Sharply 2-transitive permutation groups. 1

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Abstract

In this article a sharply 2-transitive permutation groups on some set E (finite or infinite) are studied.

Sharply 2-transitive permutation groups were described by Zassenhaus in [1,2]. He proved (see [3] too), for example, that sharply 2-transitive permutation group G on a finite set of symbols E is a group G^* of linear transformations of some near-field $< E,+,\bullet >$:

$$G^* = \{\alpha_{a,b} \mid \alpha_{a,b}(t) = a \cdot t + b,\ a \neq 0,\ a,b,t \in E\}.$$

In the case when the set E is infinite, the problem of description of sharply 2-transitive permutation groups on E is opened. Some investigations in this direction were pursued in [4,5,6,7]. The same problem was formulated by Mazurov in [8, № 11.52].

In this work we try to describe some new approach to problem mentioned above by means of transversals in groups. Necessary definitions and propositions may be found in [9] and in the author's article [13] in this issue too.

§1. Preliminary lemmas and a partition on cases

Let G be a sharply 2-transitive permutation group on an arbitrary set E.

Lemma 1. *All elements of order 2 from G are in one and the same class of conjugate elements.*

Proof was given in [3]. □

Since G is a sharply 2-transitive permutation group, then only the identity permutation **id** fixes more than one symbol from E. So we obtain the following two cases:

<u>Case 1.</u> *Every element of order 2 from G is a fixed-point-free permutation on E.*

<u>Case 2.</u> *Every element of order 2 from G has exactly one fixed point from E.*

Lemma 2. *Let α and β be distinct elements of order 2 from G. Then the permutation $\gamma = \alpha\beta$ is a fixed-point-free permutation on E.*

Proof was given in [3]. □

Let 0 and 1 be some distinguished distinct elements from E. Denote

$$H_0 = St_0(G).$$

§2. A loop transversal in group G and its properties

Lemma 3. *In both of cases 1 and 2 there exists a left transversal T in G to H_0, which consists from* **id** *and elements of order 2.*

Proof. By the definition (see [9,10] a complete system T of representatives of the left (right) cosets in G to H_0 is called a *left (right) transversal in G to H_0*.

If case 1 takes place, then we define the following set of permutations from G

$$T=\{t_j\}_{j\in E};$$

$$t_j=\begin{pmatrix} 0 & j & \dots \\ j & 0 & \dots \end{pmatrix}, \text{ if } j\neq 0; \tag{1}$$

$$t_0=\mathrm{id}.$$

Then T is a left transversal in G to H_0 and for any $j\neq 1$

$$t_j^2=\begin{pmatrix} 0 & j & \dots \\ 0 & j & \dots \end{pmatrix}=\mathrm{id},$$

since only the identity permutation **id** fixes more than one symbol from E. So all nonidentity elements from T have order 2.

Let the case 2 takes place. Note (see proof in [3]), that for any given $i_0\in E$ there exists an unique element $\alpha\in G$ of order 2 such that $\alpha(i_0)=i_0$. So there exists an unique element $\alpha_0\in G$ of order 2 such that $\alpha(0)=0$; moreover, $\alpha_0\in H_0$. Then define the following set

$$T=\{t_j\}_{j\in E};$$

$$t_j=\begin{pmatrix} 0 & j & \dots \\ j & 0 & \dots \end{pmatrix}, \text{ if } j\neq 0; \tag{2}$$

$$t_0=\alpha_0.$$

Then T is a left transversal in G to H_0 and further proof is analogous to the same in case 1. □

Lemma 4. *Transversal* T *is a normal (invariant) subset in the group* G.

Proof. Let case 1 takes place. We have for any $j \in E$ and $\pi \in G$

$$\pi t_j \pi^{-1} = t_k h, \tag{3}$$

where $k \in E, h \in H_0$ (since T is a left transversal in G to H_0). If $j = 0$, then

$$\pi t_0 \pi^{-1} = \pi \cdot \mathrm{id} \cdot \pi^{-1} = \mathrm{id} = t_0.$$

If $j \neq 0$, then we have from (3)

$$h = t_k^{-1} \cdot (\pi t_j \pi^{-1}). \tag{4}$$

By means of **Lemma 2** we obtain: product in the right part of (4) has to be equal to **id**. Then we obtain

$$h = t_k^{-1} \cdot (\pi t_j \pi^{-1}) = \mathrm{id},$$

since $h \in H_0$. Then for any $j \in E$ and $\pi \in G$ we have from (3)

$$\pi t_j \pi^{-1} = t_k,$$

i.e.

$$\pi T \pi^{-1} \subseteq T.$$

From the last equality we have

$$T \subseteq \pi^{-1} T \pi = \pi' T \pi'^{-1},$$

where $\pi' = \pi^{-1} \in G$. So, for any $\pi \in G$ we have

$$T \subseteq \pi T \pi^{-1},$$

i.e.

$$T = \pi T \pi^{-1}.$$

Then T is a normal subset in G.

Proof in case 2 is analogous to that in case 1. □

Lemma 5. *Set T is:*

a loop transversal in G to H_0 in case 1;

a stable transversal [10] in G to H_0 in case 2.

Proof. As we can see from **Lemma 4**,

$$T = \pi T \pi^{-1}$$

for any $\pi \in G$. Then for any $\pi \in G$ the set $T^{\pi} = \pi T \pi^{-1} = T$ is a left transversal in G to H_0. So by means of [10, **theorem 2.1**] we obtain that T is a loop (correspondingly, stable) transversal in G to H_0. □

We can correctly introduce (see [9,10]) the following operation on the set E:

$$i \cdot j = k \overset{def}{\Leftrightarrow} t_i t_j = t_k h, \quad h \in H_0.$$

Then we obtain from **Lemma 5** (see [9,10] too) that the system $<E, \cdot, 0>$ is:

a loop with the identity element 0 in case 1;

a quasigroup with the right identity element 0 in case 2.

Lemma 6. *Let's define the following permutation representation $\hat{G}$ of a group G by the left cosets to H_0 with the help of a left transversal in G to H_0:*

$$\hat{g}(x) = y \overset{def}{\Leftrightarrow} g t_x H_0 = t_y H_0$$

Then we have

$$\hat{G} \cong G \quad and \quad \hat{g}(x) = g(x)$$

for any $x \in E$.

Proof. Let all conditions of the **Lemma** hold. Then we have

$$\hat{g}(u) = v,$$
$$gt_u H_0 = t_v H_0,$$
$$gt_u = t_v h, \quad h \in H_0,$$
$$gt_u(0) = t_v h(0) = t_v(0),$$
$$g(u) = v,$$

i.e. $\hat{g}(u) = g(u)$ for any $u \in E$. So the reflection $\varphi: \hat{g} \to g$ is an isomorphism between groups $\hat{G}$ and G. □

Lemma 7. *The following identities hold on* $<E,\cdot,0>$:

1. $x \cdot x = 0$;
2. $x \cdot (x \cdot y) = y$;
3. $x/y = y/x$;
4. $x \cdot (y \cdot (x \cdot z)) = (x \cdot (y \cdot x)) \cdot z$; *(left **Bol** identity)*
5. *System* $<E,\cdot,0>$ *is a left G-quasigroup.*

Proof. All definitions see in [11].

1. We have for any $x \in E$

$$t_0 = \mathrm{id} = t_x^2 = t_x t_x = t_{x \cdot x} h, \quad h \in H_0,$$

i.e. $x \cdot x = 0$.

2. We have for any $x, y \in E$

$$t_x t_y = t_{x \cdot y} h, \quad h \in H_0;$$

$$t_y = t_x^{-1} t_{x\cdot y} h = t_x t_{x\cdot y} h = t_{x\cdot(x\cdot y)} h', \quad h' \in H_0;$$

i.e. $x \cdot (x \cdot y) = y$.

3. We have

$$x \cdot (x \cdot y) = y.$$

Since system $<E,\cdot,0>$ is a quasigroup (in both of cases 1 and 2) then we can replace: $x = z/y$. Then we obtain for any $y, z \in E$

$$(z/y) \cdot z = y;$$

$$z/y = y/z;$$

4. Let us denote

$$h_{x,y} \overset{def}{=} t_{x\cdot y}^{-1} t_x t_y = t_{x\cdot y} t_x t_y.$$

Therefore

$$h_{x,y}^{-1} = (t_{x\cdot y} t_x t_y)^{-1} = t_y^{-1} t_x^{-1} t_{x\cdot y}^{-1} = t_y t_x t_{x\cdot y}.$$

Then we obtain by **Lemma 6** and [9]:

$$h_{x,y}(u) = t_{x\cdot y} t_x t_y(u) = (x \cdot y) \cdot (x \cdot (y \cdot u)); \tag{5}$$

$$h_{x,y}^{-1}(u) = t_y t_x t_{x\cdot y}(u) = y \cdot (x \cdot ((x \cdot y) \cdot u)) \tag{6}$$

for any $u \in E$. From **Lemma 4** we obtain for any $x, y \in E$

$$t_x t_y t_x = t_x t_y t_x^{-1} = t_z,$$

where

$$z = t_z(0) = t_x t_y t_x(0) = x \cdot (y \cdot x)$$

Now we have

$$t_{x\cdot(y\cdot x)} = t_x t_y t_x = t_x t_{y\cdot x} h_{y,x} = t_{x\cdot(y\cdot x)} h_{x,y\cdot x} h_{y,x}.$$

So

$$h_{x,y\cdot x} = h_{y,x}^{-1}. \tag{7}$$

From (5)-(7) it follows that for any $x, y, u \in E$

$$(x \cdot (y \cdot x)) \cdot (x \cdot ((y \cdot x) \cdot u)) = x \cdot (y \cdot ((y \cdot x) \cdot u)).$$

Since system $<E,\cdot,0>$ is a quasigroup (in both of cases 1 and 2) then we can replace: $w=(y\cdot x)\cdot u$. Then we obtain

$$(x\cdot(y\cdot x))\cdot(x\cdot w)=x\cdot(y\cdot w).$$

Finally, we can replace: $z=x\cdot w$. Then we have for any $x,y,z\in E$

$$(x\cdot(y\cdot x))\cdot z=x\cdot(y\cdot(x\cdot z),$$

since

$$x\cdot z=x\cdot(x\cdot w)=w,$$

(see 2.). We proved that the system $<E,\cdot,0>$ is:

left Bol loop in case 1;

left Bol quasigroup in case 2;

(see definitions in [11, 12]).

5. We have for any $a\in E$

$$t_x t_y=t_{x\cdot y}h,\quad h\in H_0;$$

$$t_a t_x t_y t_a^{-1}=t_a t_{x\cdot y} h t_a^{-1},\quad h\in H_0;$$

$$t_a t_x t_a^{-1}\cdot t_a t_y t_a^{-1}=t_a t_{x\cdot y} t_a^{-1}\cdot t_a h t_a^{-1},\quad h\in H_0,$$

$$t_{a\cdot(x\cdot a)} t_{a\cdot(y\cdot a)} t_a=t_{a\cdot((x\cdot y)\cdot a)} t_a h,\quad h\in H_0;$$

$$\varphi_a(x)\cdot R_a(\varphi_a(y))=R_a(\varphi_a(x\cdot y)),$$

where

$$\varphi_a(u)=a\cdot(u\cdot a)\qquad(*)$$

is a permutation on E. Then φ_a is a left pseudoautomorphism with the companion a. Moreover, any element $a\in E$ is a companion of the left pseudoautomorphism φ_a of the form (*); i.e. system $<E,\cdot,0>$ is a left G-quasigroup [11]. □

Let us return to the subgroup H_0 of the group G. Any nonidentity element $h\in H_0$ is a fixed-point-free permutation on the set $E-\{0\}$. Moreover, since G is a sharply 2-transitive permutation group on E, then H_0 is a sharply transitive permutation group on

$E-\{0\}$. So for a distinguished element $1 \in E\ (1 \neq 0)$ and for any $j \in E-\{0\}$ there exists an unique element $h_j \in H_0$ such that $h_j(1) = j$. Then we can define correctly the following operation on E:

$$i * j \overset{def}{=} k \Leftrightarrow h_i(j) = k, \quad if\ i \neq 0;$$
$$0 * j \overset{def}{=} 0; \tag{8}$$

Lemma 8. *The following statements are true:*

1. $x*0 = 0; \quad x*1 = 1*x = x;$
2. $< E-\{0\}, *, 1 > \cong H_0;$
3. $x*(y \cdot z) = (x*y)\cdot(x*z);$
4. *The system* $< E, \cdot, 0 >$ *is a left special quasigroup.*

Proof. Nesessary definitions are in [11].

1. We have for any $x \in E-\{0\}$

$$x*0 = u \Leftrightarrow u = h_x(0) = 0 \Rightarrow x*0 = 0,$$
$$x*1 = v \Leftrightarrow v = h_x(1) = x \Rightarrow x*1 = x;$$
$$1*x = w \Leftrightarrow w = h_1(x);$$

But $h_1(1) = 1$, since $h_1 \equiv \mathrm{id}$. So we obtain

$$w = h_1(x) = x,$$

i.e. $1*x = x$.

2. Let us define the following reflection

$$\varphi: E-\{0\} \to H_0,$$
$$\varphi(x) \overset{def}{=} h_x.$$

It is easy to see that φ is a bijection; moreover

$$\varphi(x)\varphi(y) = h_x h_y = h_z = \varphi(z).$$

where

$$z = h_z(1) = h_x h_y(1) = h_x(y) = x * y,$$

i.e.

$$\varphi(x)\varphi(y) = \varphi(x * y),$$

and φ is an isomorphism.

3. Since T is a normal subset in G (see **Lemma 4**) then for any $i \in E$ and $h_u \in H_0$ we obtain

$$h_u t_i h_u^{-1} = t_k,$$

where

$$k = t_k(0) = h_u t_i h_u^{-1}(0) = h_u t_i(0) = h_u(i) = u * i.$$

So, for any $i \in E$ and $u \in E - \{0\}$ we obtain

$$h_u t_i h_u^{-1} = t_{u*i}. \tag{9}$$

Then we have for any $u \in E - \{0\}$:

$$t_x t_y = t_{x \cdot y} h, \quad h \in H_0,$$

$$h_u t_x t_y h_u^{-1} = h_u t_{x \cdot y} h h_u^{-1}, \quad h \in H_0,$$

$$h_u t_x h_u^{-1} h_u t_y h_u^{-1} = h_u t_{x \cdot y} h_u^{-1} h_u h h_u^{-1}, \quad h \in H_0,$$

$$t_{u*x} t_{u*y} = t_{u*(x \cdot y)} h', \quad h' \in H_0,$$

$$(u * x) \cdot (u * y) = u * (x \cdot y). \tag{10}$$

Finally,

$$(0 * x) \cdot (0 * y) = 0 \cdot 0 = 0 = 0 * (x \cdot y).$$

4. We can write the equality (10) in the following form

$$h_u(x) \cdot h_u(y) = h_u(x \cdot y)$$

for any $u, x, y \in E$. So, any permutation $h_u \in H_0$ is an automorphism of the system $<E, \cdot, 0>$. Then the permutation $h_{x,y}$ (see (2)) is an automorphism of the system $<E, \cdot, 0>$ for any $x, y \in E$. Since

$$h_{x,y} \equiv L_{x \cdot y}^{-1} L_x L_y,$$

then system $<E, \cdot, 0>$ is a left special quasigroup. □

Lemma 9.

$$G = \{\alpha_{a,b} \mid \alpha_{a,b}(x) = a \cdot ((a \cdot b) * x),\ a \neq b,\ a,b \in E\}.$$

Proof. Since T is a left loop transversal (in case 1) or a stable transversal (in case 2) in G to H_0, then we can respresent any element $g \in G$ in the form

$$g = t_a h_c = t_a h_{a \cdot b},$$

where $t_a \in T,\ h_{a \cdot b} \in H_0,\ a \neq b$. So we obtain for any $x \in E$

$$g(x) = t_a h_{a \cdot b}(x) = t_a((a \cdot b) * x) = a \cdot ((a \cdot b) * x) \equiv \alpha_{a,b}(x);$$

moreover,

$$g(0) = a \cdot 0 = a, \qquad g(1) = a \cdot (a \cdot b) = b,$$

i.e.

$$g(x) = \alpha_{g(0),g(1)}(x). \tag{11}$$

□

Let us define the following two operations on E:

$$(x,a,y) \overset{def}{=} x \cdot ((x \cdot y) * a) = \alpha_{x,y}(a), \tag{12}$$

$$(x,\infty,y) \overset{def}{=} x \cdot y. \tag{13}$$

Definition. [11] Two operations $< E, \cdot >$ and $< E, \bullet >$ are called *orthogonal*, if the system

$$\begin{cases} x \cdot y = a, \\ x \bullet y = b \end{cases}$$

has an unique solution in $E \times E$ for any given $a,b \in E$.

Lemma 10. *The following statements are true:*

1).
$$(x,0,y)=x;\quad (x,1,y)=y;$$
$$(x,t,x)=x;\quad (0,t,1)=t;$$
$$(x,\infty,0)=x;\quad (x,\infty,x)=0;$$

2). *The operations* (x,a,y) *and* (x,∞,y) *are orthogonal for any* $a\in E$;

3). *The operations* (x,a,y) *and* (x,b,y) *are orthogonal for any given* $a,b\in E, a\neq b$.

Proof. 1). We have

$$(x,0,y)=x\cdot 0=x;$$
$$(x,1,y)=x\cdot(x\cdot y)=y;$$
$$(x,\infty,0)=x\cdot 0=x;$$
$$(x,t,x)=x\cdot(0*t)=x\cdot 0=x;$$
$$(x,\infty,x)=x\cdot x=0;$$
$$(0,t,1)=0\cdot((0\cdot 1)*t)=\alpha_{0,1}(t).$$

But by means of (11) we obtain

$$(0,t,1)=\alpha_{0,1}(t)=\mathrm{id}(t)=t.$$

2). Let a be an arbitrary given element from E. Then we have for any given $b,c\in E$:

$$\begin{cases}(x,a,y)=b;\\(x,\infty,y)=c;\end{cases}\Leftrightarrow\begin{cases}x\cdot((x\cdot y)*a)=b;\\ x\cdot y=c;\end{cases}\Leftrightarrow$$
$$\Leftrightarrow\begin{cases}x\cdot(c*a)=b;\\ y=x\cdot c;\end{cases}\Leftrightarrow\begin{cases}x={}^{b}\!/_{(c*a)};\\ y=({}^{b}\!/_{(c*a)})\cdot c;\end{cases}$$

i.e. there exists an unique solution $(x,y)=({}^{b}\!/_{(c*a)},({}^{b}\!/_{(c*a)})\cdot c)$ of the initial system in $E\times E$.

3). Let $a,b\,(a\neq b)$ be an arbitrary given elements from E. Then we have for any given $c,d\in E$: if $c\neq d$, then

$$\begin{cases}(x,a,y)=c;\\(x,b,y)=d;\end{cases}\Leftrightarrow\begin{cases}x\cdot((x\cdot y)*a)=c;\\x\cdot((x\cdot y)*b)=d;\end{cases}\Leftrightarrow\begin{cases}\alpha_{x,y}(a)=c;\\\alpha_{x,y}(b)=d;\end{cases}$$

Since G is a sharply 2-transitive permutation group on E, then by **Lemma 9** we obtain that there exists an unique such pair $(x,y)\in E\times E$.

If $c=d$, then we have

$$\begin{cases}(x,a,y)=c;\\(x,b,y)=c;\end{cases}\Leftrightarrow\begin{cases}x\cdot((x\cdot y)*a)=c;\\x\cdot((x\cdot y)*b)=c;\end{cases}\Leftrightarrow\begin{cases}(x\cdot y)*a=x\cdot c;\\(x\cdot y)*b=x\cdot c;\end{cases}\Rightarrow$$

$$(x\cdot y)*a=(x\cdot y)*b$$

$$((x\cdot y)*a)\cdot((x\cdot y)*b)=0$$

$$(x\cdot y)*(a\cdot b)=0$$

Since $a\neq b$, then $a\cdot b\neq 0$. So we get

$$x\cdot y=0,$$

i.e. $x=y$. Then pair $(x,y)=(c,c)$ is an unique solution of the initial system. □

As we can see from **Lemma 10**, the system $<E,(x,t,y),0,1>$ is a DK-ternar [13] without the conditions 6a) and 6b) of **Definition 2** from [13], and the operation (x,∞,y) is a supplemented operation to it.

Lemma 11. *The following statements are true:*

1). *The operation* (x,a,y) *is a quasigroup for any given* $a\neq 0,1$;

2). $(x,(u,z,v),y)=((x,u,y),z,(x,v,y))$;

3). *The permutation* $\alpha_{a,b}$ *is an automorphism of the operation* (x,c,y) *for any given* $a,b,c\in E$, $a\neq b$; *i.e. any operation* (x,c,y) *admits the sharply 2-transitive automorphism group* G.

Proof. 1). It is proved in [13].

2). As we can see from **Lemma 9**,

$$G=\{\alpha_{a,b}|\alpha_{a,b}(t)=(a,t,b),, a\neq b, a,b\in E\}.$$

Then we have for any $x,y,u,v,z\in E, x\neq y, u\neq v$:

$$\alpha_{x,y}\cdot\alpha_{u,v}(z)=\alpha_{w,s}(z) \tag{14}$$

for some $w,s\in E, w\neq s$. We obtain from (14)

$$\alpha_{x,y}\cdot\alpha_{u,v}(z)=\alpha_{x,y}((u,z,v))=(x,(u,z,v),y);$$
$$w=\alpha_{w,s}(0)=\alpha_{x,y}\cdot\alpha_{u,v}(0)=\alpha_{x,y}(u)=(x,u,y);$$
$$s=\alpha_{w,s}(1)=\alpha_{x,y}\cdot\alpha_{u,v}(1)=\alpha_{x,y}(v)=(x,v,y).$$

So

$$(x,(u,z,v),y)=((x,u,y),z,(x,v,y)).$$

When $x=y$ or $u=v$ the last identity is a trivial corollary of 1), **Lemma 10**.

3). Let a,b,c be arbitrary given elements from E, $a\neq b$. Then we have from 2)

$$(a,(x,c,y),b)=((a,x,b),c,(a,y,b)),$$
$$\alpha_{a,b}((x,c,y))=(\alpha_{a,b}(x),c,\alpha_{a,b}(y)),$$

i.e. the permutation $\alpha_{a,b}$ is an automorphism of the operation (x,c,y). □

Lemma 12. *The operations* (x,a,y) *and* $(x,0,y)=y\cdot x$ *are orthogonal for any* $a\in E$.

Proof. We prove at first the following identity

$$((x\cdot y)\cdot x)*u^{-1}=((x\cdot u)\cdot z)*(u\cdot(x\cdot z))^{-1}, \tag{15}$$

for any $x,u,z\in E, u\neq 0, u\neq x\cdot z$. Really, we have from (5) for any $t\neq 0$

$$(x\cdot u)\cdot(x\cdot(u\cdot t))=h_{x,u}(t)=h_{h_{x,u}(1)}(t)=h_{x,u}(1)*t=((x\cdot u)\cdot(x\cdot(u\cdot 1)))*t,$$

$$(x\cdot u)\cdot(x\cdot(u\cdot 1))=((x\cdot u)\cdot(x\cdot(u\cdot t)))*t^{-1}. \tag{16}$$

If $t=u$, then we have from (16)

$$(x\cdot u)\cdot(x\cdot(u\cdot 1))=((x\cdot u)\cdot x)*u^{-1}. \tag{17}$$

If $t=u\cdot(x\cdot z)$, then we obtain from (16)

$$(x\cdot u)\cdot(x\cdot(u\cdot 1))=((x\cdot u)\cdot z)*(u\cdot(x\cdot z))^{-1}. \tag{18}$$

The identity (15) follows from (17)-(18).

Further on, we have for any given $a,b,c\in E$

a). If $c=0$, then

$$\begin{cases}(x,a,y)=b;\\(x,\Diamond,y)=0;\end{cases}\Leftrightarrow\begin{cases}x\cdot((x\cdot y)*a)=b;\\ y\cdot x=0;\end{cases}\Leftrightarrow x=y=b,$$

i.e. the pair $(x,y)=(b,b)$ is an unique solution of the initial system.

b). If $a=0$ then

$$\begin{cases}(x,a,y)=b;\\(x,\Diamond,y)=c;\end{cases}\Leftrightarrow\begin{cases}x=b;\\ y\cdot x=c;\end{cases}\Leftrightarrow(x,y)=(b,c/b).$$

c). Let $a\neq 0, c\neq 0$. Then

$$\begin{cases}(x,a,y)=b;\\(x,\Diamond,y)=c;\end{cases}\Leftrightarrow\begin{cases}x\cdot((x\cdot y)*a)=b;\\ y\cdot x=c;\end{cases}\Leftrightarrow\begin{cases}x\cdot((x\cdot y)*a)=b;\\ x=y\cdot c;\end{cases}\Leftrightarrow$$

$$\Leftrightarrow\begin{cases}(y\cdot c)\cdot(((y\cdot c)\cdot y)*a)=b;\\ x=y\cdot c;\end{cases}\Leftrightarrow\begin{cases}((y\cdot c)\cdot y)*a=(y\cdot c)\cdot b;\\ x=y\cdot c;\end{cases}$$

Let us denote: $z=c^{-1}*y$, $b'=c^{-1}*b$. Then the last system is equivalent to the following system

$$\begin{cases}(z\cdot 1)\cdot z=((z\cdot 1)\cdot b')*a^{-1};\\ x=(c*z)\cdot c;\end{cases} \tag{19}$$

If $u=1$ then we obtain from (15)

$$(x\cdot 1)\cdot x=((x\cdot 1)\cdot z)*(1\cdot(x\cdot z))^{-1}, \tag{20}$$

for any $x, z \in E, x \cdot z \neq 1$. Using (20) in (19), we obtain the following system

$$\begin{cases} ((z\cdot 1)\cdot b')*(1\cdot(z\cdot b')) = ((z\cdot 1)\cdot b')*a \ ; \\ x = (c*z)\cdot c; \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} \left[\begin{array}{l} 1\cdot(z\cdot b') = a; \\ (z\cdot 1)\cdot b' = 0; \end{array}\right. \\ x = (c*z)\cdot c; \end{cases} \Leftrightarrow \begin{cases} \left[\begin{array}{l} (c \ \ *y)\cdot(c \ \ *b) = 1\cdot a, \\ (c^{-1}*y)\cdot 1 = c^{-1}*b; \end{array}\right. \\ x = y\cdot c; \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} \left[\begin{array}{l} y\cdot b = c*(1\cdot a); \\ y\cdot c = b; \end{array}\right. \\ x = y\cdot c; \end{cases} \Leftrightarrow \left[\begin{array}{l} \begin{cases} x = ((c*(1\cdot a))/b)\cdot c, \\ y = (c*(1\cdot a))/b; \end{cases} \\ \begin{cases} x = b; \\ y = b/c; \end{cases} \end{array}\right.$$

Assume that $(x,y) = (b, b/c)$, then we obtain from the initial system:

$$b\cdot((b\cdot(b/c))*a) = b,$$
$$(b\cdot(b/c))*a = 0.$$

Since $a \neq 0$, then

$$b\cdot(b/c) = 0,$$
$$b/c = b,$$
$$b = c\cdot b,$$
$$c = 0.$$

But $c \neq 0$ by the conditions of the case c). Then we obtain: the pair $(x,y) = (((c*(1\cdot a))/b)\cdot c, (c*(1\cdot a))/b)$ is an unique solution of the initial system in the case c). Proof is completed. □

Remark. Note that the collection P of operations $(x,a,y), a \in E$ and (x,∞,y) (or $(x,\Diamond,y)$) is a complete system of orthogonal operations, i.e. there is no such an operation $x \otimes y$ which is orthogonal to all operations from P.

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