Quasigroups and related systems, vol. 1, no. 1(1), 1994

LOOPS WITH UNIVERSAL ELASTICITY

Paraskowya N. Syrbu

Abstract

An identity is called universal for a loop Q(·), if it holds in this loop and in each principal isotope of Q(·). Loops with universal law of elasticity, i.e.

$$xy \cdot x = x \cdot yx,$$

are investigated in this note.

Invariant properties on isotopy of quasigroups, i.e. universal properties of quasigroups, represent an important part in the theory of quasigroups and loops. The theory of algebraic nets gives, in particular, examples of such properties: all loops which coordinate the same net are isotopic between themselves and the identities which follow from the closure conditions of this net are universal for each of these loops. For example, Bol and Moufang identities are universal for loops. Our subject of investigations is the loop with universal law of elasticity (or symply, with universal elasticity), its properties and connections with some classes of well known loops such as Bol and Moufang loops.

Let $Q(\cdot)$ be a loop with universal elasticity, i.e. a loop for which the law of elasticity

$$x \cdot yx = xy \cdot x$$

is universal. Denote by $Q(*)$ a principal isotope of $Q(\cdot)$, i.e.

$$x * y = R_a^{-1}x \cdot L_b^{-1}y,$$

where

$$T = (R_a^{-1}, L_b^{-1}, 1)$$

is the isotopy, R_a (L_a) is the right (left) multiplication by the element a. The universality of the identity

$$x \cdot yx = xy \cdot x$$

for $Q(\cdot)$ involves the fact that the law of elasticity

$$(x * y) * x = x * (y * x)$$

holds in every LP-isotope $Q(*)$ of $Q(\cdot)$. Now, replace $(*)$ by $(\cdot)$ in the last identity. We get the following identity in $Q(\cdot)$:

$$R_a^{-1}(R_a^{-1}x \cdot L_b^{-1}y) \cdot L_b^{-1}x = R_a^{-1}x \cdot L_b^{-1}(R_a^{-1}y \cdot L_b^{-1}x)$$

or

$$(xy/z)(b \backslash xz) = x(b \backslash [(by/z)(b \backslash xz)]) \qquad (1)$$

and

$$(bx/z)(b \backslash yx) = ([(bx/z)(b \backslash yz)]/z)x \qquad (2)$$

where we replaced a by z.

Proposition 1. *The law of elasticity is universal for the loop $Q(\cdot)$ iff the identity* (1) ((2)) *holds in the primitive loop $Q(\cdot,/,\backslash)$.*

The following proposition gives some properties of loops with universal elasticity.

Proposition 2. *If $Q(\cdot)$ is a loop with universal elasticity, then*

(i) *$Q(\cdot)$ is strong power-associative (i.e. every its element generates an associative subloop);*

(ii) *$N_l = N_r$, where N_l is the left and N_r is the right nucleus of $Q(\cdot)$;*

(iii) *All (three) nuclei of $Q(\cdot)$ coincide iff each element of the middle nucleus is a Bol element.*

Proof. **(i)** It is known (see [1], pp.46-47) that if the identity

$$x \cdot x^2 = x^2 \cdot x$$

is universal for the loop $Q(\cdot)$, then $Q(\cdot)$ is strong power-associative.

(ii) To prove this we will need the following identity:

$$(x/z) \cdot yx = ([(x/z) \cdot yx]/z)x$$

(it follows from (2) taking $b=e$, where e is the unit of $Q(\cdot)$, $y \to yz, x \to bx$; here and below by "$x \to y$" we will mean "x is replased by y"). If $x \in N_r$ in the last identity, then

$$(x/z)y = [(x/z) \cdot yz]/z,$$

or

$$xy \cdot z = x \cdot yz,$$

i.e. $x \in N_l$ and $N_r \subseteq N_l$. Conversely, taking $z=e$ in (1) we have:

$$xy \cdot (b \backslash x) = x(b \backslash [by \cdot (b \backslash x)]).$$

If $x \in N_l$ in the last identity, then

$$y(b\backslash x) = b\backslash[by\cdot(b\backslash x)],$$

or

$$b\cdot yx = by\cdot x,$$

i.e. $x \in N_r$, and so $N_r = N_l$.

(iii) Let us remind that an element a of the loop $Q(\cdot)$ is called a *Bol element* if the equality

$$a(x\cdot ay) = (a\cdot xa)y$$

is valid for every $x, y \in Q$. Suppose that a is a Bol element of the loop $Q(\cdot)$ and $a \in N_m$, where by N_m we denote the middle nucleus of $Q(\cdot)$:

$$N_m = \{a \in Q | xa\cdot y = x\cdot ay \text{ for every } x, y \in Q\}.$$

Then

$$a(xa\cdot y) = a(x\cdot ay) = (a\cdot xa)y = (ax\cdot a)y,$$

so

$$a(xa\cdot y) = (a\cdot xa)y,$$

or, after replacing $x \to xa$,

$$a\cdot xy = ax\cdot y,$$

i.e. $a \in N_l = N_r$, and $N_m \subseteq N_l = N_r$. Further, if a is a Bol element of $Q(\cdot)$ and $a \in N_l = N_r$, then

$$ax\cdot ay = a(x\cdot ay) = (a\cdot xa)y = (ax\cdot a)y.$$

So

$$ax\cdot ay = (ax\cdot a)y,$$

or after replacing $x \to a\backslash x$,

$$x\cdot ay = xa\cdot y,$$

i.e. $a \in N_m$ and

$$N_l = N_r \subseteq N_m.$$

Now this proposition follows from the next result of Florea (see [3]): all (three) nuclei of a loop coincide if and only if each element of these nuclei is a Moufang and Bol element at the same time.

Remark 1. A loop with universal elasticity satisfies the equality

$$x^p y\cdot x^q = x^p\cdot yx^q$$

for every $x, y \in Q$, where p and q are arbitrary integers (for the proof of this proposition see [2]).

A loop $Q(\cdot)$ is called a *middle Bol loop* if it satisfies the identity

$$(z/x)(y\backslash z) = z(yx\backslash z).$$

Proposition 3. *A loop $Q(\cdot)$ with universal elasticity and such that*

$$yb \cdot b = yz \cdot z$$

for all $y, b, z \in Q$, is a middle Bol loop.

Proof. Make the replacement $y \to x \backslash y$ in (1), then

$$[b(x \backslash y)/z](b \backslash xz) = b(x \backslash [(y/z)(b \backslash xz)]).$$

Now replace $y \to yz$ in the last identity:

$$[b(x \backslash yz)/z](b \backslash xz) = b(x \backslash [y(b \backslash xz)]),$$

or, if $x = yz$:

$$(b/z)(b \backslash (yz \cdot z)) = b(yz \backslash [y(b \backslash (yz \cdot z))]).$$

Suppose that

$$yb \cdot b = yz \cdot z,$$

for all $y, b, z \in Q$. Then

$$yb = (yz \cdot z)/b,$$

or

$$b = (yz \cdot z)/(y \backslash b),$$

if we replace $b \to y \backslash b$. So,

$$b \backslash (yz \cdot z) = y \backslash b,$$

and

$$(b/z)(y \backslash b) = b(yz \backslash b)$$

for all $b, y, z \in Q$, i.e. $Q(\cdot)$ is a middle Bol loop.

Remark 2. We will see in what follows that the loop $Q(\cdot)$ under the conditions of **Proposition 3** is associative and so is an abelian group.

A loop $Q(\cdot)$ is called a *LIP-loop* (a *RIP-loop*), i.e. a *loop with left inverse property* (*right inverse property*) if

$$x^{-1} \cdot xy = y \quad (yx \cdot x^{-1} = y)$$

holds for every $x, y \in Q$. A loop is called an *IP-loop* if it is a *LIP*-loop and a *RIP*-loop at the same time.

Proposition 4. *A loop with universal elasticity satisfies the left alternative law*

$$x \cdot xy = x^2 \cdot y$$

iff it has the right inverse property.

Proof. Denote by e the unit element of the loop $Q(\cdot)$ and replace in (1) $z \to x^{-1}$ and $y \to e$. As $Q(\cdot)$ is strong power-associative, we get

$$x^2b^{-1} = x(b\backslash[(b/x^{-1})b^{-1}]). \tag{3}$$

If $Q(\cdot)$ satisfies the left alternative law, then

$$x^2b^{-1} = x\cdot xb^{-1}$$

and from (3) we get

$$xb^{-1} = b\backslash[(b/x^{-1})b^{-1}],$$

that is

$$b/x^{-1} = bx$$

or

$$bx\cdot x^{-1} = b$$

for every $x,b\in Q$ (see **Remark 1**). Conversely, if $Q(\cdot)$ has the right inverse property, then

$$b/x^{-1} = bx$$

and using **Remark 1** and the identity (3), we have:

$$x^2b^{-1} = x[b\backslash(bx\cdot b^{-1})] = x[b\backslash(b\cdot xb^{-1})] = x\cdot xb^{-1},$$

i.e. $Q(\cdot)$ satisfies the left alternative law.

Analogously, but begining with (2), we can prove the following

Proposition 5. *A loop with universal elasticity satisfies the right alternative law*

$$yx\cdot x = yx^2,$$

iff it has the left inverse property.

Corollary. *A loop with universal elasticity is a IP-loop iff it satisfies the left and right alternative laws.*

Proposition 6. *A loop with universal elasticity has the left inverse property iff it has the right inverse property.*

Proof. Let $Q(\cdot)$ be a *LIP*-loop with universal elasticity. From the identity

$$bx\cdot(b\backslash(by\cdot x)) = (bx\cdot y)x \tag{4}$$

(this identity follows from (2) when $y\to by, z=e$), if $y=(bx)^{-1}$, we get

$$b[(bx)^{-1}x] = b(bx)^{-1}x \tag{5}$$

(the elasticity involvs ${}^{-1}a = a^{-1}$). Now let us make the replacement $y\to yz$ and $b=e$ in (1):

$$x(y\cdot xz) = [(x\cdot yz)/z]\cdot xz \tag{6}$$

and suppose that $y=(xz)^{-1}$ in (6). Then

$$[x/xz]z = x((xz)^{-1}z),$$

and if $x = b, z = x$:

$$[b/bx]x = b[(bx)^{-1}x]. \tag{7}$$

From (5) and (7) we have

$$b/bx = b(bx)^{-1}$$

and so

$$bx^{-1} \cdot x = b$$

for every $x, b \in Q$, i.e. $Q(\cdot)$ is a RIP-loop. Conversely, let $Q(\cdot)$ be a RIP-loop with universal elasticity. Substitute $y \to (xz)^{-1}$ in the identity (6):

$$[x(xz)^{-1}]z = x((xz)^{-1}z). \tag{8}$$

Take in (4) $y = (bx)^{-1}$:

$$b(bx \backslash x) = b(bx)^{-1} \cdot x.$$

Now, because of the last identity and (8) we get:

$$b(bx \backslash x) = b((bx)^{-1}x),$$

or

$$b(b^{-1}x) = x,$$

for all $x, b \in Q$, i.e. $Q(\cdot)$ is a LIP-loop.

Corollary. *If $Q(\cdot)$ is a loop with universal elasticity, then the following properties are equivalent in $Q(\cdot)$:*

right inverse property;

left inverse property;

right alternative law;

left alternative law.

Now, let us consider IP-loops with universal elasticity. Each of the identities (1) and (2) is equivalent in such a loop to the following identity:

$$(xz \cdot by)z \cdot bx = xz \cdot b(yz \cdot bx) \tag{9}$$

(Indeed, in a IP-loop are valid the equalities:

$$x \backslash y = x^{-1}y$$

and

$$x/y = xy^{-1}$$

for every $x, y \in Q$. Now, to prove this use the last two equalities in (1) and (2) and make the replacements $x \to xz^{-1}, y \to b^{-1}y, z^{-1} \to z, b^{-1} \to b$ in (1) and $x \to b^{-1}x, y \to yz^{-1}$ in (2)).

It is known (see [4]) that in a IP-loop all (three) nuclei coincide. Denote by N the nucleus of the IP-loop $Q(\cdot)$.

Proposition 7. *The commutative IP-loop $Q(\cdot)$ with universal elasticity and $x^2 \in N$ for all $x \in Q$, is associative.*

Proof. To prove this, make the replacement $z \to xz$ and $b \to bx$ in (9) and use **Propositions 4,5** and the fact that $x^2 \in N$, for all $x \in Q$:

$$[z(bx \cdot y) \cdot xz]b = z[bx \cdot (y \cdot xz)b].$$

Now, if $y=z$ in the last identity then

$$(bx \cdot xz)b = z(bx)^2,$$

or, using the commutativity of $Q(\cdot)$ and substituting $x \to b^{-1}x$:

$$x(b^{-1}x \cdot z) \cdot b = zx^2,$$

$$x(b^{-1}x \cdot z) = zx^2 \cdot b^{-1} = x^2 z \cdot b^{-1} = x^2 \cdot zb^{-1} = x(x \cdot zb^{-1}),$$

$$b^{-1}x \cdot z = x \cdot zb^{-1},$$

$$xb^{-1} \cdot z = x \cdot b^{-1}z,$$

so, the loop $Q(\cdot)$ is an abelian group.

Corollary. *A IP-loop $Q(\cdot)$ with universal elasticity and such that $x^2 = 1$ for all $x \in Q$ is associative.*

Proposition 8. *The identity (9) involves both inverse properties in the loop.*

Proof. Substitute $b = e$ and $y = z^{-1}$ in (9). Then we have:

$$(xz \cdot z^{-1})z \cdot x = xz \cdot x,$$

or

$$xz \cdot z^{-1} = x$$

for all $x, z \in Q$, i.e. $Q(\cdot)$ is a RIP-loop. Analogously, from (9) (making the replacement $z = e, y = b^{-1}$) we have

$$x \cdot bx = x \cdot b(b^{-1} \cdot bx),$$

or

$$x = b^{-1} \cdot bx$$

for every $x, b \in Q$; hence, $Q(\cdot)$ is a LIP-loop.

Corollary 1. *The loop which satisfies the identity (9) is a loop with universal elasticity.*

Corollary 2. *The identity* (9) *is universal for* $Q(\cdot)$ *iff* $Q(\cdot)$ *is a Moufang loop.*

Indeed, according to the information given up, the identities (1) and (9) are equivalent in an *IP*-loop.

Remark 3. The identity (1) follows from the identity (9) but, in general, these two identities are not equivalent. For instance, the loop from the example given below satisfies the identity (1) but it is not an *IP*-loop, hence it does not satisfy (9).

	1	2	3	4	5	6	7	8
1	1	2	3	4	5	6	7	8
2	2	1	4	3	6	5	8	7
3	3	4	1	2	7	8	6	5
4	4	3	2	1	8	7	5	6
5	5	6	7	8	1	2	3	4
6	6	5	8	7	2	1	4	3
7	7	8	6	5	3	4	1	2
8	8	7	5	6	4	3	2	1

Using the computer the author proved that in loops of order n, where $n \leq 6$, the identity (1) and Bol's middle law

$$(z/x)(y\backslash z) = z(yx\backslash z)$$

are equivalent. For other loop orders this question is open.

References

1. *Belousov V.D.* Algebraic nets and quasigroups. (Russian), Kishinev, "Shtiintsa", 1971.

2. *Shelekhov A.M., Shestakova M.A.* About identities in weakly associative loops. (Russian), Probl. teorii tkanei i kvazigrupp. Kalinin, 1985, p.115-121.

3. *Florea I.A.* Loops with one of the inverse properties. (Russian), Izv. AN MSSR, v.7(1965), p.68-79.

4. *Belousov V.D.* Foundations of the theory of quasigroups and loops. (Russian), Moskow, "Nauka", 1967.

Syrbu P.N. **Ph.D.**
department of quasigroup theory,
Institute of Mathematics,
Academy of Sciences of Moldova,
5, Academiei str.,
Kishinau, 277028,
Moldova.

Received August 10, 1993