ONE-SIDED $T$-QUASIGROUPS AND IRREDUCIBLE BALANCED IDENTITIES

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Abstract

Left and right $T$-quasigroups are considered. It is proved that all primitive left (right) $T$-quasigroups form the variety which can be characterized by two identities. Some varieties of primitive left (right) $T$-quasigroups and $T$-quasigroups characterized by irreducible balanced identities are picked out.

Introduction.

It is known that all primitive quasigroups isotopic to groups form the variety characterized by one identity [1].

The class of linear quasigroups plays the important role in this variety. As V.D.Belousov has shown in [1] these quasigroups are closely connected with irreducible balanced identities in quasigroups.

A quasigroup $Q(\cdot)$ is called linear (over a group) if a group $Q(+)$, its automorphisms $\varphi, \psi$ and an element $c \in Q$ exists such that

$$xy = \varphi x + c + \psi y$$

for all $x, y \in Q$.

The automorphisms $\varphi, \psi$ are called determining automorphisms for the quasigroup $Q(\cdot)$.

In [2] the concept of linear quasigroup was generalized as follows.

A quasigroup $Q(\cdot)$ is called a left (right) linear quasigroup if there exist group $Q(+)$, its automorphism $\varphi (\psi)$ and an one-to-one mapping $\beta (\alpha)$ of $Q$ onto $Q$ such that

$$xy = \varphi x + \beta y \quad (xy = \alpha x + \psi y)$$

for all $x, y \in Q$. 

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As it was shown in [2], left (right) linear quasigroups are closely connected with the left (right) nucleus in quasigroups. They also arised in [1] in the investigation of irreducible balanced identities in quasigroups.

All primitive left linear quasigroups form the variety characterized by the following identity:

$$[x(u \setminus y)]z = [x(u \setminus u)] \cdot (u \setminus yz).$$  \hspace{1cm} (2)

Analogously, all primitive right linear quasigroups are characterized by the identity

$$x[(y / u)z] = (xy / u) \cdot [(u / u)z]$$  \hspace{1cm} (3)

(\textbf{Corollary 2} [2]).

All primitive linear quasigroups also form the variety which can be characterized by the identities (2) and (3) (\textbf{Corollary 3} [2]) or the unique identity

$$xy \cdot uv = xu \cdot (\alpha_u y \cdot v)$$  \hspace{1cm} (4)

where $\alpha_u$ is a mapping of $Q$ in $Q$ depending on $u$ (\textbf{Theorem 1} [3]). It is easy to see that $\alpha_u$ is an one-to-one mapping of $Q$ onto $Q$:

$$\alpha_u y = [u \setminus (u / u)y \cdot u] / (u \setminus u).$$

The $T$-quasigroups, i.e. the quasigroups linear over abelian groups, are the special case of linear quasigroups. These quasigroups were introduced and studied in detail in [4,5]. The well known medial quasigroups are a special case of $T$-quasigroups.

In [6] it was proved that the $T$-quasigroups play a role in the theory of quasigroups comparable to that of abelian groups among groups. Namely, a quasigroup coincides with its centre iff it is a $T$-quasigroup (see \textbf{Theorem 6} [6]).

In [6] the variety of all primitive $T$-quasigroups is characterized by two identities: (4) and the identity

$$xy \cdot uv = (\beta_x v \cdot y) \cdot ux,$$  \hspace{1cm} (5)

where

$$\beta_x v = [(x((x / x)v)) / x] / (x \setminus x).$$

In this article we consider the one-sided $T$-quasigroups (left and right $T$-quasigroups) and prove that all primitive left (right) $T$-quasigroups form the variety, which can be characterized by two identities. We also pick out a number of varieties of primitive left (right) $T$-quasigroups and $T$-quasigroups characterized by irreducible balanced identities.
1. Left (right) $T$-quasigroups and their characterization.

The following case of a left linear quasigroup $Q(\cdot)$ arose in [1] due to V.D. Belousov when he studied quasigroups with irreducible balanced identities:

$$xy = \varphi x + \beta y,$$

where $Q(+) \text{ is an abelian group, } \varphi \text{ is its automorphism, } \beta \text{ is an one-to-one mapping of } Q \text{ onto } Q.$ Using this we say that a quasigroup $Q(\cdot)$ is a left (right) $T$-quasigroup, briefly, a $LT$-quasigroup ($RT$-quasigroup) if $Q(\cdot)$ is a left (right) linear quasigroup over an abelian group.

First, we recall that the primitive quasigroup $Q(, \backslash, /)$ corresponds to each quasigroup $Q(\cdot)$, where

$$xy = z \iff x \backslash z = y \iff z / y = x.$$

We also note that according to Lemma 1 [2] a left linear quasigroup, which is simultaneously a right linear quasigroup, is a linear quasigroup. From this Lemma it immediately follows that if a $LT$-quasigroup is a $RT$-quasigroup, then it is a $T$-quasigroup.

**Theorem 1.** All primitive $LT$-quasigroup form the variety characterized by the following two identities

$$[x(u \backslash y)]z = [x(u \backslash u)](u \backslash yz),$$  

(6)

$$x(u \backslash u)(u \backslash y) = (y / u)(u \backslash x).$$  

(7)

All primitive $RT$-quasigroups are characterized by the identity (7) and the following identity

$$xy / u[(u / u)z].$$  

(8)

**Proof.** According to Corollary 2 [2] the identity (6) means that $Q(\cdot)$ is left linear over a group $Q(+)$. But (7) implies that $Q(+) \text{ is an abelian group.}$ Really, write (7) as follows

$$R_u^{-1}x \cdot L_u^{-1}y = R_u^{-1}y \cdot L_u^{-1}x,$$

(9)

where $R_u, L_u \text{ are the translations of } Q(\cdot) \text{ with respect to an element } u \in Q.$

$$R_u x = xu, \quad L_u x = ux.$$

Fixing in (9) the element $u$, we obtain that

$$xoy = yox,$$
where \( Q(\cdot) \) is a loop principally isotopic to \( Q(\cdot) \). Hence, the loop \( Q(\cdot) \) is commutative. By the *Albert's theorem* (see, for example, Theorem 1.4 [7]) the loop \( Q(\cdot) \) is an abelian group. Thus, \( Q(\cdot) \) is a \( LT \)-quasigroup.

Conversely, if \( Q(\cdot) \) is a \( LT \)-quasigroup, then it is left linear over an abelian group \( Q(\cdot) \) and by Corollary 2 [2] \( Q(\cdot) \) satisfies the identity (6). Next, since the group \( Q(\cdot) \) is abelian, then by the *Albert's theorem* each loop, isotopic to \( Q(\cdot) \), is commutative. Hence, the equality (9) is satisfied for all \( x, y, u \in Q \), i.e. the identity (7) holds. This completes the proof for the \( LT \)-quasigroups.

The proof for the \( RT \)-quasigroups is similar if we take into account that the identity (8) characterizes the variety of all right linear quasigroups (see Corollary 2 [2]).

In the introduction it was noted that the variety of all primitive \( T \)-quasigroups is characterized by two identities (4) and (5). From Theorem 1 an another characterization of \( T \)-quasigroups follows.

**Corollary 1.** The variety of all primitive \( T \)-quasigroups can be characterized by three identities (6),(7) and (8).

Indeed, it follows from above that if a \( LT \)-quasigroup \( Q(\cdot) \) is also a \( RT \)-quasigroup, then \( Q(\cdot) \) is a \( T \)-quasigroup. The converse follows from Theorem 1.

2. \( LT \)-quasigroups, \( RT \)-quasigroups, \( T \)-quasigroups and balanced identities.

Now we recall that an identity

\[
w_1 = w_2
\]

defined on a quasigroup \( Q(\cdot) \) is called balanced if each variable \( x \), which occurs on one side \( w_1 \) of the identity, occurs on the another side \( w_2 \) too and if no variable occurs in \( w_1 \) or \( w_2 \) more than once. This definition is due to A.Sade (see [8]). All balanced identities can be separated on two kinds. An identity \( w_1 = w_2 \) is kind 1 if the elements in \( w_1 \) and \( w_2 \) are equally ordered and is kind 2 otherwise.

An identity \( w_1 = w_2 \) is called reducible [1] if either
(i) each of $w_1$ and $w_2$ contains a "free element" $x$ so that $w_1$ is of the form $u_1x$ or $xv_1$ and $w_2$ likewise is the form $u_2x$ or $xv_2$ (where $u_i$ or $v_i$ represents a subword of the word $w_i$ for $i = 1,2$); or

(ii) $w_1$ has the product $xy$ of two free elements $x$ and $y$ as a subword and $w_2$ has one of the product $xy$ or $yx$ as a subword, or the dual of this statement.

An identity which is not reducible is called irreducible.

V.D. Belousov has proved the following remarkable theorem (Theorem 3 [1]): a quasigroup which satisfies an irreducible balanced identity is isotopic to a group.

Let

$$(x_1, x_2, ..., x_k) = (((((x_1x_2)x_3)...)x_k),$$

$$[x_1x_2...x_k] = x_1(x_2(...)x_{k-2}(x_{k-1}x_k)...))$$

and $m|n$ means that $m$ is a divisor of $n$. By $|\phi|$ we denote the order of the automorphism $\phi$ and let $S_Q$ denotes the set of all one-to-one mappings of $Q$ onto $Q$.

A mapping $\gamma \in S_Q$ is called a quasiautomorphism of a quasigroup $Q(\cdot)$ if there exist one-to-one mappings $\alpha, \beta \in S_Q$ such that

$$\gamma(xy) = \alpha x \cdot \beta y.$$

According to Lemma 2.5 [7] if $\gamma$ is a quasiautomorphism of a group $Q(+)$, then

$$\gamma x = R_\gamma \gamma_1 x = L_\gamma \gamma_2 x,$$

where $\gamma_1, \gamma_2$ are automorphisms of $Q(+)$;

$$R_\gamma x = x + s, \quad L_\gamma x = s + x.$$

V.D. Belousov in [1, p.79] has proved the following important for us statement, which can be formulated as follows

**Theorem 2** [1]. Let $Q(\cdot)$ be a JT-quasigroup:

$$\varphi x = \varphi x + \beta y,$$

$\varphi$ is an automorphism of the group $Q(\cdot)$ of the order $m$, $\theta$ is a permutation of the set $M = \{0,1,...,n\}$, where $m|n$, satisfying the conditions:

1. $\theta 0 \neq 0$,
2. $\theta n \neq n$,
3. $\theta i = (i \text{ mod } m)$

for each $i \in M$. Then the following irreducible balanced identity of kind 2

$$(\alpha y_0 y_1...y_{n-1} y_n) = (\alpha y_{\theta_0} y_{\theta_1}...y_{\theta_{(n-1)}} y_{\theta_n})$$

is satisfied in $Q(\cdot)$. 


Conversely, if the identity (10) holds in a quasigroup $Q(\cdot)$ for a nonidentity permutation $\Theta$ of $M$, then $Q(\cdot)$ is a LT-quasigroup:

$$xy = \varphi x + \beta y,$$

the automorphism $\varphi$ has a finite order $m$ which is a divisor of $(\Theta i - i)$ for each $i = 0, 1, \ldots, n$ and the permutation $\Theta$ satisfies the conditions (1), (2), and (3).

For our aims the next special case of Theorem 2 [1] is useful.

**Theorem 3.** Let $Q(\cdot)$ be a LT-quasigroup:

$$xy = \varphi x + \beta y,$$

$|\varphi| = m, m|n$. Then $Q(\cdot)$ satisfies the following irreducible balanced identity of kind 2:

$$(xy_0y_1\ldots y_{n-1}y_n) = (xy_ny_1\ldots y_{n-1}y_0). \quad (11)$$

Conversely, if a quasigroup $Q(\cdot)$ satisfies the identity (11), then $Q(\cdot)$ is a LT-quasigroup:

$$xy = \varphi x + \beta y,$$

and the order $m$ of the automorphism is a divisor of $n$.

For the proof it is enough to observe that the identity (11) is (10) if $\Theta = (0n)$, where $(0n)$ is a transposition (a cycle of the length two). Evidently, $\Theta = (0n)$ satisfies each of conditions (1), (2), (3).

Remark, that the case $m = n$ corresponds to the identity (11) of a "minimal length".

The analogue of Theorem 2 [1] is true for RT-quasigroups if we take the identity

$$[y_ny_{n-1}\ldots y_1y_0x] = [y_0y_{(n-1)}\ldots y_0y_0x]$$

instead of (10), but we shall formulate and prove the analog of Theorem 3 changing a little the outline of the proof of the corresponding statement from [1].

**Theorem 4.** Let $Q(\cdot)$ be a RT-quasigroup:

$$xy = \alpha x + \psi y,$$

$|\psi| = k, k|l$. Then the following irreducible balanced identity of kind 2:

$$[y_1y_{l-1}\ldots y_1y_0x] = [y_0y_{l-1}\ldots y_1y_1x] \quad (12)$$

is satisfied in $Q(\cdot)$.

Conversely, if the identity (12) is satisfied in a quasigroup $Q(\cdot)$ for some $l \geq 1$, then $Q(\cdot)$ is a RT-quasigroup:
and the order $k$ of the automorphism $\psi$ is a divisor of $l$.

**Proof.** Let $Q(\cdot)$ be a RT-quasigroup:

\[ xy = \alpha x + \psi y, \]

$|\psi| = k$, $k|l$. Then

\[
[y_i y_{i-1} \ldots y_1 y_0 x] = y_i (y_{i-1} \ldots (y_1 (y_0 x)) \ldots) = \\
= \alpha y_i + \psi \alpha y_{i-1} + \psi^2 \alpha y_{i-2} + \ldots + \psi^{i+1} x = \\
= \alpha y_i + \psi \alpha y_{i-1} + \psi^2 \alpha y_{i-2} + \ldots + \alpha y_0 + \psi x = \\
= y_0 (y_{i-1} \ldots (y_1 (y_1 x)) \ldots) = [y_0 y_{i-1} \ldots y_1 y_i x].
\]

Conversely, let the identity (12) be satisfied in a quasigroup $Q(\cdot)$ for some $l \geq 1$. By **Theorem 3** from [1] $Q(\cdot)$ is isotopic to a group $Q(\cdot)$:

\[ xy = \lambda x + \delta y \tag{13} \]

where $\lambda, \delta \in S_Q$. That is why from (12) we have

\[
[y_i y_{i-1} \ldots y_1 y_0 x] = y_i [y_{i-1} \ldots y_1 y_0 x] = \\
= \lambda y_i + \delta [y_{i-1} \ldots y_1 y_0 x] = \lambda y_0 + \delta [y_{i-1} \ldots y_1 y_i x].
\]

Fix $x$ and all $y_j, j \neq 0, l$, in this equality:

\[
\lambda y_i + \delta_1 y_0 = \lambda y_0 + \delta_1 y_i
\]

for some $\delta_1 \in S_Q$. But by **Lemma 11** from [1] a group $Q(\cdot)$ is abelian if the equality

\[
\alpha x + \beta y = \gamma y + \delta x
\]

is satisfied in $Q(\cdot)$ for some $\alpha, \beta, \gamma, \delta \in S_Q$.

Next show that $\delta$ from (13) is a quasiautomorphism of the abelian group $Q(\cdot)$. The identity (12) means that

\[ y_i (y_{i-1} \ldots (y_1 (y_0 x)) \ldots) = y_0 (y_{i-1} \ldots (y_1 (y_1 x)) \ldots). \tag{14} \]

Let $l \geq 3$, then (14) can be written as follows

\[
\lambda y_i + \delta (\lambda y_{i-1} + \delta [y_{i-2} \ldots y_1 y_0 x]) = \\
= \lambda y_0 + \delta (\lambda y_{i-1} + \delta [y_{i-2} \ldots y_1 y_i x]).
\]

Put in this equality

\[ x = \lambda y_0 = y_1 = y_{l-2} = 0, \]

where $0$ is the identity element of $Q(\cdot)$, then

\[ \lambda y_i + \delta_1 y_{i-1} = \delta (\lambda y_{i-1} + \delta_2 y_i) \]

for the corresponding $\delta_1, \delta_2 \in S_Q$. Hence, $\delta$ is a quasiautomorphism of $Q(\cdot)$.

Let now $l = 2$, then (14) implies

\[ \lambda y_2 + \delta (\lambda y_1 + \delta (y_0 x)) = \lambda y_0 + \delta (\lambda y_1 + \delta (y_2 x)). \]

Put here $\lambda y_0 = x = 0$, then
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\[ \lambda y_2 + \delta_3 y_1 = \delta(\lambda y_1 + \delta_4 y_2) \]

for $\delta_3, \delta_4 \in S_Q$. At last, let $l=1$, then from (14) we have

\[ \lambda y_1 + \delta(y_0 x) = \lambda y_0 + \delta(y_1 x), \]

or

\[ \lambda y_1 + \delta' x = \delta(y_1 x), \]

if we put $\lambda y_0 = 0$.

Thus, in all cases we obtain that $\delta$ is a quasiautomorphism of $Q(+)$. According to Lemma 2.5 [7]

\[ \delta x = s + \psi x, \]

where $\psi$ is an automorphism of $Q(+)$. Hence,

\[ xy = \lambda x + \delta y = \alpha x + \psi y, \tag{15} \]

where

\[ \alpha x = \lambda x + s. \]

Using (15) in (14) we have

\[ \alpha y_1 + \psi \alpha y_{l-1} + \psi^2 \alpha y_{l-2} + \ldots + \psi^{l-1} \alpha y_1 + \psi^l \alpha y_0 + \psi^{l+1} x = \]

\[ = \alpha y_0 + \psi \alpha y_{l-1} + \psi^2 \alpha y_{l-2} + \ldots + \psi^{l-1} \alpha y_1 + \psi^l \alpha y_l + \psi^{l+1} x \]

whence

\[ \alpha y_1 + \psi' \alpha y_0 = \alpha y_0 + \psi' \alpha y_1, \]

\[ \psi'(\alpha y_0 - \alpha y_l) = \alpha y_0 - \alpha y_l. \]

Therefore, $\psi' x = x$ for every $x \in Q$, so the order $|\psi|$ of the automorphism $\psi$ is a divisor of $l$. This completes the proof.

**Theorems 3 and 4 imply**

**Corollary 2.** Let $Q(\cdot)$ be a $T$-quasigroup:

\[ xy = \varphi x + c + \psi y, \]

$|\varphi| = m$, $|\psi| = k$, $m | n$, $k | l$. Then the identities (11), (12) are satisfied in $Q(\cdot)$. Conversely, if the identities (11) and (12) hold for certain $n, l \geq 1$ in quasigroup $Q(\cdot)$, then $Q(\cdot)$ is a $T$-quasigroup:

\[ xy = \varphi x + c + \psi y, \]

$|\varphi| | n$ and $|\psi| | l$.

**Proof.** Since every $T$-quasigroup is a $LT$-quasigroup and a $RT$-quasigroup, the first statement follows at once from **Theorems 3** and **4**. Conversely, according to **Theorem 4** if (12) is satisfied in a quasigroup $Q(\cdot)$ for some $|\varphi|$, then $Q(\cdot)$ is a $RT$-quasigroup:
\[ xy = \lambda x + \delta y = \alpha x + \psi y,\]

(see (15)) and \(|\psi|\) is a divisor of \(l\). Next, using the equalities (11) and (13), we can prove that \(\lambda\) is a quasiautomorphism of \(Q(+)\):
\[ \lambda x = \varphi x + t,\]

where \(t \in Q\), \(\varphi \in \text{Aut}Q(+)\) \(\text{and } |\varphi| \mid n\). The proof is similar to that of the case for \(\text{Theorem } 4\). Thus,
\[ xy = \lambda x + \delta y = \varphi x + t + s + \psi y = \varphi x + c + \psi y,\]

where \(c = t + s\), \(|\varphi| \mid n\), \(|\psi| \mid l\). This completes the proof.

3. Some subvarieties of the varieties of \(LT\)- (\(RT\)-) quasigroups and \(T\)-quasigroups.

The above proved results present the possibility to pick out some varieties of primitive \(LT\)-quasigroups, \(RT\)-quasigroups and \(T\)-quasigroups, which are characterized by irreducible balanced identities of kind \(2\) and depend on the orders of their determining automorphisms.

We begin with the following Lemma which means that the order of a determining automorphism \(\varphi\) (\(\psi\)) of a \(LT\)-quasigroup (\(RT\)-quasigroup) \(Q(\cdot)\) is its invariant and does not depend on a group over which \(Q(\cdot)\) is left (right) linear.

**Lemma 1.**

(i) Let \(Q(\cdot)\) be a \(LT\)-quasigroup and
\[ xy = \varphi x + \beta y = \varphi x \varphi \beta y,\]

where \(\varphi\) (\(\bar{\varphi}\)) is an automorphism of the abelian group \(Q(+)\) (\(Q(0)\)), \(\beta, \bar{\beta} \in S_Q\). Then \(\varphi x = R_a \varphi R_a^{-1} x\) for certain \(a \in Q\) (\(R_a x = x + a\)), i.e. \(|\varphi| = |\bar{\varphi}|\).

(ii) Let \(Q(\cdot)\) be a \(RT\)-quasigroup and
\[ xy = \alpha x + \psi y = \alpha x \alpha \psi y,\]

where \(\psi \in \text{Aut}Q(+)\), \(\bar{\psi} \in \text{Aut}Q(0)\). Then \(\psi y = R_a \bar{\psi} R_a^{-1} y\) for some \(a \in Q\), i.e. \(|\psi| = |\bar{\psi}|\).

**Proof.** Let
\[ xy = \varphi x + \beta y = \bar{\varphi} \alpha \bar{\beta} y,\]
\(\varphi \in \text{Aut}Q(+)\), \(\bar{\varphi} \in \text{Aut}Q(0)\).
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In this case the group $Q(o)$ is principally isotopic to the group $Q(+)$ By **Albert's Theorem** $Q(o)$ is isomorphic to $Q(+)$ Moreover, there exists such an element $a \in Q$ that

$$R_a(xo)y = R_ax + R_ay, \quad R_ax = x + a$$

(see the proof of **Albert's Theorem** in [7], p.17). Hence, using the equality

$$R_a^{-1}x = x - a,$$

we have

$$xy = \overline{\phi xo \overline{\beta} y} = R_a^{-1}(R_a\overline{\phi} x + R_a\overline{\beta} y) = R_a\overline{\phi} R_a^{-1}(x + a) + \overline{\beta} y = R_a\overline{\phi} R_a^{-1}x + \overline{\beta_1} y$$

($\overline{\beta_1} y = R_a\overline{\phi} 0 + \overline{\beta} y, 0$ is the identity element of $Q(+)$), since

$$\phi_1 = R_a\overline{\phi} R_a^{-1}$$

is an automorphism of $Q(+)$ Thus,

$$xy = \phi x + \beta y = \phi_1 x + \overline{\beta_1} y$$

whence by $x = 0$ have

$$\beta = \overline{\beta_1}, \phi = \phi_1, |\phi| = |\overline{\phi}|$$

The second part of Lemma 1 is proved analogously.

**Corollary 3.** If $Q(\cdot)$ is a T-quasigroup and

$$xy = \phi x + c + \psi y = \overline{\phi x + \overline{c} + \overline{\psi} y},$$

then

$$\phi x = R_a\overline{\phi} R_a^{-1},$$

$$\psi y = R_a\overline{\psi} R_a^{-1},$$

i.e. $|\phi| = |\overline{\phi}|, |\psi| = |\overline{\psi}|$.

The proof follows immediately from Lemma 2.

Now let $m,n$ be natural numbers Denote by $R_m^l (R_n^r)$ the class of all LT-quasigroups (RT-quasigroups) with determining automorphisms whose orders are devisors of $m$ (of $n$) In other words, a LT-quasigroup (a RT-quasigroup) $Q(\cdot)$ lies in $R_m^l (R_n^r)$ iff $xy = \phi x + \beta y$ ($xy = \alpha x + \psi y$) for certain abelian group $Q(+)$, its automorphism $\phi$ ($\psi$) such that $\phi^m = \varepsilon$ ($\psi^n = \varepsilon$), i.e. $|\phi| = m \quad |\psi| = n$. Here $\varepsilon$ is the identity mapping of $Q$.

By $R_{m,n}$ we denote the class of all T-quasigroups with a pair $(\phi, \psi)$ of the determining automorphisms such that
\[ \varphi^m = \psi^n = \varepsilon. \]

Hence, a \( T \)-quasigroup \( Q(\cdot) \) belongs to \( R_{m,n} \) iff

\[ xy = \varphi x + c + \psi y, \]

\[ |\varphi| \mid m \quad \text{and} \quad |\psi| \mid n \]

for some abelian group \( Q(+) \).

From Lemma 1 it follows at once that

\[ R^l_m \cap R^l_n = R^l_{(m,n)}, \quad (R^r_m \cap R^r_n = R^r_{(m,n)}) \]

where \((m,n)\) is the greatest common divisor of \( m,n \). In particular, if \( p,q \) are prime numbers, then

\[ R^l_p \cap R^l_q = R^l_1, \quad (R^r_p \cap R^r_q = R^r_1). \]

Next we prove

**Lemma 2.** \( R_{m,n} = R^l_m \cap R^r_n. \)

**Proof.** It is clear, that

\[ R_{m,n} \subseteq R^l_m \cap R^r_n. \]

Let \( Q(\cdot) \) occurs in \( R^l_m \) and \( R^r_n \). Then there exists abelian groups \( Q(+) \) and \( Q(0) \), their automorphisms \( \varphi \) and \( \overline{\psi} \), such that

\[ \varphi^m = \overline{\psi}^n = \varepsilon \]

and

\[ xy = \varphi x + \beta y = \alpha x \overline{\psi} y \]  \hspace{1cm} (16)

for some \( \alpha, \beta \in S_Q \). In this case there exists such an element \( a \in Q \) that

\[ R_a(x \overline{y}) = R_a x + R_a \overline{y} \]

(see the proof of Lemma 1). Hence, from (16) by \( x = 0 \) we have

\[ \beta y = \alpha \overline{\psi} y = R_a^{-1}(R_a \alpha 0 + R_a \overline{\psi} y) = \]

\[ = -a + a + \alpha 0 + R_a \overline{\psi} R_a^{-1}(a + y) = c + \psi y, \]

where

\[ c = \alpha 0 + R_a \overline{\psi} 0, \]

since \( \overline{\psi} R_a^{-1} \) is an automorphism of \( Q(+) \). Thus,

\[ |\psi| = |\overline{\psi}|, \]

\[ xy = \varphi x + c + \psi y, \]

and \( Q(\cdot) = R_{m,n} \), as required.
Now denote by \( \mathcal{R}_m^l \), \( \mathcal{R}_m^r \), \( \mathcal{R}_{m,n} \) the classes of corresponding primitive LT-quasigroups, RT-quasigroups and T-quasigroups.

**Theorem 5.**

(i) \( \mathcal{R}_m^l \) is a variety of primitive LT-quasigroups characterized by the identity

\[
(xy_0 y_1 \ldots y_n) = (x y_0 y_1 y_2 \ldots y_{n-1} y_0).
\]

(ii) \( \mathcal{R}_m^r \) is a variety of primitive RT-quasigroups characterized by the identity

\[
[y_n y_{n-1} \ldots y_0 x] = [y_0 y_{n-1} \ldots y_1 y_n x].
\]

(iii) \( \mathcal{R}_{m,n} \) is a variety of primitive T-quasigroups characterized by the identities (17) and (18).

**Proof.**

(i) Let \( Q(\cdot) \in \mathcal{R}_m^l \):

\[
xy = \varphi x + \beta y, \quad |\varphi| \leq m,
\]

then \( Q(\cdot) \) satisfies (17) by the first part of Theorem 2. Conversely, if \( Q(\cdot) \) satisfies (17), then it is a LT-quasigroup by the second part of Theorem 2 and

\[
xy = \varphi x + \beta y, \quad |\varphi| \leq m,
\]

i.e. \( Q(\cdot) \in \mathcal{R}_m^l \).

(ii) follows similarly from Theorem 3.

(iii) is a consequence of Lemma 2, (i) and (ii).

Next we consider some special cases of the above varieties.

The variety \( \mathcal{R}_1^l \) (\( \mathcal{R}_1^r \)) includes all quasigroups such that

\[
xy = x + \beta y \quad (xy = \alpha x + y), \quad \alpha, \beta \in S_Q
\]

over all abelian groups \( Q(+) \) (\( Q \) is a nonfixed set). These varieties are characterized by the identities

\[
y_0 \cdot y_1 = xy_1 \cdot y_0 \quad (y_1 \cdot y_0 x = y_0 \cdot y_1 x),
\]

respectively.

The variety \( \mathcal{R}_2^l \) (\( \mathcal{R}_2^r \)) includes all quasigroups from \( \mathcal{R}_1^l \) (\( \mathcal{R}_1^r \)) and quasigroups of the form

\[
xy = \varphi x + \beta y, \quad |\varphi| = 2
\]

\[
(xy = \alpha x + \psi y, \quad |\psi| = 2)
\]

If \( Q(\cdot) \in \mathcal{R}_2^l \) (\( Q(\cdot) \in \mathcal{R}_2^r \)), then \( Q(\cdot) \) satisfies the identity

\[
(y_0 \cdot y_1) y_2 = (x y_2 \cdot y_1) y_0,
\]

\[
(y_2 \cdot y_0 x) = y_0 (y_1 \cdot y_2 x),
\]
and conversely

Let \( p, q \) be simple numbers. Then \( \mathfrak{R}^I_p (\mathfrak{R}^r_q) \) contains all quasigroups from \( \mathfrak{R}^I_1 (\mathfrak{R}^r_1) \) and all LT-quasigroups (RT-quasigroups) with the determining automorphisms of the order \( p \) (of the order \( q \)). If \( Q(\cdot) \in \mathfrak{R}_{p,q} \), then it has one of the next forms:

\[
xy = \varphi x + c + \psi y, \quad |\varphi| = p, \quad |\psi| = q,
xy = \varphi x + c + y, \quad |\varphi| = p,
xy = x + c + \psi y, \quad |\psi| = q,
xy = x + c + y.
\]

Finally we note that the variety of all abelian groups is contained in every variety from \( \mathfrak{R}^I_m, \mathfrak{R}^r_n, \mathfrak{R}^r_{m,n} \) for any \( m, n \).

References


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