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THE INTERPRETATION AND EQUIVALENCE OF THE VARIETY OF n-GROUPS

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Abstract

In this paper the varieties of n-ary groups are studied from the point of view of the interpretation and equivalence.

In this paper we deal with the interpretation and equivalence of the variety of n-groups. First of all, we shall recall some definitions and theorems from the theory of universal algebras (cf. [4]). We use the letter A to denote the universe of the algebra $\underline{A}$. Let F be a similarity type of an algebra $\underline{A}$. The subset of all n-ary function symbols in F is denoted by F_n for $n = 0,1,2,\ldots$

Definition 1. Let F be a similarity type. Let $\omega = \{0,1,2,\ldots,n,\ldots\}$. By a term (of type F) we mean an element of the term algebra $T_F(\omega)$. We put $z_n = (n)$, and the terms z_n $(n \in \omega)$ are called *variables*.

For every $n \geq 0$ the term algebra $T_F(n)$ is the subalgebra of $T_F(\omega)$ generated by $Z_n = \{z_0, z_1, \ldots, z_{n-1}\}$. If $n = 0$, then $Z_0 = \varnothing$. If $F_0 = \varnothing$, then the algebra $T_F(0)$ does not exist.

Assume that $\underline{A}$ is an algebra of type F. If $p \in T_F(\omega)$ then $p^{\underline{A}}$ denotes the term operation of the algebra $\underline{A}$ determined by the term p. We use the symbols $\mathrm{Clo}(\underline{A})$ and $\mathrm{Pol}(\underline{A})$ to denote the clones of term operations and polynomial operations of an algebra $\underline{A}$, respectively.

Definition 2. Two algebras $\underline{A}$ and $\underline{B}$ are called *term equivalent* if

(i) $A = B$,

(ii) $\mathrm{Clo}(\underline{A}) = \mathrm{Clo}(\underline{B})$.

Definition 3. Two algebras $\underline{A}$ and $\underline{B}$ are called *polynomially equivalent* if

(i) $A = B$,

(ii) $\mathrm{Pol}(\underline{A}) = \mathrm{Pol}(\underline{B})$..

Definition 4. Let V and W be varieties of respective similarity types F and G. By an *interpretation of* V *in* W a mapping $D: F \to T_G(\omega)$ is meant satisfying the following conditions:

(D1) If $f \in F_n$ and $n > 0$, then $D(f) = f_D \in T_G(n)$;

(D2) If $f \in F_0$, then $D(f) = f_D \in T_G(1)$ and the equation $f_D(z_0) \approx f_D(z_1)$ is valid in W;

(D3) For every algebra $\underline{A} \in W$, the algebra $\underline{A}^D = (A, f_D^{\underline{A}}\ (f \in F))$ belongs to the variety V.

Definition 5. By an *equivalence of varieties* V *and* W a pair of interpretations (D, E) is meant satisfying the following conditions:

1) D is an interpretation of V in W;

2) E is an interpretation of W in V;

3) $\forall \underline{A} \in W: \quad \underline{A}^{DE} = \underline{A}$;

4) $\forall \underline{B} \in V: \quad \underline{B}^{ED} = \underline{B}$.

Now we present the fundamental definitions and theorems from the theory of n-groups (cf. [1], [5], [6]). For simplicity of notation, it will be convenient to abbreviate $x_1, \ldots, x_k$, as x_1^k. If $x_1 = x_2 = \ldots = x_k = x$, then we write x^k.

Definition 6. An algebra $\underline{A} = (A, f)$ endowed with an n-ary operation $f: A^n \to A$ $(n \geq 2)$ is called an *n-groupoid* (a *groupoid* for $n = 2$).

Definition 7. An n-groupoid $\underline{A} = (A, f)$ is said to be an *n-group* if the following conditions hold:

(1) for all $x_1, \ldots, x_{2n-1} \in A$ the associative law

$$f(x_1^{i-1}, f(x_i^{n+i-1}), x_{n+i}^{2n-1}) = f(x_1^{j-1}, f(x_j^{n+j-1}), x_{n+j}^{2n-1})$$

holds for all $i, j \in \{1, \ldots, n\}$;

(2) for every $k \in \{1, \ldots, n\}$ and for all $x_0, x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n \in A$ there exists an uniquely element $z_k \in A$ such that

$$f(x_1^{k-1}, z_k, x_{k+1}^n) = x_0.$$

Theorem 1 (cf. [2]). *An n-groupoid $\underline{A} = (A, f)$ is an n-group if and only if*

$$f(x_1^n) = x_1 \cdot \alpha(x_2) \cdot \alpha^2(x_3) \cdot \ldots \cdot \alpha^{n-2}(x_{n-1}) \cdot a \cdot x_n \tag{3}$$

for all $x_1, \ldots, x_n \in A$, where the following conditions hold:

(i) $(A, \cdot, ^{-1}, e)$ *is a group;*

(ii) $a \in A$ *is a fixed element;*

(iii) $\alpha \in Aut(A,\cdot,^{-1},e)$, $\alpha(a)=a$, $\alpha^{n-1}(x)=a\cdot x\cdot a^{-1}$ *for every* $x \in A$.

Definition 8. An algebra $\underline{A}^{(n)}=(A,\cdot,^{-1},e,\alpha,\alpha^{-1},a)$ fulfilling the conditions (3) and (i)-(iii) of **Theorem 1** is called an *α-algebra associated with the n-group* $\underline{A}=(A,f)$

For simplicity, this algebra will be denoted by $\underline{A}^{(n)}=(A,\cdot,\alpha,a)$.

From now on, we consider n-groups with $n \geq 3$.

Theorem 2 (cf. [1]). *If* $\underline{A}=(A,f)$ *is an n-group, then there is a unary operation* $x \to \bar{x}$ *on the set* A *such that*

$$f(\bar{x},x^{n-2},y)=y \quad \text{and} \quad f(y,x^{n-2},\bar{x})=y \tag{4}$$

for all $x,y \in A$. *Conversely, if a nonempty set A carries an n-ary operation* f *and a unary operation* $x \to \bar{x}$ *satisfying the conditions (1) and (4), then* $\underline{A}=(A,f)$ *is an n-group.*

In view of **Theorem 2**, an n-group will often be defined as an algebra $\underline{A}=(A,f,-)$ equipped with an n-ary operation f and an unary operation $x \to \bar{x}$ satisfying the conditions (1) and (4). The element $\bar{x}$ is usually called a *skew element*.

Let $\underline{A}=(A,f,-)$ be an n-group. Let $p \in A$ be an arbitrary fixed element. Using the Sokolov method (cf. [6]) we can construct an α-algebra $\underline{A}^{(n)}=(A,\cdot,\alpha,a)$ associated with the n-group $\underline{A}=(A,f,-)$ in the following way.

Define the binary operation according to the formula

$$x\cdot y=f(x,p^{n-2},y)$$

for all $x, y \in A$. Let $\alpha: A \to A$ be the mapping defined by

$$\alpha(x) = f(\overline{p}, x, p^{n-2})$$

for $x \in A$. Finally, let us take

$$a = f(\overline{p}^{n}).$$

The α-algebra $\underline{A}^{(n)} = (A, \cdot, \alpha, a)$ associated with the n-group $\underline{A} = (A, f, -)$ and constructed by means of the above method will be denoted by $\underline{A}_p^{(n)} = (A, \cdot, \alpha, a)$.

From **Theorem 2** it follows that the class of all n-groups is a variety which will be denoted by V_n. It is easy to verify that

$$p(x, y, z) = f(x, \overline{y}, y^{n-3}, z)$$

is a Malcev term for V_n. Thus the variety V_n is congruence-permutable.

For $n \geq 3$ we define an algebra

$$\underline{A}^{(n)} = (A, \cdot, ^{-1}, e, \alpha, \alpha^{-1}, a) \tag{5}$$

fulfilling the following conditions:

(i) $(A, \cdot, ^{-1}, e)$ is a group;

(ii) $\forall x, y \in A: \; \alpha(x \cdot y) = \alpha(x) \cdot \alpha(y)$;

(iii) $\forall x \in A: \; \alpha(\alpha^{-1}(x)) = \alpha^{-1}(\alpha(x)) = x$;

(iv) $\alpha(a) = a$;

(v) $\forall x \in A: \; \alpha^{n-1}(x) = a \cdot x \cdot a^{-1}$.

For simplicity, this algebra will be denoted by $\underline{A}^{(n)} = (A, \cdot, \alpha, a)$.

Definition 9. The algebra $\underline{A}^{(n)}$ defined by (5) is called an *α-algebra.*

For an arbitrary fixed $n \geq 3$ the class of all α-algebras $\underline{A}^{(n)}$ is a Malcev variety and it will be denoted by W_n.

Theorem 3. *The variety* V_n *is interpretable into the variety* W_n.

Proof. Let us consider the similarity types $F = \{f, -\}$ and $G = \{\cdot, ^{-1}, e, \alpha, \alpha^{-1}, a\}$ of the varieties V_n and W_n, respectively. We define the mapping $D: F \to T_G(\omega)$ according to the formulas:

$$D(f) = z_1 \cdot \alpha(z_2) \cdot \alpha^2(z_3) \cdot \ldots \cdot \alpha^{n-2}(z_{n-1}) \cdot a \cdot z_n$$

where $z_1, z_2, \ldots, z_n \in T_G(n)$ are variables, and

$$D(-) = a^{-1} \cdot \alpha^{n-2}(z^{-1}) \cdot \ldots \cdot \alpha^2(z^{-1}) \cdot \alpha(z^{-1})$$

where $z \in T_G(1)$ is a variable. It follows immediately that conditions (D1) and (D2) of **Definition 4** are satisfied. Assume that $\underline{A} = (A, \cdot, ^{-1}, e, \alpha, \alpha^{-1}, a) \in W_n$. Let us take

$$f(x_1^n) = x_1 \cdot \alpha(x_2) \cdot \alpha^2(x_3) \cdot \ldots \cdot \alpha^{n-2}(x_{n-1}) \cdot a \cdot x_n,$$

$$\overline{x} = a^{-1} \cdot \alpha^{n-2}(x^{-1}) \cdot \ldots \cdot \alpha^2(x^{-1}) \cdot \alpha(x^{-1})$$

for all $x, x_1, \ldots, x_n \in A$. It is easy to check that $\underline{A}^D = (A, f, -) \in V_n$. Consequently, the condition (D3) holds.

Corollary 1. *For every α-algebra* $\underline{A} \in W_n$ *there exists an n-group* $\underline{A}^D \in V_n$ *such that* $\mathrm{Clo}(\underline{A}^D) \subset \mathrm{Clo}(\underline{A})$.

The variety W_n is not interpretable into the variety V_n.

Example 1. Consider the **Klein** group with the universe $A = \{e, a, b, c\}$. The mapping

$$\alpha(e) = e, \quad \alpha(a) = b, \quad \alpha(b) = a, \quad \alpha(c) = c$$

is an automorphism of the above group. We define the 3-group $\underline{A} = (A, f, {}^{-})$ as follows

$$f(x_1, x_2, x_3) = x_1 \alpha(x_2) x_3$$

for all $x_1, x_2, x_3 \in A$. The 3-group $\underline{A}$ has the two different one-element 3-subgroups with the subinverses $\{e\}$ and $\{c\}$. On the other hand, there are no α-algebras of the variety W_3 with two different one-element α-subalgebras.

Definition 10. Let us consider an algebra $\underline{A} = (A, f, {}^{-}, p)$ fulfilling the following conditions:

(i) the reduct $(A, f, {}^{-})$ is an n-group,

(ii) p is a constant 0-ary operation such that $\bar{p} = p$.

An algebra $\underline{A} = (A, f, {}^{-}, p)$ is called an *n-group with constant.*

The class V_n^0 of all n-groups with constant is a Malcev variety.

Let W_n^0 be a class of all α-algebras $\underline{A}^{(n)} = (A, \cdot, {}^{-1}, e, \alpha, \alpha^{-1}, a)$ of the variety W_n such that $a = e$. In this case we write $\underline{A}^{(n)} = (A, \cdot, {}^{-1}, e, \alpha, \alpha^{-1})$. Since any α-algebra $\underline{A}^{(n)} = (A, \cdot, {}^{-1}, e, \alpha, \alpha^{-1})$ belongs to the class of Ω-groups (cf. [3]) with $\Omega = \{\alpha, \alpha^{-1}\}$, it will be referred to as an α-group. The class W_n^0 is a Malcev variety.

Theorem 4. *The varieties V_n^0 and W_n^0 are equivalent.*

Proof. Let $F=\{f,^{-},p\}$ and $G=\{\cdot,^{-1},e,\alpha,\alpha^{-1}\}$ be similarity types of the varieties V_n^0 and W_n^0, respectively. We define the mapping $D: F\to T_G(\omega)$ as follows:

$$D(f)=z_1\cdot\alpha(z_2)\cdot\alpha^2(z_3)\cdot\ldots\cdot\alpha^{n-2}(z_{n-1})\cdot z_n,$$

where $z_1,\ldots,z_n\in T_G(n)$ are variables,

$$D(^{-})=\alpha^{n-2}(z^{-1})\cdot\ldots\cdot\alpha^2(z^{-1})\cdot\alpha(z^{-1}),$$

$$D(p)=z\cdot z^{-1},$$

where $z\in T_G(1)$ is a variable.

The conditions (D1) and (D2) are satisfied. Suppose that $\underline{A}^{(n)}=(A,\cdot,^{-1},e,\alpha,\alpha^{-1})\in W_n^0$. Let us put:

$$f(x_1^n)=x_1\cdot\alpha(x_2)\cdot\alpha^2(x_3)\cdot\ldots\cdot\alpha^{n-2}(x_{n-1})\cdot x_n,$$

$$\bar{x}=\alpha^{n-2}(x^{-1})\cdot\ldots\cdot\alpha^2(x^{-1})\cdot\alpha(x^{-1}),$$

$$p=e$$

for all $x,x_1,\ldots,x_n\in A$. It is easy to check that $(\underline{A}^{(n)})^D=(A,f,^{-},e)\in V_n^0$. Thus, D is an interpretation of V_n^0 in W_n^0. Next, we define the mapping $E: G\to T_F(\omega)$ as follows:

$$E(\cdot)=f(z_1,p^{n-2},z_2)$$

where $z_1,z_2\in T_F(2)$ are variables,

$$E(^{-1})=f(p,\bar{z},z^{n-3},p),$$

$$E(e)=f(\bar{z},z^{n-2},p),$$

$$E(\alpha)=f(p,z,p^{n-2}),$$

$$E(\alpha^{-1})=f(p^{n-2},z,p)$$

where $z\in T_F(1)$ is a variable.

The mapping E satisfies the conditions (D1) and (D2).

Suppose that $\underline{A}=(A,f,^{-},p)\in V_n^0$. Let us put:

$$x_1\cdot x_2=f(x_1,p^{n-2},x_2),$$

$$x^{-1} = f(p, \bar{x}, x^{n-3}, p),$$
$$e = p,$$
$$\alpha(x) = f(p, x, p^{n-2}),$$
$$\alpha^{-1}(x) = f(p^{n-2}, x, p)$$

for all $x, x_1, x_2 \in A$.

Using the Sokolov method it is easy to check that $\underline{A}^E = (A, \cdot, ^{-1}, e, \alpha, \alpha^{-1}) \in W_n^0$. Thus E is an interpretation of W_n^0 in V_n^0. A straightforward computation shows that $(\underline{A}^{(n)})^{DE} = \underline{A}^{(n)}$ and $\underline{A}^{ED} = \underline{A}$. The pair of interpretations (D, E) is an equivalence of varieties V_n^0 and W_n^0.

As an immediate consequence of **Theorem 4** we obtain the following corollary.

Corollary 2. *If* $\underline{A}$ *is an n-group with constant* p, *then the algebras* $\underline{A}$ *and* $\underline{A}_p^{(n)}$ *are term equivalent.*

Proposition 1. Let $\underline{A} = (A, f, \bar{\ }, p)$ *be a 3-group with an arbitrary fixed constant* p *(not necessarily* $p = \bar{p}$*). The algebras* $\underline{A}$ *and* $\underline{A}_p^{(3)} = (A, \cdot, \alpha, a)$ *are term equivalent.*

Proof. Since

$$x \cdot y = f(x, p, y),$$

$$x^{-1} = f(\overline{p}, \overline{x}, \overline{p}),$$
$$e = \overline{p},$$
$$\alpha(x) = f(\overline{p}, x, p),$$
$$\alpha^{-1}(x) = f(p, x, \overline{p}),$$
$$a = f(\overline{p}, \overline{p}, \overline{p})$$

for all $x, y \in A$, we conclude that $\mathrm{Clo}(\underline{A}_p^{(3)}) \subset \mathrm{Clo}(\underline{A})$. We have

$$f(x_1, x_2, x_3) = x_1 \cdot \alpha(x_2) \cdot a \cdot x_3,$$
$$\overline{x} = a^{-1} \cdot \alpha(x^{-1}),$$
$$p = a^{-1}$$

for all $x, x_1, x_2, x_3 \in A$. Consequently, $\mathrm{Clo}(\underline{A}) \subset \mathrm{Clo}(\underline{A}_p^{(3)})$. □

On the whole, an n-group $\underline{A}$ with an arbitrary fixed constant p (not necessarily $p = \overline{p}$) is not term equivalent to an α-algebra $A_p^{(n)}$ for $n > 3$. Consider a suitable example.

Example 2. Let Z_4 be the additive group of the integers modulo 4. Consider the 4-group $\underline{A} = (Z_4, f, -, 1)$ with constant 1 defined according to the formula

$$f(x_1^4) = x_1 + x_2 + x_3 + x_4 + 2$$

for all $x_1, x_2, x_3, x_4 \in Z_4$. Define the mapping

$$\varphi(0) = 0, \quad \varphi(1) = 3, \quad \varphi(2) = 2, \quad \varphi(3) = 1.$$

The mapping φ is an automorphism of the algebra $\underline{A}_1^{(4)} = (Z_4, +, \mathrm{id}_{Z_4}, 2)$, but it is not an automorphism of the 4-group $\underline{A}$. Thus the algebras $\underline{A}$ and $\underline{A}_1^{(4)}$ are not term equivalent.

Proposition 2. *Let* $\underline{A} = (A, f, {}^{-})$ *be an* n*-group, and let* $\underline{A}^{(n)} = (A, \cdot, \alpha, a)$ *be an* α*-algebra associated with the* n*-group* $\underline{A}$*. Then the algebras* $\underline{A}$ *and* $\underline{A}^{(n)}$ *are polynomially equivalent.*

Proof. Since

$$f(x_1^n) = x_1 \cdot \alpha(x_2) \cdot \alpha^2(x_3) \cdot \ldots \cdot \alpha^{n-2}(x_{n-1}) \cdot a \cdot x_n,$$
$$\bar{x} = a^{-1} \cdot \alpha^{n-2}(x^{-1}) \cdot \ldots \cdot \alpha^2(x^{-1}) \cdot \alpha(x^{-1}),$$

for all $x, x_1, \ldots, x_n \in A$, we conclude that $\mathrm{Pol}(\underline{A}) \subset \mathrm{Pol}(\underline{A}^{(n)})$. Note that $\bar{e} = a^{-1}$. It is easy to prove that

$$x \cdot y = f(x, e^{n-3}, \bar{e}, y),$$
$$x^{-1} = f(e, \bar{x}, x^{n-3}, e),$$
$$\alpha(x) = f(e, x, e^{n-3}, \bar{e}),$$
$$\alpha^{-1}(x) = f(\bar{e}, e^{n-3}, x, e)$$

for all $x, y \in A$. Therefore $\mathrm{Pol}(\underline{A}^{(n)}) \subset \mathrm{Pol}(\underline{A})$. □

Corollary 3. *Let* $\underline{A} = (A, f, {}^{-})$ *be an* n*-group, and let* $\underline{A}^{(n)} = (A, \cdot, \alpha, a)$ *be an* α*-algebra associated with the* n*-group* $\underline{A}$*. Then there exists a constant* $p \in A$ *such that the algebra* $\underline{A}' = (A, f, {}^{-}, p)$ *is term equivalent to the* α*-algebra* $\underline{A}^{(n)}$*.*

It is sufficient to take $p = e$. □

References

1. ***W.A. Dudek, K. Glazek, B. Gleichgewicht***, *A note on the axioms of n-groups*, Colloq. math. Soc. J. Bolyai, 29, Universal Algebra, Esztergom 1977, p. 195-202.

2. ***M. Hosszu***, *On the explicit form of n-group operations*, Publ. Math., Debrecen, **10**(1963), p. 88-92.

3. ***A.G. Kurosh***, *Lectures on General Algebra*, Chelsea, New York, 1963.

4. ***R.N. McKenzie, G.F. McNulty, W.F. Taylor***, *Algebras, lattices, varieties*, vol. **1**, The Wadsworth and Brooks/Cole Mathematics Series, Monterey, California, 1987.

5. ***J.D. Monk, F.M. Sioson***, *On the general theory of m-groups*, Fund. Math. **72**(1971), p. 233-244.

6. ***E.I. Sokolov***, *On the Gluskin-Hosszu theorem on the Dornte n-groups* (Russian) Mat. Issled. **39**(1976), p. 187-189.

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