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OSBORN'S G-LOOPS

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Abstract

It is proved, that if in a loop Q(·) the equality

$$(\cdot)_{I^{-1}x} = {}_{Ix}(\cdot)$$

holds for every $x \in Q$, then Q(·) is a G-loop. From this result it follows that:

a) An Osborn's loop Q(·) in which $x^2 \in N$ for every $x \in Q$ is a G-loop;
b) Every *i*-loop is a G-loop.

In the present work we continue the study of the class of *G*-loops which sprang from works [1], [2]. It is proved that if in the loop Q(·) the equality

$$(\cdot)_{I^{-1}x} = {}_{Ix}(\cdot)$$

holds for every $x \in Q$, then Q(·) is a *G*-loop.

Firstly we remind for some definitions and results which are necessary for proof of the main result of the present work.

The operation $(\cdot)_a$, defined by the equality

$$(\cdot)_a = (\cdot)^{(L_a, 1, L_a)}$$

is called the *right derivative operation of* (·). Analogously, the operation

$${}_a(\cdot) = (\cdot)^{(1, R_a, R_a)}$$

is called the *left derivative operation of* (·) (a is a fixed element of Q, $Q(\cdot)$ is a loop).

A loop $Q(\cdot)$ is called a *G-loop*, if all the right and left derivative operations of the loop $Q(\cdot)$ are isomorphic to the operation (·).

A loop Q(·) is a *G*-loop if and only if every loop $Q(\cdot)$ which is isotopic to Q(·) will be isomorphic to Q(·) (see [3]).

A loop $Q(\cdot)$ in which the equality

$$I(xy) \cdot I^2x = Iy$$

holds is called an WIP_1-loop. In a WIP_1-loop translations L_x and R_x ar connected as follows:

$$IL_{I^{-1}x}I^{-1} = R_{Ix}^{-1}, \quad I^{-1}R_{Ix}I = L_{I^{-1}x}^{-1}. \tag{1}$$

If $T=(\alpha,\beta,\gamma)$ is an autotopy of an WIP_1-loop $Q(\cdot)$, then

$$T_1 = (I\gamma I^{-1}, I^2\alpha I^{-2}, I\beta I^{-1}) \quad \text{and} \quad T_2 = (I^{-2}\beta I^2, I^{-1}\gamma I, I^{-1}\alpha I) \tag{2}$$

are also autotopies of the loop $Q(\cdot)$.

Theorem 1. *A loop $Q(\cdot)$ in which the equality*

$$(\cdot)_{I^{-1}x} = {}_{Ix}(\cdot) \tag{3}$$

holds for every $x \in Q$ is a G-loop.

Proof. Let (3) be fulfilled in the loop $Q(\cdot)$, then

$$(\cdot)^{(L_{I^{-1}x},1,L_{I^{-1}x})} = (\cdot)^{(1,R_{Ix},R_{Ix})},$$

whence we get the autotopy

$$T = (L_{I^{-1}x}, R_{Ix}^{-1}, L_{I^{-1}x}R_{Ix}^{-1}).$$

From T the equality

$$(I^{-1}x \cdot y) \cdot R_{Ix}^{-1}z = I^{-1}x \cdot R_{Ix}^{-1}(y \cdot z) \tag{4}$$

follows. In (4) putting $R_{Ix}z$ instead of z and after that Ix instead of x we get

$$xy \cdot z = x \cdot R_{I^2x}^{-1}(y \cdot zI^2x). \tag{5}$$

Let $z = I(x \cdot y)$ in (5), then

$$1 = x \cdot R_{I^2x}^{-1}(y \cdot I(x \cdot y) \cdot I^2x),$$

whence

$$Ix = R_{I^2x}^{-1}(y \cdot I(xy)I^2x),$$

$$y \cdot I(xy)I^2x = R_{I^2x}Ix,$$

$$y \cdot I(xy)I^2x = 1,$$

$$I(x \cdot y) \cdot I^2x = Iy,$$

i.e. $Q(\cdot)$ is an WIP_1-loop. Applying (2) and (1) to T we obtain

$$T_1 = (IL_{I^{-1}x}R_{Ix}^{-1}I^{-1}, I^2L_{I^{-1}x}I^2, IR_{Ix}^{-1}I^{-1}) =$$

$$= (R_{Ix}^{-1}IR_{Ix}^{-1}I^{-1}, R_{Ix}R_{Ix}^{-1}IR_{Ix}^{-1}I^{-1}, R_{Ix}R_{Ix}^{-1}IR_{Ix}^{-1}I^{-1}) =$$

$$= (\alpha^{-1}, R_{Ix}\alpha^{-1}, R_{Ix}\alpha^{-1}),$$

$$\alpha^{-1}1 = 1,$$

whence it follows

$$ {}_{Ix}(\cdot) = (\cdot)^{\alpha}. \tag{6} $$

From (3) and (6) it follows

$$ (\cdot)_{I^{-1}x} = {}_{Ix}(\cdot) = (\cdot)^{\alpha}, $$

i.e. the loop $Q(\cdot)$ is a G-loop.

A loop in which the identity

$$ xy \cdot \theta_x zx = (x \cdot yz) \cdot x $$

is fulfilled, where θ_x is a substitution depending on x, is called *Osborn's loop*.

It is proved in [4], that a loop $Q(\cdot)$ is an Osborn's loop if and only if

$$ (\cdot)_x = {}_{Ix}(\cdot) \tag{7} $$

for every $x \in Q$.

Statement 1. *A Osborn's loop $Q(\cdot)$ in which $x^2 \in N$ for every $x \in Q$ is a G-loop.*

Proof. Let in an Osborn's loop $x^2 \in N$ for every $x \in Q$, then $x^2 = n$, where $n \in N$ or $n^{-1}x \cdot x = 1$, whence

$$ n^{-1}x = I^{-1}x, $$
$$ x = nI^{-1}x, \tag{8} $$

Using (8) in (7) we get

$$ {}_{Ix}(\cdot) = (\cdot)_x = (\cdot)_{nI^{-1}x} = ((\cdot)_n)_{I^{-1}x} = (\cdot)_{I^{-1}x}, $$

so

$$ (\cdot)_{I^{-1}x} = {}_{Ix}(\cdot), $$

i.e. we have got (3). By **Theorem 1** the loop $Q(\cdot)$ is a G-loop.

In the work [5] *i*-loops have been studied. A loop $Q(\cdot)$ in which the equality

$$ xy \backslash ((xy) \cdot u)v = u(v \cdot (y \cdot x)) / yx \tag{9} $$

holds for arbitrary $x,y,u,v \in Q$ is called an *i-loop*. If $a \cdot b = c$, then $a \backslash c = b$, but $b = L_a^{-1}c$, so $a \backslash c = L_a^{-1}c$; similarly, if $ba = c$, then $c / a = R_a^{-1}c$. Now the equality (9) can be written as

$$ L_{xy}^{-1}((xy \cdot u) \cdot v) = R_{yx}^{-1}(u \cdot (v \cdot yx)) $$

or changing v by $R_{yx}^{-1}v$ as

$$ (xy \cdot u)R_{yx}^{-1}v = xy \cdot R_{yx}^{-1}(u \cdot v). \tag{10} $$

At the end of [5] the author notes: "It seems to be difficult to answer the question, are i-loops G-loops".

Statement 2. *Every i-loop is a G-loop.*

Proof. Let $Q(\cdot)$ be an i-loop, then (10) holds, whence it follows that

$$T_3 = (L_{xy}, R_{yx}^{-1}, L_{xy}R_{yx}^{-1})$$

is an autotopy of the loop $Q(\cdot)$ and then

$$(\cdot)_{xy} = {}_{yx}(\cdot). \tag{11}$$

Put in (11) $y = e$, then

$$(\cdot)_x = {}_x(\cdot). \quad 12)$$

Using (12) in (11) we get

$$(\cdot)_{xy} = {}_{xy}(\cdot) = {}_{yx}(\cdot),$$

i.e.

$${}_{xy}(\cdot) = {}_{yx}(\cdot). \tag{13}$$

Let $y = Ix$ in (13), then

$$(\cdot) = {}_{Ix\cdot x}(\cdot),$$

then $Ix \cdot x = n$, where $n \in N$ or $n^{-1}Ix \cdot x = 1$, but $I^{-1}x \cdot x = 1$ and then $n^{-1}Ix = I^{-1}x$ or $nI^{-1}x = Ix$. Change in (12) x by Ix, then

$${}_{Ix}(\cdot) = (\cdot)_{Ix} = (\cdot)_{nI^{-1}x} = ((\cdot)_n)_{I^{-1}x} = (\cdot)_{I^{-1}x},$$

i.e.

$$(\cdot)_{I^{-1}x} = {}_{Ix}(\cdot),$$

and we again obtain (3). By **Theorem 1** $Q(\cdot)$ is a G-loop.

Statement 3. *If in an Osborn's loop $Q(\cdot)$ $x^2 = 1$ for every $x \in Q$, then $Q(\cdot)$ is an abelian group (1 is the identity element of the loop $Q(\cdot)$).*

Proof. If $Q(\cdot)$ is an Osborn's loop and $x^2 = 1$ for every $x \in Q$, then $x = x^{-1} = Ix$ and

$$R_{Ix} = R_x. \tag{14}$$

But in the Osborn's loop

$$R_{Ix}^{-1} = L_x^{-1}R_xL_x. \tag{15}$$

From (14) and (15) it follows

$$L_xR_x^{-1} = R_xL_x$$

and then the autotopy

$$T=(L_x, R_{Ix}^{-1}, L_x R_{Ix}^{-1})$$

of the loop $Q(\cdot)$ takes the form:

$$T=(L_x, R_x^{-1}, L_x R_x^{-1})=(L_x, R_x^{-1}, R_x L_x),$$

whence

$$L_x y \cdot R_x^{-1} z = R_x L_x (y \cdot z). \tag{16}$$

Let $z=y$ in (16), then

$$L_x y \cdot R_x^{-1} y = 1,$$
$$L_x y = R_x^{-1} y,$$
$$R_x L_x y = y,$$

and then (16) has the form:

$$L_x y \cdot z = y \cdot R_x z,$$
$$xy \cdot z = y \cdot zx. \tag{17}$$

Let $z=1$ in (17), then

$$xy = yx.$$

From (17) and (18) it follows that $Q(\cdot)$ is an abelian group.

Statement 4. *An Osborn's loop $Q(\cdot)$ in which $x^2 \in N$ for every $x \in Q$ and $N \neq \{1\}$ is an extension of a group by means of an abelian group.*

Proof. The kernel N of the loop $Q(\cdot)$ is nontrivial and is a normal subloop of $Q(\cdot)$. The factor-loop $Q/N(\cdot)$ is an Osborn's loop in which $\bar{x}^2 = \bar{1}$ for every $\bar{x} \in Q/N$ ($\bar{1}$ is the identity of the loop $Q/N(\cdot)$). By **Statement 3** the loop $Q/N(\cdot)$ is an abelian group.

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