Quasigroups and related systems, vol. 1, no. 1(1), 1994

SHARPLY k-TRANSITIVE SETS OF PERMUTATIONS AND LOOP TRANSVERSALS IN S_n

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Abstract

The work is devoted to the investigation of sharply k-transitive sets of permutations which are a natural generalization of sharply k-transitive groups. Its main result is the establishment of the connection between such notions as sharply k-transitive sets of permutations, sharply k-transitive loops of permutations (introduced by F.Bonetti, G.Lunardon and K.Strambach) and loop transversals.

One special class of sets of permutations - sharply multiple transitive permutation sets - is studied in this work. These sets are the natural generalization of sharply multiple transitive permutation groups [3,4,5], but in contrast to the former are not described in the mathematical literature. On the other side, various known algebraical and geometrical problems are reduced to the research of conditions of existance and properties of sets mentioned above ([1], the end of §1.7; [2]).

To find the connection between the sets mentioned above, sharply k-transitive loops of permutations [9] and loop transversals in groups [6,7] is the main aim of this investigation. This connection allows us to describe the pure combinatorical objects - sharply multiple transitive sets of permutations - in terms of loop transversals of subgroups in groups. Moreover, since there exists a connection between finite projective planes and sharply 2-transitive sets of permutations (see [2,8]), the well-known problems of existence and of the number of projective planes of given order may be reformulated and studied in terms of loop transversals in the symmetric group S_n.

The main result of this work is

Theorem 1. *Let X be a set, cardX=n and 1≤k≤n. Then the following conditions are equivalent:*

1. *there exists a sharply k-transitive set of permutations on X;*
2. *there exists a sharply k-transitive loop of permutations on X;*
3. *there exists a loop transversal in* S_n *to* $St_{1,\dots,k}(S_n)$*;*

It is supposed that 1,2,... ∈*X.*

Using the results from [2,8] and the **Theorem 1**, we have

Theorem 2. *The following conditions are equivalent:*

1. *there exists a finite projective plane of order n;*
2. *there exists a loop transversal in* S_n *to* $St_{1,2}(S_n)$*;*

Let us pass to the detailed description of the paper results with all necessary definitions and notations.

1. The necessary definitions.

Definition 1.[1] A set M of permutations on X is called *sharply (strongly) k-transitive* (1≤k≤cardX), if for any two k-tuples $(a_1,\dots,a_k)$ and $(b_1,\dots,b_k)$ of different elements from X there exist the unique permutation $\alpha \in M$ satisfying the following conditions:

$$\alpha(a_i) = b_i$$

for any $i=1,\dots,k$. If the set M is closed relative to the multiplication of permutations, then M is a *sharply k-transitive group of permutations on* X.

Definition 2.[9] A loop G is called a *loop of permutations on the set X*, if there exists a map (action)

$$f: G \times X \to X,$$
$$f(g,x) = g(x),$$

which satisfies the following conditions:

1. $e(x) = x$ for any $x \in X$, where e is the unit of the loop G;
2. If b lies in the kernel of the loop G ([10], p.13), then
$$(ab)(x) = a(b(x))$$
for any $a \in G$ and $x \in X$;

3. There exists an element $x_0 \in X$ such that the set

$$G_{x_0} = \{a \in G | a(x_0) = x_0\}$$

is a subloop in G, and:

a) If $b \in G_{x_0}$ and $a \in G$, then

$$(ba)(x_0) = b(a(x_0));$$

b) If $a_1, a_2 \in G$ and $a_1(x_0) \neq x_0$, then

$$(a_2 a_1)(x_0) \neq a_2(x_0);$$

c) If $a_2 \notin G_{a_1(x_0)}$, then

$$(a_2 a_1)(x_0) \neq a_1(x_0);$$

If a loop of permutations on X is sharply k-transitive (as a set of permutations), then it is called a *sharply k-transitive loop of permutations on X.*

Definition 3. Let G be a group and H a subgroup of G. A complete system T of representatives of the left (right) cosets in G to H (unit $e \in T$) is called the *left (right) transversal in G to H.*

We can correctly introduce on T the following operation:

$$t_1 * t_2 = t_3 \Leftrightarrow t_1 t_2 = t_3 h, \quad h \in H. \tag{1}$$

If the system $< T, *, e >$ is a loop, then the transversal T in G to H is called the *loop transversal.*

In [6] it was proved

Lemma 1. *The following conditions are equivalent for left (right) transversal T in G to H:*

1. *T is a loop transversal;*
2. *T is a left (right) transversal in G to $\pi H \pi^{-1}$ for any $\pi \in G$;*
3. *$\pi H \pi^{-1}$ is a left (right) transversal in G to H for any $\pi \in G$.*

2. Proof of the Theorem 1.

Definition 4. Let M be a set of permutations on X. If **id**$\in M$ (where **id** is the identity permutation on X), then the set M is called a *reduced set* of permutations.

Lemma 2. *The following conditions are equivalent:*

1. *There exists a sharply k-transitive set of permutations on X;*
2. *There exists a reduced sharply k-transitive set of permutations on X.*

Proof. 2.⇒1. It is evident.

1.⇒2. Let M be a sharply k-transitive set of permutations on X. If $\mathbf{id} \in M$ then all is proved. Let $\mathbf{id} \notin M$. Let us take an arbitrary element $\alpha_0 \in M$ and introduce the following set:

$$M_0 = \{\alpha_0^{-1}\beta \mid \beta \in M\} = \alpha_0^{-1}M.$$

Since $\alpha_0 \in M$ then $\mathbf{id} \in M_0$, i.e. M_0 is a reduced set of permutations. Let $(a_1,\dots,a_k)$ and $(b_1,\dots,b_k)$ be any two k-tuples of different elements from X. Using the sharply k-transitivity of M we have that there exists the unique permutation $\beta_0 \in M$ such that

$$b_0(a_i) = a_0(b_i)$$

for any $i=1,\dots,k$, i.e.

$$(\alpha_0^{-1}\beta_0)(a_i) = b_i$$

for any $i=1,\dots,k$. Therefore there exists the unique permutation $\gamma_0 = \alpha_0^{-1}\beta_0 \in M$ such that

$$\gamma_0(a_i) = b_i$$

for any $i=1,\dots,k$, i.e. M_0 is a sharply k-transitive set of permutations.

Lemma 3. *The following conditions are equivalent:*

1. *M_0 is a reduced sharply k-transitive set of permutations on X;*
2. *M_0 is a loop transversal in S_n to $St_{1,\dots,k}(S_n)$; n=cardX.*

Proof. 1.⇒2. Let M_0 be a reduced sharply k-transitive set of permutations on X, *cardX=n*. The left (right) cosets in S_n to subgroup $H_0 = St_{1,\dots,k}(S_n)$ are sets of the following kind:

$$G_{a_1,\dots,a_k} = \{\alpha \in S_n \mid \alpha(i) = a_i, i = 1,\dots,k\},$$

where $(a_1,\dots,a_k)$ may be any k-tuples of different elements from X. Using the sharply k-transitivity of M_0, obtain that every coset of S_n to H_0 contains exactly one element from M_0, $\mathbf{id} \in M_0$ too. It means M_0 is the left transversal in S_n to H_0.

We must prove that M_0 is a loop transversal in S_n to H_0. It is sufficient (see **Lemma 1**) to prove that M_0 is a left transversal in S_n to $\pi H_0 \pi^{-1}$, where $\pi \in S_n$. Let π be an arbitrary element from S_n and

$$\pi(i) = a_i, \quad i = 1,\dots,k.$$

Then we have

$$(\pi h \pi^{-1})(a_i) = \pi h(i) = \pi(i) = a_i$$

for any $h \in H_0$, i.e.

$$\pi H_0 \pi^{-1} = St_{a_1,\dots,a_k}(S_n) = H_\pi.$$

Let us define the following sets:

$$G_\alpha = \alpha H_\pi,$$

where $\alpha \in M_0$. It is obvious that $G_{id} = H_\pi$. If we assume

$$\gamma \in (G_\alpha \cap G_\beta) \neq \varnothing, \quad \alpha \neq \beta,$$

then

$$\gamma = \alpha h_1 = \beta h_2, \quad h_1, h_2 \in H_\pi,$$

$$\alpha^{-1}\beta = h_1 h_2^{-1} \in H_\pi,$$

i.e.

$$\beta = \alpha H_\pi,$$

and

$$\alpha_2(a_i) = \alpha_1(a_i)$$

for any $i=1,\dots,k$. It's impossible, since M_0 is sharply k-transitive set. So

$$G_\alpha \cap G_\beta = \varnothing, \quad \text{if } \alpha \neq \beta.$$

Let g be an arbitrary element from S_n and

$$g(a_i) = c_i, \quad i = 1,\dots,k.$$

Then since M_0 is a k-transitive set of permutations, there exists an element $\alpha_0 \in M_0$ such that

$$\alpha_0(a_i) = c_i, \quad i = 1,\dots,k.$$

Therefore

$$(\alpha_0^{-1} g)(a_i) = \alpha_0^{-1}(c_i) = a_i$$

for any $i=1,\dots,k$, i.e.

$$(\alpha_0^{-1} g) \in H_\pi$$

and

$$g \in \alpha_0 H_\pi = G_{\alpha_0}.$$

It is proved that $\{G_\alpha\}_{\alpha \in M_0}$ is a complete system of left cosets in S_n to $St_{a_1,\dots,a_k}(S_n)$; therefore M_0 is a left transversal in S_n to $St_{a_1,\dots,a_k}(S_n)$.

2.⇒1. The reasoning is carried out in the opposite direction.

The proof is complete.

Lemma 4. *The following conditions are equivalent:*

1. *There exists a sharply k-transitive loop of permutations on X;*
2. *There exists a loop transversal in S_n to $St_{1,\dots,k}(S_n)$; n=cardX.*

Proof. 1.⇒2. A sharply k-transitive loop of permutations on **X** is a sharply k-transitive set of permutations on X. Therefore this implication is a corollary of **Lemma 3.**

2.⇒1. Let T be a loop transversal in S_n to $St_{1,\dots,k}(S_n)$. Then the system A=<T,*,**id**> is a loop, where "*" is defined in (1). We shall show that this loop is a sharply k-transitive loop of permutations on X.

The reflection f (see **Definition 2**) is defined naturally, because $T\subset S_n$. It is necessary to prove that the conditions **1.-3.** from **Definition 2** are satisfied. Let us denote $H=St_{1,\dots,k}(S_n)$.

The condition **1.** is satisfied, because **id**$\in T$ and **id** is the unit of the loop A.

Let us verify condition **2.** Let b_0 lie in the kernel of the loop A, i.e. for any $x,y\in A$:

$$(b_0*x)*y=b_0*(x*y),$$
$$(x*y)*b_0=x*(y*b_0),$$
$$(x*b_0)*y=x*(b_0*y).$$

From the last equality we have

$$xb_0h_1^{-1}yh_2^{-1}=xb_0yh_3^{-1}h_4^{-1},$$

where

$$x*b_0=xb_0h_1,\qquad b_0*y=b_0yh_3,$$
$$(x*b_0)*y=(x*b_0)yh_2,\quad x*(b_0*y)=x(b_0*y)h_4,\quad h_1,h_2,h_3,h_4\in H.$$

Then for any $y\in A$

$$h_1^{-1}=(yh^*y^{-1})\in(yHy^{-1}),$$

where $h^*=(h_3^{-1}h_4^{-1}h_2)\in H$. Therefore

$$h_1^{-1}\in\bigcap_{y\in A}(yHy^{-1})=\bigcap_{g\in S_n}(gHg^{-1})=Core_{S_n}(H)=<\mathbf{id}>,$$

because T is a transversal in S_n to H. Therefore $h_1=\mathbf{id}$, and

$$x*b_0=xb_0$$

for any $x\in A$. Then for any $x\in X$ and $a\in A$

$$(a*b_0)(x)=(ab_0)(x)=a(b_0(x)).$$

Let us show that condition **3.** from **Definition 2** is satisfied. We will take $x_0=1\in X$. Then the set

$$G_1 = \{\alpha \in A | \alpha(1) = 1\}$$

is a subloop in A. Indeed, $\mathbf{id} \in G_1$, and if $\alpha, \beta \in G_1$ then we have for $\gamma = \alpha * \beta$:

$$\gamma(1) = (\alpha * \beta)(1) = (\alpha\beta h)(1) = (\alpha\beta)(1) = \alpha(1) = 1,$$

where $h \in H$; i.e. $\gamma \in G_1$, and G_1 is closed under the operation "*" in A.

If $b \in G_1$ and $a \in A$ then

$$(b * a)(1) = (bah)(1) = (ba)(1) = b(a(1)),$$

where $h \in H$, i.e. condition **3a.** is satisfied.

If $a_1, a_2 \in A$ and $a_1 \neq 1$ then

$$(a_2 * a_1)(1) = (a_2 a_1 h)(1) = (a_2 a_1)(1) = a_2(a_1(1)) \neq a_2(1),$$

because a_2 is a permutation from S_n, $h \in H$. Therefore the condition **3b.** is satisfied.

If

$$a_2 \notin G_{a_1(1)} = \{\alpha \in A | \alpha(a_1(1)) = a_1(1)\},$$

then

$$(a_2 * a_1)(1) = (a_2 a_1 h)(1) = (a_2 a_1)(1) = a_2(a_1(1)) \neq a_1(1),$$

where $h \in H$; i.e. condition **3c.** is satisfied.

It means that A is a loop of permutations on X. Using **Lemma 3** we have that A is a sharply k-transitive set of permutations on X. Therefore A is a sharply k-transitive loop of permutations on X. The proof is complete.

Theorem 1 is a simple corollary from **Lemmas 2-4**.

3. Proof of Theorem 2.

The author of this paper has proved in [8] the existence of correspondence between a projective plane of order n and the DK-ternar of order n coordinatizing this plane. Up to certain four fixed points in general position on the projective plane this is a 1-1 correspondence. In **Theorem 5** from [8] is proved that the cell permutations of a DK-ternar form a sharply 2-transitive set of permutations of degree n. It means that a projective plane of order n exists if and only if a sharply 2-transitive set of permutations of degree n exists (see [2] too). Using this reasonings we get **Theorem 2** from the **Theorem 1** when k=2.

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Received August 10, 1993.