e},\bar{e},\overset{(k-2)}{e},\ldots,\overset{(k-2)}{e},\bar{e})=e \tag{6}$$

implies $0=1+\alpha+\alpha^2+\ldots+\alpha^{k-2}$. Thus $\alpha^{k-1}=1$ and $k-1=tm$, because α is a primitive mth root of unity. Hence $pm=n-1=stm$, and in the consequence $s=p$. Therefore $n=p(k-1)+1$. Thus from (6) for $a=e$ we obtain $e=(\bar{e})^p\bullet b$, which is impossible because $py+1=0$ has no any solutions in **K**. This contradiction proves that our n-ary group is not reducible to any k-group. This completes the proof. □

From the above proof it follows that non-reducible non-idempotent distributive n-groups which are not autodistributive exist for some $n\geq 7$. For $n=4,5,6$ this problem is open.

Corollary 15. *For any $n\geq 3$ there exist an autodistributive n-group which is not derived from any group of the arity $m<n$.*

Proof. Such n-groups are constructed in the proof of the **Theorem 4**. □

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